The Bogomolny Equations and Solutions for Einstein-Yang-Mills-Dilaton-σ Models

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Abstract

We derive Bogomolny equations for an Einstein-Yang-Mills-dilaton-σ model (EYMD-σ) on a static spacetime, showing that the Einstein equations are satisfied if and only if the associated (conformally scaled) three-metric is flat. These are precisely the static metrics for which which super-covariantly constant spinors exist. We study some general properties of these equations and then consider the problem of obtaining axially symmetric solutions for the gauge group $SU(2)$. 

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1 Introduction

The Bogomolny equations [1] have played a central role in the search for solutions of the Yang-Mills-Higgs (YMH) system of equations in the limit of vanishing self-interactions of the Higgs field. Because of their lower order, the Bogomolny differential equations prove a major simplification to the problem of finding analytical solutions for the YMH equations. It is natural therefore to seek similar equations in the study of Einstein-Yang-Mills-Higgs and related systems in which the coupling to gravity increases the (already high) non-linearities of the coupled, second order equations. This paper presents several new results along such lines.

The investigation of curved-space Bogomolny equations has interested many people. A modified version of the Euclidean Bogomolny equations was considered by Comtet [2] very early on while studying Yang-Mills-Higgs (YMH) systems (in the Prasad-Sommerfield limit) on fixed, static, curved-space backgrounds utilizing a spherically symmetric ansatz. Here a system of first order Bogomolny type equations was proposed that implied the YMH equations of motion provided the metric satisfied a differential constraint. After this, in an important work, Comtet, Forgács and Horváthy [3] treated the general fixed, static, curved-space background with no (spatial) symmetry assumptions. Again first order Bogomolny type equations were found that yielded the YMH equations of motion provided the metric satisfied the differential constraint:

\[ \Delta \ln \sqrt{|g_{00}|} = 0. \]  

(1)

Going beyond Comtet’s earlier work, these authors now asked about the compatibility of this constraint with the Einstein equations (here only considering the spherically symmetric case). Compatibility was found possible only for a very special physical solution: the Higgs field was required to take its vacuum value \( \text{Tr} \phi^2 = 1 \) and the metric described an extreme Reissner-Nordström black hole. As we shall see, this first constraint (that \( \text{Tr} \phi^2 = 1 \), which may be viewed as a sigma model constraint) is fundamental, and appears either implicitly or explicitly in most authors work.

Balakrishna and Wali [4], working with an Einstein-Yang-Mills-Higgs (EYMH) system with non-minimal \( R \text{Tr} \phi^2 \) coupling, and assuming a conformstatic metric with flat three-geometry, also obtained the same Bogomolny equations of Comtet et al. Here the Einstein equations were found to be satisfied as a consequence of the Bogomolny equations provided \( \text{Tr} \phi^2 = 1 \) and the choice of units \( 4\pi G = 1 \) was made. The sigma model constraint in fact means that there is no difference between minimal and non-minimal gravitational coupling for the solutions being considered; the choice of units corresponds to a balance between the attractive gravitational and repulsive gauge force of like particles, and will
be commented upon further in the sequel. Balakrishna and Wali also examined the large and small $r$ asymptotics of the resulting metric, assuming spherical symmetry. In a similar vein Cho et al.\cite{5} consider a (now minimally coupled) Einstein–Yang-Mills-Higgs-Dilaton (EYMHD) system for a static metric with flat three-geometry, their work assuming the constant $\text{Tr} \phi^2$ sigma model constraint. Analogous Bogomolny equations to those found earlier (but incorporating the dilaton) were obtained. Various multipole solutions generalising the Majumdar-Papapetrou \cite{6,7} electrovac solutions were found, as well as the Gross-Perry-Sorkin multi-monopole solution obtained from a five dimensional Kaluza-Klein theory \cite{8,9,10}.

Most recently Forgács \textit{et al.} \cite{11} have made explicit the recurring sigma model constraint noted above by considering the coupling of an Einstein-Yang-Mills system to a gauged sigma model (EYM-$\sigma$) or an Einstein-Yang-Mills-dilaton system with analogous coupling (EYMD-$\sigma$). They view this as the infinite mass limit of an EYMH system: the infinite mass forcing the length of the Higgs to assume its vacuum minimum. The key point is that when coupling to a sigma model field $n$, the differential constraint (1) is replaced by

$$\Delta \ln \sqrt{|g_{00}|} = \text{Tr}(D_inD^in),$$

and the Yang-Mills and sigma model equations of motion follow from a Bogomolny equation if and only if the metric satisfies this equation. Now (again with units $4\pi G = 1$) they show that (2) is equivalent to the (00) component of the Einstein equations for a general static metric. Although these authors do not prove the compatibility of their Bogomolny equation with the remainder of the Einstein equations they do present a particular case for which this compatibility holds. This particular case corresponds to spherical symmetry and the t’Hooft ansatz: the resulting ODE’s corresponds with those found in [4] noted above, which shows that the issue of the compatibility is nontrivial. Most recently Viet and Wali \cite{12} adopting a spherically symmetric ansatz show the compatibility of the Bogomolny equation with the equations of motion and Einstein equations of a non-minimally coupled EYMH system. Here again a sigma model constraint is imposed, and we again encounter the same equations of Forgács \textit{et al.} but now the full compatibility with the Einstein equations is shown within their ansatz.

In this paper we follow Forgács \textit{et al.} by initially considering an EYM-$\sigma$ model on an arbitrary, static spacetime. Our first result is to show that, with the assumption of static fields and units so that $4\pi G = 1$, the Bogomolny equations (see (19) or (20)) yield all of the field equations if and only if the (appropriately conformally scaled) three-metric is flat. This at once answers the question as to the compatibility of the Bogomolny equations with the remaining Einstein equations and justifies the ansatz of the various authors mentioned above. Interestingly the space-times we find Bogomolny equations for are precisely those
static space-times for which super-covariantly constant spinors exist [13]. This is consistent with the arguments of [14] who associate supersymmetric extensions to theories exhibiting Bogomolny bounds and suggests a supersymmetric extension exists to this theory. Having obtained our Bogomolny equations we proceed to investigate these in section three. First we show that an auxiliary magneto-static problem may be associated with the equations for any gauge group. This is particularly relevant for situations where the sigma model field is covariantly constant and leads to the Majumdar-Papapetrou electrovac solutions noted above. To examine the case of non-covariantly constant solutions we focus on the case of axially symmetric charge one solutions for the gauge group \( SU(2) \); the ansatz we adopt readily incorporates the spherically symmetric ansatz of the earlier works and we recover these. Here we also give various improved asymptotic solutions. Section four extends our earlier analysis to include a dilaton coupling: up to some physical redefinitions we find that the Bogomolny equations (and consequently their solutions) are unaltered. Our penultimate section considers axially symmetric \( SU(2) \) solutions to the Bogomolny equations with charge (or winding number) greater than one. Here we can illustrate covariantly constant solutions with arbitrary charge but are unable to find solutions to the ansatz of various authors for the non-covariantly constant case. We conclude with a brief discussion of our results.

2 The Bogomolny Equations

Our study of the EYM-\( \sigma \) system is based upon the action

\[
S = \int d^4x \sqrt{|g|} \left[ \frac{1}{16\pi G} R - \frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \text{Tr} D_\mu n D^\mu n + \frac{\lambda}{2} (\text{Tr} n^2 - 1) \right].
\]

(3)

Here \( G \) is the gravitational constant, \( R \) the Ricci scalar associated to the spacetime metric \( g_{\mu\nu} \) and the scalar field \( n \) is in the adjoint representation of the gauge group with associated field strength \( F \). Indices \( \mu, \nu, \ldots \) run from 0 to 3 and we are working with a signature \((- + + +)\). The Lagrange multiplier \( \lambda \) of the final term in the action imposes the \( \sigma \)-model constraint. This action has been considered previously in Ref.[11] as the infinite mass limit of spontaneously broken gauge theories with adjoint Higgs fields.

The field equations derived from (3) are

\[
\frac{1}{\sqrt{|g|}} D_\mu \left( \sqrt{|g|} F^{\mu\nu} \right) = [n, D^\nu n],
\]

(4)

\[
\frac{1}{\sqrt{|g|}} D_\mu \left( \sqrt{|g|} D^\mu n \right) = - \left( \text{Tr} D^\mu n D_\mu n \right) n,
\]

(5)
\[ R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \]  
(6)

where \( R_{\mu\nu} \) is the Ricci tensor, \( T_{\mu\nu} \) is the total energy-momentum tensor associated to the gauge field and the \( \sigma \)-model, and \( T = T_{\nu}^{\mu} \). (See [11] for details.)

We restrict our attention to the static, purely magnetic case \( (A_0 = 0) \) of this theory and for the purpose of this paper assume a static metric parameterised as

\[ ds^2 = -g_{00} dt^2 + g_{ij} dx^i dx^j = -V^2 dt^2 + h_{ij} V^2 dx^i dx^j. \]  
(7)

Comment on the extension to a stationary metric will be made in the sequel and here \( i, j \) run from 1 to 3. For such a metric \( R_{0i} = 0 \), and we note the identity \[ R_{00} = -V \Box_g V = -V^3 \Box_h V + V^2 h^{ij} \partial_i V \partial_j V, \]  
(8)

where \( \Box_{g(h)} \) is the scalar Laplacian with respect to the three-metric \( g_{ij} \) (or \( h_{ij} \)). Since the time derivative of every field is assumed to be zero we may construct a reduced action \( S \), with \( S = \int dt S \), and where

\[ S = \int d^3 x \sqrt{\hbar} \left[ \frac{1}{16\pi G} \left( R^{(3)}(h) + 2 \nabla^i \nabla_i \ln V - 2 \nabla^i \ln V \nabla_i \ln V \right) \right. \]
\[ -\left. \frac{1}{4} V^2 \text{Tr} F_{ij} F^{ij} - \frac{1}{2} \text{Tr} n D_i n D^i n + \frac{\lambda V^{-2}}{2} \left( \text{Tr} n^2 - 1 \right) \right]. \]  
(9)

In this reduced action, and indeed from here onwards, all covariant derivatives are constructed from and indices lowered etc. with respect to the three metric \( h_{ij} \), and \( R^{(3)}(h) \) is the scalar curvature of this metric. We obtain (9) by expressing \( R(g) \) in terms of \( R^{(3)}(h) \) and the function \( V \). For the reduced action the equations of motion for \( V \) correspond to the (00)-Einstein equations (6).

Now let us implement the Lagrange multiplier constraint so that \( \text{Tr} n^2 = 1 \). Then

\[ \text{Tr} n D_i n = 0. \]  
(10)

Let \( \tau_{ijk} = \sqrt{\hbar} \epsilon_{ijk} \) where \( \epsilon_{ijk} \) is the Levi-Civita symbol (satisfying \( \epsilon_{123} = +1 \)). Then the volume form is \( \tau_h = \sqrt{\hbar} d^3 x = \tau_{ijk} dx^i dx^j dx^k / 3! \) and the Hodge star operation is given in terms of \( \tau_{ijk} \). For example \((*F)_i = \tau_{ijk} F^{jk}/2\) for a two form

\[ \text{With our assumptions of static, purely magnetic, fields we have an energy density} \ T_0^0 = \mathcal{L}_M, \text{ these being the final three terms of (9).} \]
Upon using the Bianchi identity, the $\sigma$-model constraint and (10) one may verify that
\[
\sqrt{h} \text{Tr} h^{ij} \left[ \frac{1}{2} \tau_{ikl} e^u F^{kl} \pm (n \partial_i u + D_i n) \right] \left[ \frac{1}{2} \tau_{jmn} e^u F^{mn} \pm (n \partial_j u + D_j n) \right] = \sqrt{h} \left( \frac{1}{2} e^{2u} \text{Tr} F_{ij} F^{ij} + \text{Tr} D_i n D^i n + \nabla^i u \nabla_i u \right) \pm \partial_i \text{Tr} \left( \sqrt{h} e^u \tau^{ijk} F_{jk} n \right).
\]
(11)

Comparison of (11) with (9) shows that upon setting
\[
V(x^1, x^2, x^3) = e^u,
\]
and with units so that $4\pi G = 1$, we may rewrite the reduced action as
\[
S = \int d^3 x \sqrt{h} \left[ \frac{1}{4} R^{(3)}(h) - \frac{1}{2} \text{Tr} h^{ij} v_i^\pm v_j^\pm \right] + B^\pm,
\]
(13)
where $B^\pm$ is the surface term
\[
B^\pm = \frac{1}{2} \int d^3 x \partial_i \left( \sqrt{h} h^{ij} \partial_j u \pm \text{Tr} \left( \sqrt{h} e^u \tau^{ijk} F_{jk} n \right) \right),
\]
(14)
and we have set
\[
v_i^\pm = \frac{1}{2} \tau_{ikl} e^u F^{kl} \pm (n \partial_i u + D_i n).
\]
(15)
These equations may be succinctly written in terms of differential forms as
\[
S = \int \left[ \frac{1}{4} R^{(3)}(h) \tau_n - \frac{1}{2} \text{Tr} v^\pm \wedge * v^\pm + \frac{1}{2} d (* du \pm \text{Tr} (2 e^u F n)) \right],
\]
(16)
with
\[
v^\pm = v_i dx^i = * e^u F \pm (n du + Dn).
\]
(17)
We remark that the requirement $4\pi G = 1$ sets a mass scale for the model. It has the same effect as in classical Newtonian mechanics of allowing static configurations of self-gravitating point charges when the masses and charges agree in suitable units. It is analogous to the choice of a critical coupling when constructing Bogomolny equations for the Abelian-Higgs model of superconductors.

At this stage we may derive Bogomolny equations for the action (9) (or equivalently (16)). The key point is that the action is quadratic in $v^\pm$ (and this quadratic is of definite signature say for a semisimple gauge group). Thus for
any field variation this term of the action has extremal when \( \nu_i^{a\pm} = 0 \) for every \( i \) and gauge index \( a \). In particular, supposing \( \nu_i^{a\pm} = 0 \), then varying the action with respect to the three-metric \( h_{ij} \) yields,

\[
\delta \int d^3x \sqrt{h} \ R^{(3)}(h) = 0.
\]

This shows \( h_{ij} \) is a solution of the three dimensional Einstein equations

\[
R^{(3)}_{ij}(h) - \frac{1}{2} h_{ij} R^{(3)}(h) = 0,
\]

which in turn implies

\[
R^{(3)}(h) = 0, \quad R^{(3)}_{ij}(h) = 0.
\]

Using the three dimensional identity

\[
R_{ijkl} = h_{ik} R_{jl} - h_{il} R_{jk} + h_{jl} R_{ik} - h_{jk} R_{il} + \frac{1}{2} (h_{il} h_{jk} - h_{ik} h_{jl}) R
\]

we deduce that the three-metric \( h_{ij} \) is necessarily flat. This shows the various flat models assumed by previous authors were in fact necessary. If space is topologically \( \mathbb{R}^3 \) then this flatness means it is isometric to Euclidean space and we may take \( h_{ij} = \delta_{ij} \), in which case the four-metric is

\[
ds^2 = -V^2 dt^2 + \frac{1}{V^2} d\mathbf{x} \cdot d\mathbf{x}.
\]

Let us summarise our calculations thus far. Assuming \( n \) satisfies \( \text{Tr} \, n^2 = 1 \) (and with a choice of units such that \( 4\pi G = 1 \)) we find

\[
n^a \partial_i u + D_in^a = \mp \frac{1}{2} \tau_{ikl} e^u F^{a\,kl},
\]

or equivalently

\[
ndu + Dn = \mp \ast e^u F,
\]

yield solutions of the field equations (4-6) if and only if the three-metric \( h_{ij} \) defined by (7) is flat. These are our Bogomolny equations for the action (3); one can readily begin with (19) or (20) and a flat three-metric and derive (4-6).

As we mentioned in the introduction, the spacetimes with metric (18) are precisely those static space-times for which super-covariantly constant spinors exist [13]. (Tod [15] extended this classification to stationary metrics.) Hlousek and Spector [14] have advanced arguments associating supersymmetric extensions to theories exhibiting Bogomolny bounds. This suggests a supersymmetric extension exists to the theory here under consideration. We will not pursue the construction of such a theory here.
3 Solutions

It remains to discuss solutions of the Bogomolny equations (19, 20). Before focusing on the case of axially symmetric solutions for the gauge group $SU(2)$ we first show that an auxiliary magnetostatic problem may be associated with the equations and then discuss the case of covariantly constant solutions.

3.1 General Properties

First then let us consider projecting the Bogomolny equations in the direction of the $\sigma$-model field. This is analogous to projecting the usual Bogomolny equations of Yang-Mills-Higgs theory in the direction of the Higgs fields. Using (10) we find

$$\partial_i (e^{-u}) = \pm \frac{1}{2} \tau_{kli} \text{Tr}(F^{kl} n),$$

(21)

(equivalently $de^{-u} = \pm * \text{Tr}(F n)$) and this together with the Bianchi identities yields

$$\nabla^k \nabla_k e^{-u} = \pm \frac{1}{2} \tau^{ijk} \text{Tr}(F_{ij} D_k n).$$

(22)

Upon setting $b_i = \partial_i (e^{-u})$ and $\vec{b} = (b_1, b_2, b_3)$ we may recast our equations in the form

$$\nabla \cdot \vec{b} = \rho, \quad \nabla \times \vec{b} = 0, \quad \nabla^2 \Phi = \rho.$$  

(23)

Here

$$\rho = \pm \frac{1}{2} \tau^{ijk} \text{Tr}(F_{ij} D_k n), \quad \vec{b} = \nabla \Phi, \quad \Phi = e^{-u}.$$  

(24)

We see then from (23) that there is an Abelian magnetostatic problem associated with (19) in which $\Phi$, $\vec{b}$, and $\rho$ play the role of the magnetic scalar potential, the magnetic field and the magnetic charge density, respectively. In particular the original metric coefficient $g_{00}$ is entirely determined by the Poisson equation of (23).

We observe that for the special sub-class of covariantly constant scalar solutions, those characterised by

$$D_i n^a = 0,$$  

(25)

\footnote{Equally one may consider this to be an auxiliary electrostatic problem. Because we have in mind the nonabelian problem where the sources are magnetic monopoles, we adopt the magnetostatic perspective.}
the projected Bogomolny equations (21) in fact encode the full Bogomolny equations. In this situation the magnetic charge density \( \rho \) (the source in the Poisson equation above) vanishes and the metric coefficient \( g_{00} \) is determined by the solution of a Laplace equation. Further, \( \vec{b} \) is divergenceless in this particular situation and so it is the curl of an Abelian vector potential which has string singularities defined in the associated three-dimensional Euclidean space. The general solution for \( V \) in this situation describes a system of point-like monopoles in static equilibrium interacting with the gravitational field. These are the Majumdar-Papapetrou solutions [6, 7]. With only one monopole placed at the origin of coordinates this (positive mass) solution takes the form

\[
e^u = \frac{1}{1 + \frac{|Q|}{r}},
\]

where \( Q \) is the magnetic charge and \( r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \). The asymptotic form of the metric is given by

\[
g_{00} \approx 1 - \frac{2|Q|}{r},
\]

with a total gravitational mass \( M = 4\pi|Q| \). Such a solution corresponds to an extreme Reissner-Nordström black hole in which the null hypersurface \( r = 0 \) corresponds to the horizon with radius \( |Q| \) in Schwarzschild coordinates.

### 3.2 \( su(2) \) Solutions

In order to proceed further with our analysis of the Bogomolny equations (19) we restrict our attention to the \( su(2) \) case. When the scalar field is non-covariantly constant the full nonlinearity of the Bogomolny equations becomes apparent: in order to solve these equations we start by constructing a static, magnetic ansatz for the components of the Yang-Mills connection

In the \( su(2) \) setting a non-covariantly constant \( D_i n^a \) may be expressed as

\[
D_i n^a = \alpha_i p^a + \beta_i q^a,
\]

where \( \alpha_i \) and \( \beta_i \) are functions of the space coordinates and \( p^a \) and \( q^a \) are chosen to satisfy \( p^a n^a = q^a n^a = 0 \), so that (10) holds. Imposing the additional conditions \( p^a p^a = q^a q^a = 1 \) and \( q^a = \epsilon_{abc} p^b q^c \), the triad \((n, p, q)\) becomes a rotating, orthonormal base for \( su(2) \). Working in spherical coordinates, we adopt an ansatz in which we assume that \( n, p, \) and \( q \) take the special forms

\[
\begin{align*}
n &= \sin \theta \cos \phi \ T_1 + \sin \theta \sin \phi \ T_2 + \cos \theta \ T_3 = \hat{x}^a T_a, \\
p &= \cos \theta \cos \phi \ T_1 + \cos \theta \sin \phi \ T_2 - \sin \theta \ T_3, \\
q &= -\sin \phi \ T_1 + \cos \phi \ T_2,
\end{align*}
\]

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where $T_1, T_2, T_3$ are the generators of the adjoint representation of $su(2)$.\(^3\)

Now upon combining the definition of the covariant derivative $D_i n^a$ with (28) we may solve (extending [17]) for the Yang-Mills potential $A = A_i \, dx^i = A_i^a \, T_a \, dx^i$, to obtain

$$A_i^a = \alpha_i q^a - \beta_i p^a - \epsilon_a^\text{bc} n^b \partial_i n^c - \delta_i n^a,$$  \hfill (30)

where $\delta_i$ are three arbitrary functions of $x^i$. This means the components of the gauge field may be expressed in polar coordinates as

$$A^a = q^a(\alpha - d\theta) + p^a(\sin \theta d\phi - \beta) - n^a \delta.$$  \hfill (31)

By projecting (20) in the $(n, p, q)$ directions we find the equations connecting these various unknowns ($\alpha = \alpha_i \, dx^i$ etc.) to be

$$du = \mp \star e^u \left(-d\delta + \alpha \wedge \beta - \sin \theta d\theta \wedge d\phi\right),$$

$$\alpha = \mp \star e^u \left(-d\beta + \delta \wedge \alpha + \alpha \wedge \cos \theta d\phi\right),$$

$$\beta = \mp \star e^u \left(d\alpha - \beta \wedge \delta + \beta \wedge \cos \theta d\phi\right).$$  \hfill (32)

Further, we have that

$$[L_3 - T_3, A] = q \partial_\phi \alpha - p \partial_\phi \beta - n \partial_\phi \delta,$$  \hfill (33)

where $L_3$ is the generator of space-rotations around the $x^3$ axis. By taking $\alpha$, $\beta$ and $\delta$ independent of $\phi$ we then have

$$[L_3 - T_3, A_i] = 0.$$  \hfill (34)

This property is characteristic of an axially symmetric, $su(2)$ connection [18]. It is perhaps worth remarking that an axially symmetric connection does not necessitate an axially symmetric space-time. This will occur when $V$ (or equivalently $e^u$) is independent of $\phi$. From (32) we see this means

$$0 = \partial_\phi \delta_2 - \partial_\theta \delta_1 - \alpha_1 \beta_2 + \alpha_2 \beta_1,$$

and so for (7) to admit $\partial/\partial \phi$ as a Killing vector implies a constraint on some of the connection parameters. Similarly further constraints are needed for the metric to be spherically symmetric.

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\(^3\)With $[T_a, T_b] = \epsilon_{abc} T_c$ we have

$$q = [n, p], \quad n = [p, q], \quad p = [q, n],$$

and

$$dn = p \, d\theta + \sin \theta q \, d\phi, \quad dp = -n \, d\theta + \cos \theta q \, d\phi, \quad dq = -\left(\sin \theta n + \cos \theta p\right) d\phi.$$
At this stage we have expressed the Bogomolny equations (19) for the gauge group \( su(2) \) in the form (32). No approximations have so far been introduced and the problem remains of either solving (32) or finding ansatz that enable their solution.

Our ansatz for the study of (19) is now based on the following choice for \( \alpha_i, \beta_i, \) and \( \delta_i \):

\[
\begin{align*}
\alpha_1 &= 0, \quad \alpha_2 = \chi, \quad \alpha_3 = 0, \\
\beta_1 &= 0, \quad \beta_2 = 0, \quad \beta_3 = \sin \theta \chi \\
\delta_1 &= 0, \quad \delta_2 = 0, \quad \delta_3 = \psi.
\end{align*}
\]

(35)

where \( \chi \) and \( \psi \) are in general functions of \( r \) and \( \theta \). These choices mean (34) is satisfied and we have an axially symmetric, \( su(2) \) connection.

With this ansatz and taking the (-) sign in (19) we find five distinct equations amongst the components of the Bogomolny equations (32):

\[
\begin{align*}
\frac{\partial}{\partial r} \ln |\chi| &= -e^{-u} \\
\frac{\partial}{\partial \theta} \ln |\chi| &= -\frac{\psi}{\sin \theta}.
\end{align*}
\]

(36)

(37)

\[
\frac{\partial}{\partial r} (e^{-u}) = -\frac{1}{r^2} \left( 1 - \chi^2 + \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right),
\]

(38)

\[
\frac{\partial}{\partial \theta} (e^{-u}) = \frac{1}{\sin \theta} \frac{\partial \psi}{\partial r},
\]

(39)

\[
\frac{\partial}{\partial \phi} (e^{-u}) = 0.
\]

(40)

Not all of these equations are independent: (39) is a consequence of (36) and (37). Now equation (40) implies that our conformstatic metric (7) admits \( \frac{\partial}{\partial \phi} \) as a Killing vector. Within our ansatz the \( \sigma \)-model equations (5) reduce to (37) and the axisymmetric charge density \( \rho \) (24) simplifies to

\[
\rho = \frac{1}{r^2} \frac{\partial (\chi^2)}{\partial r}.
\]

(41)

\(^4\)Choosing the opposite sign leads to the same class of solutions. The sign ambiguity can be absorbed by working with a quantity \( \pm e^u \); this sign is responsible for the existence of solutions with either positive or negative magnetic charge and positive gravitational mass.
The latter result indicates that \( e^{-u} = \frac{1}{V} \) is not in general an harmonic function. Combining (36-38) yields the equation
\[
r^2 \frac{\partial^2}{\partial r^2} \ln |\chi| + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \ln |\chi| \right) = 1 - \chi^2. \tag{42}
\]
This equation governs the solutions of our Bogomolny equations. Once we have a solution for (42), the remaining unknowns \( V \) and \( \psi \) can be simply determined using (12), (36) and (37). With \( \ln |\chi(r, \theta, \phi)| = -v(\varrho, \theta, \phi) - \ln r \) and \( \varrho = 1/r \) we may rewrite (42) in terms of the three-dimensional Laplacian,
\[
\Delta_3 v = \frac{\partial^2}{\partial \varrho^2} v + \frac{2}{\varrho} \frac{\partial}{\partial \varrho} v + \frac{1}{\varrho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} v \right) + \frac{1}{\varrho^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} v = e^{-2v}. \tag{43}
\]
We remark that spherically symmetric solutions are covered within our ansatz by \( \psi = 0, \chi = \chi(r) \). In this case
\[
A^a_i = \epsilon_{ab} \hat{x}_b \frac{1}{r}(1 - \chi(r)),
\]
as in the t’Hooft ansatz. For such a restriction the \( \sigma \)-model equations (5) become trivial (as (37) vanishes) and equations (36-40) and (42) reduce to a form of Emden’s isothermal gas equation. With \( \ln |\chi(r)| = -F(r) - \ln r \) this is equation (21) of [5],
\[
r^4 F'' = e^{-2F}. \tag{43}
\]
Such Emden type equations may be rewritten in the form of an Abel equation
\[
\frac{d\chi}{d\bar{r}} = -\frac{\bar{r} \chi}{\bar{r} - 1 + \chi^2}, \tag{44}
\]
where \( \bar{r} = r F'(r) - 1 = r e^{-u} \) (and \( \chi = e^{-F}/r \)). Indeed the variable \( \bar{r} \) naturally arising in this transformation is such that the metric takes the form
\[
d\bar{s}^2 = -e^{2u} dt^2 + e^{-2u} dr^2 + \bar{r}^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]
Unfortunately, even in this spherically symmetric case, no nontrivial, closed-form solutions are known to (43) or (44). If \( |\chi(r)| \) is small for large \( r \), then (42) and (36) have the approximate (asymptotically flat) solution
\[
\chi(r) \approx B \frac{e^{-r}}{r}, \quad e^u \approx 1 - \frac{1}{r}, \tag{45}
\]
This is to be compared with the static Abelian-Higgs model of vortices which is governed outside of the vortices by an equation of the form
\[
\Delta_{3v} = e^v - 1.
\]
where $B$ is an arbitrary constant. Comparing this asymptotic solution with (27) we conclude that (45) corresponds to a magnetic monopole with unit magnetic charge and total gravitational mass $M = 4\pi$. Another approximate solution for (42) is

$$\ln |\chi| \approx C \sqrt{r} \sin \left( \frac{\sqrt{7}}{2} \ln r + \Omega \right),$$

(46)

where $C$ and $\Omega$ are integration constants. As discussed by Balakrishna and Wali [4], (46) is valid for $|\chi| \approx 1$, which is the case when $r \to 0$. The oscillatory behaviour of (46) implies a countable, infinite set of (naked) singular spheres surrounding the origin, which is the locus of an event horizon of zero surface area. (See [4] for details.) In the sequel, we shall see how the radial dependence of the solutions near $r = 0$ is affected in the non-spherically symmetric case. We also note that the transition between the (asymptotic) monotone and oscillatory regimes of these solutions (45) and (46) can be conveniently analysed in terms of the new variable $\xi(x) = \ln \chi^2$, where $x = \ln r$, which satisfies the equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial \xi}{\partial x} + 2 e^\xi = 2,$$

(47)

This transformation enables both simple qualitative and numerical analysis near $x = 0$. The numerical solution shown in Figure 1, satisfies $\xi(0) = 1$, $\xi'(0) = 0$. It evidences the different behaviours of (45) and (46), and displays a smooth regime transition at $x = 0$ ($r = 1$). As a consequence of (36), $V^2$ is infinite at every $x$ where $\xi'(x) = 0$.

In the non-spherically symmetric case, we may begin with the approximation (for large $r$) to (42),

$$\chi \approx B e^{-r} \left[ 1 + \epsilon \sum_{l=1}^{\infty} \frac{f_l}{r^l} P_l(\cos \theta) \right].$$

(48)

Here $B$ is is arbitrary and $\epsilon$ is a small parameter. Upon using (36) and dropping terms of order $\epsilon^2$ we find

$$e^a \approx 1 - \left[ \frac{1}{r} + \epsilon \sum_{l=1}^{\infty} \frac{l f_l}{r^{l+1}} P_l(\cos \theta) \right].$$

(49)

Comparing with (12) we see this corresponds to an asymptotically flat solution.

Similarly using (37) we obtain

$$\psi \approx -\epsilon \sin \theta \sum_{l=1}^{\infty} \frac{f_l}{r^l} \frac{dP_l(\cos \theta)}{d\theta}.$$
Observe that, in contrast to $\chi$, $\psi$ is a long range potential. It is also possible to find an approximate solution of (42) for small $r$, such that $\chi$ is finite at $r = 0$:

$$|\chi| \approx 1 + C \sqrt{r} \sin \left( \frac{\sqrt{7}}{2} \ln r + \Omega \right) + \sum_{l=1}^{\infty} c_l \, r^{\frac{1}{2} + \sqrt{l(l+1)-\frac{1}{4}}} P_l(\cos \theta),$$

where $C$, $\Omega$, and $c_1, c_2, c_3, \ldots$ are arbitrary constants. This expression shows a remarkable difference between the radial dependence of the monopole and higher multipole terms in $\chi$. We see that (48-51) reduce to the previous spherically-symmetric results when every $f_l$ and $c_l$ vanishes.

A simple perturbative method can be used in order to improve the asymptotic solutions (45) and (48-50). We illustrate the procedure in the spherically symmetric case only, but it can be extended straightforwardly to the case in which $\chi$ depends on $\theta$ as well. We have for large $r$ that $\chi(r) \approx Be^{-r}/r$ which gives the correct asymptotic behaviour for the metric. Now suppose that

$$\chi(r) = B \frac{e^{-r}}{r} (1 + \varepsilon)$$

where $\varepsilon \ll 1$. Upon substituting in (42) we find the approximate expression

$$\varepsilon = -\frac{B^2 \, e^{-2r}}{4 \, r^4}.$$  \hspace{1cm} (52)

As a consequence, our new asymptotic solutions are

$$\chi \approx B \frac{e^{-r}}{r} \left( 1 - \frac{B^2 \, e^{-2r}}{4 \, r^4} \right),$$

$$g_{00} \approx 1 - \frac{2m(r)}{r},$$

where

$$m(r) = 1 - \frac{B^2 \, e^{-2r}}{2 \, r^3}.$$  \hspace{1cm} (54)

This improved asymptotic approximation exhibits a very small correction in the distribution of gravitational mass, suggesting the existence of a massive, extended magnetic core. Such exponential corrections also arise from instanton corrections.
4 Inclusion of a Dilaton

We now extend our earlier analysis to include a dilaton. Up to a redefinition of \( u \) we will find that the Bogomolny equations are unaltered. Consider the EYMD\( \sigma \) system given by the action

\[
S_d = \int d^4x \sqrt{|g|} \left[ \frac{1}{16\pi G} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{4} \text{Tr} (e^{\gamma \phi} F_{\mu\nu} F^{\mu\nu}) \right. \\
\left. - \frac{1}{2} \text{Tr} (D_\mu n D^\mu n) + \frac{\lambda}{2} \left( \text{Tr} n^2 - 1 \right) \right].
\]

(56)

The field equations are now (5) and (6) together with the modified gauge equation

\[
\frac{1}{\sqrt{|g|}} D_\mu \left( \sqrt{|g|} e^{\gamma \phi} F^{\mu\nu} \right) = [n, D^\nu n],
\]

(57)

and the new dilaton field equation

\[
\frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} \partial^\mu \phi \right) = (4\pi G \gamma) e^{\gamma \phi} \text{Tr} F_{\mu\nu} F^{\mu\nu}.
\]

(58)

Upon making use of (8) one observes [11] that with the identification

\[
\phi = 2\gamma \ln \sqrt{g_{00}}
\]

(59)

the dilaton equation of motion (58) coincides with the (00)-Einstein equation. Our construction of a reduced action now proceeds as before. To employ the identity (11) we want

\[
e^{2u} = V^2 e^{\gamma \phi},
\]

which, upon using (59), requires

\[
u = \left( 1 + \gamma^2 \right) \ln \sqrt{g_{00}} \; \Rightarrow \; \text{i.e. } e^u = V^{1+\gamma^2}.
\]

(60)

Finally, the choice

\[
4\pi G \left( 1 + \gamma^2 \right) = 1
\]

(61)

means we have a reduced action

\[
S_d = \int d^3x \sqrt{h} \left[ \frac{1+\gamma^2}{4} R^{(3)}(h) - \frac{1}{2} \text{Tr} h^{ij} v^i_+ v^j_+ \right] + B_d^\pm,
\]

(62)

where \( B_d^\pm \) is again a surface term and \( v^i_\pm \) is given by (15). Up to the scaling of the scalar curvature this is of an identical form to (13). In particular this means
we obtain the same Bogomolny equations as previously with the same conclusion that the three-metric is flat.

Therefore, assuming $n$ satisfies $\text{Tr} \, n^2 = 1$ and with a choice of units (61), we find that the Bogomolny equations (19-20) again provide solutions to the field equations for (56) if and only if the three-metric $h_{ij}$ is flat. The dilaton field is given by (58) for such solutions.

An important consequence of having the same Bogomolny equations is that the solutions discussed in the previous section may be directly used in the present setting, though with the metric function $V$ now related to $e^u$ via (60) rather than (12). Thus in the absence of a dilaton the solution (26) to Laplace's equation describing a single monopole led to the asymptotic form of the metric (27) we now find

$$g_{00} \approx 1 - \frac{2m}{r}, \quad m = \frac{|Q|}{1+\gamma^2}.$$  

Whereas without the dilaton we have an extreme Reissner-Nordström black hole, according to Ref. [5] the solution with dilaton has a naked, point-like singularity at $r = 0$ for $\gamma \neq 0$. The energy integral may be determined analytically for this solution with arbitrary $\gamma$ yielding

$$E = \int d^3 \sqrt{|g|} \, T^0_0 = \frac{2\pi (1 + 2\gamma^2)|Q|}{1 + \gamma^2}.$$  

The fact that $E$ is finite in the case $\gamma \neq 0$ -in which the integration region includes the naked singularity- can be considered as the extension to curved spacetime of a result previously found by Bizon [16] in the the context of a Yang-Mills-dilaton theory.

Similarly with our axially symmetric solutions the metric function $V$ is modified accordingly, based on the same function $e^u$. Thus to a first approximation (45) leads to

$$V \approx 1 - \frac{1}{1+\gamma^2} \frac{1}{r},$$  

which may be improved via (54) with

$$m(r) = \frac{1}{1+\gamma^2} \left[ 1 - \frac{B^2 \, e^{-2r}}{2 \, r^3} \right].$$  

5 Solutions with Higher Winding Numbers

In this Section we study axisymmetric solutions of the modified Bogomolny equations with winding number greater than one. Again the problem lies in finding tractable ansatz for the exact Bogomolny equations. To this end, let
us consider the following [20, 18] magnetic, static prescription for the SU(2) potentials:

\[ A_\mu \, dx^\mu = q^{(k)} \left[ -\frac{H_1}{r} \, dr + (H_2 - 1) \, d\theta \right] + k \left[ n^{(k)} \, H_3 + p^{(k)} \, (1 - H_4) \right] \sin \theta d\phi, \tag{63} \]

where \( n^k, p^k \) and \( q^k \) are defined by

\[ n^{(k)} = \sin \theta \cos k \phi \, T_1 + \sin \theta \sin k \phi \, T_2 + \cos \theta \, T_3, \]
\[ p^{(k)} = \cos \theta \cos k \phi \, T_1 + \cos \theta \sin k \phi \, T_2 - \sin \theta \, T_3, \tag{64} \]
\[ q^{(k)} = -\sin k \phi \, T_1 + \cos k \phi \, T_2, \]

and the four functions \( H_i \) depend on \( r \) and \( \theta \) only. The integer \( k \) will be interpreted as the winding number of the solutions. (Equations (63) and (64) have been inspired by the ansatz considered in [20], which is set up in the fundamental representation of SU(2).) This new prescription for the connection can also be obtained as a consequence of (30) if we replace \( n, p, q \) by \( n^{(k)}, p^{(k)}, q^{(k)} \), respectively, and make the following choice of parameters:

\[ \alpha_1 = -\frac{H_3}{r}, \quad \alpha_2 = H_2, \quad \alpha_3 = 0, \]
\[ \beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = k \sin \theta \, H_4, \]
\[ \delta_1 = 0, \quad \delta_2 = 0, \quad \delta_3 = -k \sin \theta \, H_3. \tag{65} \]

This ansatz satisfies the axial symmetry condition

\[ [L_3 - k \, T_3, A_i] = 0, \]

which generalises (34) to higher winding numbers. We also observe that, for \( k = 1 \) and \( H_1 = 0 \), \( H_2 = H_4 = \chi(r, \theta) \), \( H_3 = -\frac{\psi(r, \theta)}{\sin \theta} \), our original axially symmetric ansatz (35) is recovered.

Combining this new ansatz for the connection with the projections of the Bogomolny equations on the rotating basis (64), we obtain the following system of equations:

\[ H_1 \, e^{-u} = \frac{\mp k}{r} \left[ \frac{\partial H_4}{\partial \theta} + \cot \theta \, (H_4 - H_2) - H_3 H_3 \right], \tag{66} \]
\[ H_2 \, e^{-u} = \frac{\mp k}{r} \left[ r \frac{\partial H_4}{\partial r} + \cot \theta \, H_4 + H_1 H_3 \right], \tag{67} \]

\(^6\)Now \( dn^{(k)} = p^{(k)} \, d\theta + k \sin \theta q^{(k)} \, d\phi \), \( dp^{(k)} = -n^{(k)} \, d\theta + k \cos \theta q^{(k)} \, d\phi \), \( dq^{(k)} = -k (\sin \theta n^{(k)} + \cos \theta p^{(k)}) \, d\phi \) and \( A = q^{(k)} (\alpha - d\theta) + p^{(k)} (k \sin \theta d\phi - \beta) - n^{(k)} \delta. \)
\[ \mp k H_4 e^{-u} = \frac{\partial H_2}{\partial r} + \frac{1}{r} \frac{\partial H_1}{\partial \theta}, \]  
(68)

\[ \frac{\partial}{\partial r} (e^{-u}) = \mp k \left[ 1 - \frac{\partial H_3}{\partial \theta} - \cot \theta H_3 - H_2 H_4 \right], \]  
(69)

\[ \frac{\partial}{\partial \theta} (e^{-u}) = \mp k \left[ \frac{\partial H_3}{\partial r} - \frac{1}{r} H_1 H_4 \right], \]  
(70)

\[ \frac{\partial}{\partial \phi} (e^{-u}) = 0. \]  
(71)

An asymptotically flat, covariantly constant solution of (66-71) can be easily obtained if we assume \( H_1 = H_2 = H_4 = 0 \) and \( H_3 = -\frac{\psi_{(r,\theta)}}{\sin \theta} \). The harmonic function \( e^{-u} \) is given by

\[ e^{-u} = 1 + \frac{|Q|}{r} + \sum_{l=1}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta), \]

where \( Q = k \) is the total magnetic charge. The corresponding solution for \( \psi \) is

\[ \psi = \psi_\infty - \frac{\sin \theta}{k} \sum_{l=1}^{\infty} \frac{C_l}{lr^l} \frac{dP_l(\cos \theta)}{d\theta}. \]  
(72)

The identification of total magnetic charge with the winding number in this solution implies that the gravitational mass is conserved for topological reasons. (See [12] for a discussion of this point in the case with no dilaton present.) We have yet to solve this ansatz for non-covariantly constant \( \sigma \)-model solutions.

### 6 Discussion

This paper has examined the Bogomolny equations and their solutions for Einstein-Yang-Mills-\( \sigma \) models (with possible dilaton couplings) on static space-times. Our derivation of the Bogomolny equations leads to several new observations. In particular the Bogomolny equations are consistent with the Einstein equations if and only if the associated (conformally scaled) three-metric is flat. These are precisely the static metrics for which super-covariantly constant spinors exists, and this class of metrics includes all of the particular ansatz considered by previous authors. (Stationary metrics for which super-covariantly constant spinors exist...
have also been classified, and the extension of such models to these space-times
will be dealt with elsewhere.) The connection with super-covariantly constant
spinors suggests a supersymmetric extension of these models that we have not
so far pursued.

Having obtained the Bogomolny equations we have also considered their solu-
tions. Two cases arise depending on whether the $\sigma$-model field is covariantly
constant or not. The former situation is somewhat easier to analyse and leads
to a set of (Euclidean) Abelian magnetostatic equations, valid for any gauge
group. These equations are sufficient to determine solutions of the Bogomolny
equations in this case.

When the $\sigma$-model field is not covariantly constant the Bogomolny equations
are rather more complicated. Here we have focussed on axially symmetric solu-
tions for the case of $su(2)$, though the situation involving larger gauge groups
is very interesting. By projecting onto the various gauge components we may
express the Bogomolny as the coupled system of first-order equations (32). The
imposition of axial symmetry on space places additional constraints on the pa-
rameters of an axially symmetric gauge connection. At this stage we adopted an
ansatz\(^7\) to simplify the analysis of (32) and we recovered the results of previous
authors who studied the spherically symmetric case. Our work also shows a to-
tally different type of radial dependence between the higher multipoles and the
monopole terms in our axially symmetric solution. Interestingly an improved
asymptotic approximation suggested a ‘massive magnetic core’, the asymptotic
corrections having the same form as instanton corrections. In our final section
we extended the original ansatz to include higher winding numbers finding an
exact, covariantly constant solution for arbitrary winding number. We have not
succeeded thus far in finding solutions to this ansatz for non-covariantly constant
solutions with higher winding numbers.

Clearly obtaining the Bogomolny equations is far from the end of the matter.
Although this system of equations is of lower order and simpler than the original
field equations, they are still complicated. Our analysis has presented some of
the features of these equations and we look forward to their wider study.

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\(^7\)One hope is to find an integrable system in these equations paralleling the appearance of
the Toda equations in the flat space monopole equations.
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References

Figure 1: Numerical solution for equation (47).