On the $SL(2, Z)$ covariant World-Sheet Action with Sources

Sudipta Mukherji

*Instituto de Física Teórica, C XVI,
Universidad Autónoma de Madrid, Madrid 28049, Spain*

e-mail: mukherji@delta.ft.uam.es

We analyse various world-sheet properties of the $SL(2, Z)$ covariant type IIB string action by coupling it with $SL(2, Z)$ covariant source.
1. Ten dimensional type IIB string theory is conjectured to have $SL(2,\mathbb{Z})$ symmetry. Under generic $SL(2,\mathbb{Z})$ transformations, the NS-NS and the R-R charges of the theory mix among each other. As a result, in particular, a macroscopic fundamental string (F-string), under $SL(2,\mathbb{Z})$, gets transformed to a D-string or various bound states of F and D-strings [1,2]. Various such objects can be classified by identifying their tensions as

$$T_{(p,q)} = \sqrt{(p - q\chi)^2 + \frac{q^2}{g^2}},$$

where $p, q$ are relatively prime integers, $g$ is the type IIB string coupling related to the expectation value of the dilaton and $\chi$ is the R-R scalar. The F-string is identified with tension $T_{(1,0)}$ and D-string with $T_{(0,1)}$. The rest can, in turn, be thought of as the bound states of these two and, following the literature, will be called $(p,q)$ strings. It is known that, for a macroscopic F or D-strings solutions, the world sheet F and D-string actions act as sources. However, since F and D-string solutions are related by $SL(2,\mathbb{Z})$ transformation, one would expect to see a version of this transformation relating world-sheet actions of F and D-strings. This is indeed the case as has been discussed in [3,4,5,6]. Moreover, in subsequent development, a manifest covariant action has been proposed which describes the entire orbit of $(p,q)$ string [7,8]. Here, in the rest of the note, we will use this action in order to explore some aspects of world-sheet physics of the $(p,q)$ strings.

It is, by now, well known that [9,10,11], if certain rules are satisfied, the type IIB string theory in ten dimension admits stable configurations of three or more string junctions. Furthermore, combining these junctions in a suitable way, it is possible to construct stable string network [12]. Whenever a string ends on another, the end point of one acts as a source for the other on its world-sheet. This naturally leads us to the investigation of string world-sheet action in presence of sources. In particular, we would analyse the $SL(2,\mathbb{Z})$ covariant action in presence of such sources and discuss the consequences.

2. As mentioned above, the $SL(2,\mathbb{Z})$ covariant IIB string action has been recently proposed in [7] and the $\kappa$-symmetry of the corresponding supergravity backgrounds has been analyzed subsequently in [8]. The action is constructed by introducing two 1-form gauge potentials, $A_\mu$ and $\tilde{A}_\mu$. $A_\mu$ is the usual Born-Infeld (BI) field that appear on the standard D-brane world-sheet action. On the other hand, $\tilde{A}_\mu$ plays the role of BI field on the fundamental type IIB string. This field, however, does not appear in standard world-sheet action of fundamental string but, as we will see, $S$-duality covariance, in various ways, requires
such a field. Specific choices of \((A_\mu, \tilde{A}_\mu)\) lead to D- or F-strings. This, in turn, breaks \(SL(2, Z)\) covariance on the world-sheet.

The duality covariant action is then straightforward to construct. As usual, first one defines modified two-form fields

\[
\mathcal{F} = \epsilon^{\mu\nu} F_{\mu\nu} = \epsilon^{\mu\nu}(\partial_\mu A_\nu - B_{\mu\nu})
\]

\[
\tilde{\mathcal{F}} = \epsilon^{\mu\nu} \tilde{F}_{\mu\nu} = \epsilon^{\mu\nu}(\partial_\mu \tilde{A}_\nu - \tilde{B}_{\mu\nu}),
\]

with \(B_{\mu\nu}\) and \(\tilde{B}_{\mu\nu}\) being the pullbacks to the world-sheet of the NS-NS and R-R two form gauge potentials respectively. With these set of fields, the \(SL(2, Z)\) covariant IIB action is

\[
S = \int d\tau d\sigma \frac{1}{2v} [\det g + e^{-\phi} \mathcal{F}^2 + e^{\phi}(\tilde{\mathcal{F}} - \chi \mathcal{F})^2].
\]

Here \(g\) is the induced metric in the Einstein frame. The string frame metric \(g_s\) is related to \(g\) as \(g_s = e^{2\phi} g\). Hence (3) in string frame will have an extra factor of \(e^{-\phi}\) in front of \(\det g_s\). Now, if we define

\[
\tau = \chi + ie^{-\phi}, \quad \mathbf{F} = \begin{pmatrix} \mathcal{F} \\ \tilde{\mathcal{F}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, \text{ with } ps - qr = 1,
\]

we see that the action is invariant if the fields transform as

\[
\tau' = \frac{p\tau + r}{q\tau + s}, \quad \mathbf{F}' = \Lambda \mathbf{F}.
\]

In (4) \(\chi\) is the usual R-R scalar, \(\phi\) is the dilaton and \(\Lambda\) is an \(SL(2, Z)\) matrix. Moreover, we note that in (3), \(v\) is an auxiliary field and, as we will see below, its expectation value determines the tension of the string.

The action (3) can be written in a manifestly covariant way by introducing a matrix \(\mathcal{M}\)

\[
\mathcal{M} = e^{\phi} \begin{pmatrix} \tau^2 & -\chi \\ -\chi & 1 \end{pmatrix}
\]

as

\[
S = \int d^2\sigma \frac{1}{2v} [\det g + \mathbf{F}^T \mathcal{M} \mathbf{F}].
\]

Using the first equation of (5), it is easy to check that under \(SL(2, Z)\), \(\mathcal{M}\) transforms as \(\mathcal{M} \rightarrow (\Lambda^T)^{-1} \mathcal{M} \Lambda^{-1}\). Thus the second term in (7) is indeed \(SL(2, Z)\) invariant. The equation of motion for \(\tilde{A}\) that follows from (3) is

\[
\tilde{\mathcal{F}} - \chi \mathcal{F} = e^{-\phi} v T,
\]

\[2\]
where \( T \) is a constant. If we now substitute (8) in (3) and add a total derivative term

\[-\int T d\hat{A} = -T \int (\hat{F} + \hat{B}), \tag{9}\]

we get

\[S = -\int \left[ \frac{e^{-\phi} \det g_s}{2v} + \frac{e^{-\phi}F^2}{2v} - \frac{e^{-\phi}vT^2}{2} \right] - T \int \chi F - T \int \hat{B}. \tag{10}\]

In writing down the last equation, we have used (8). Now solving for \( v \) and substituting it back to (10), we get

\[S = -T \int e^{-\phi} \sqrt{-\det(g_s + F) + (\chi F + \hat{B})}, \tag{11}\]

which is the standard D-string action.

It is also easy to get a \((p, q)\) string for the above action. For that, one needs to simply integrate over \( A \) and \( \tilde{A} \). A detail discussion on this can be found in the original papers [7,8].

Before we go further, let us note that we can fix a static gauge. The action (7) in the string metric takes the following form:

\[S = \int d^2\sigma \frac{1}{2v} [e^{-\phi} \det(\eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j) + F^T M F]. \tag{12}\]

Here, in writing down the action, we have identified the space time co-ordinates \( X^0 \) and \( X^1 \) with \( \tau \) and \( \sigma \) respectively. On the other hand, \( \phi^i \) corresponds to the fluctuation of eight transverse directions to the world-sheet. Now we notice that any constant scaling of the world sheet metric can be absorbed by simply rotating the world-sheet by a constant angle with respect to the target space co-ordinates \( X^0 \) and \( X^1 \). Since field strengths corresponding to gauge fields are constants in two dimension, such overall scaling can be obtained simply by turning on the gauge fields. In what follows, we analyse this in detail by exploiting certain properties of the covariant action in the presence of \( SL(2, Z) \) covariant source term. As we will see, among others, the effect of the source term is to induce variations in \( F_{\mu\nu} \) and/or \( \tilde{F}_{\mu\nu} \). This, in turn, changes the shape of the world-sheet. We hope to bring out the advantages of working in a \( SL(2, Z) \) covariant frame work as we go along.

3. Since (7) contains charged fields, it is natural to study the action in presence of sources. However, as mentioned before, such sources appear naturally if a F string (D-string) ends
on a D-string (F-string). With the simplest covariant choice of such a source term, the action takes the following form:

$$S = \int d\sigma d\tau \frac{1}{2v} [\det g + F^T \mathcal{M} F] + \int d\tau J^T A_\tau. \quad (13)$$

Here the matrix $J$ has the form

$$J = \begin{pmatrix} p \\ q \end{pmatrix}. \quad (14)$$

We also note that under $SL(2,\mathbb{Z})$, $J$ transforms as $J^T \rightarrow J^T \Lambda^{-1}$. The last term in $(13)$ acts as a source term to the original action. This term can in turn be written as an integral over world sheet with a delta function support along $\sigma$.

In order to study the effect of the source term, we start with the simplest case. We set the R-R scalar $\chi$, the R-R and NS-NS two forms to zero. The action for a D-string in this case is given by $(7)$ with

$$\mathcal{F} = 0, \quad \tilde{\mathcal{F}} = ve^{-\phi}. \quad (15)$$

We suppose that a D-string interacts with a source term corresponding to $J = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Now, the equations of motion for $A$ and $\tilde{A}$ that follows from $(13)$ have the forms

$$\mathcal{F} = -e^{\phi} v, \quad \tilde{\mathcal{F}} = ve^{-\phi}. \quad (16)$$

It is now easy to check that if we substitute $(16)$ in $(7)$ and integrate over $v$, we get an action for $(1,1)$ string with tension $T_{(1,1)} = \sqrt{1 + e^{-2\phi}}$.

One benefit of working with a covariant action is immediate. We start with a (1,0) string and $J = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as source instead. It is easy to check that (1,0) string, given by $\mathcal{F} = ve^{-\phi}, \quad \tilde{\mathcal{F}} = 0$ will again turn into a (1,1) string with $\mathcal{F}$ and $\tilde{\mathcal{F}}$ given by $(16)$. Notice that the above two examples are related to each other by a $SL(2,\mathbb{Z})$ metrix. The above result generalizes trivially to arbitrary integer $(p, q)$.

Similar story repeats when one includes the R-R scalar $\chi$. We study one example here. For a (0,1) string,

$$\mathcal{F} = v\chi e^{\phi}, \quad \tilde{\mathcal{F}} = ve^{-\phi} + v\chi^2 e^{\phi}. \quad (17)$$

In the presence of a source with $J = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it again turns in to a (1,1) string as before but with modified fields

$$\mathcal{F} = v(\chi - 1)e^{\phi}, \quad \tilde{\mathcal{F}} = ve^{-\phi} + v\chi(\chi - 1)e^{\phi}. \quad (18)$$
as can be seen from (13). Substituting thus in (7), we get a string action with tension 
$$\sqrt{(1-\chi)^2 + e^{-2\phi}}$$ as expected.

Next, we consider various field configurations which describe strings around the source. Again for simplicity, we consider a (0, 1) string turning into a (1, 1) string.

We take the (0, 1) string with non-vanishing $\phi$ and $\chi$ oriented along the $y$ axis of the two dimensional $x - y$ plane. The vector fields that describe this string are easy to find.

$$F^{(0,1)} = \left( \begin{array}{c}
\frac{v\chi_0 e^\phi}{\sqrt{\chi_0^2 + e^{-2\phi_0}}} \\
v(e^{-\phi_0} + \frac{\chi_0^2 e^\phi}{\chi_0^2 + e^{-2\phi_0}})
\end{array} \right).$$  \hspace{1cm} (19)

Here, in the second expression, we have substituted the value of $v$ that follows from its equation of motion. Moreover, the subscript on $\phi$ and $\chi$ denotes their constant background expectation values.

Since we have taken the D-string to be oriented along the $y$ axis, it is easy to get the gauge fields by integrating (19) along the axis.

$$A^{(0,1)} = \left( \begin{array}{c}
A^{(0,1)}_0 \\
\tilde{A}^{(0,1)}_0
\end{array} \right) = \left( \begin{array}{c}
\frac{y\chi_0}{\sqrt{\chi_0^2 + e^{-2\phi_0}}} \\
y \sqrt{\chi_0^4 + e^{-2\phi_0}}
\end{array} \right).$$  \hspace{1cm} (20)

Similarly, the (1, 1), that is produced because of the presence of the source, have the following field configuration (follows from (18))

$$A^{(1,1)} = \left( \begin{array}{c}
A^{(1,1)}_0 \\
\tilde{A}^{(1,1)}_0
\end{array} \right) = \left( \begin{array}{c}
\frac{z(\chi_0-1)}{\sqrt{(1-\chi_0)^2 + e^{-2\phi_0}}} \\
z \frac{e^{-2\phi_0} + (\chi_0 - 1)\chi_0}{\sqrt{(1-\chi_0)^2 + e^{-2\phi_0}}}
\end{array} \right).$$  \hspace{1cm} (21)

Here we have chosen $z$ to be the direction of the (1, 1) string in the $x - y$ plane. It is now clear that since there is no source for $\tilde{A}$, $\tilde{A}^{(1,1)}_0 = \tilde{A}^{(0,1)}_0$. Comparing now with (20) and (21), we see that the (1, 1) string is bent making an angle $\theta_1$ with the original D-string. The angle is given by

$$\cos \theta_1 = \frac{e^{-2\phi_0} + (\chi_0 - 1)\chi_0}{\sqrt{[(1-\chi_0)^2 + e^{-2\phi_0}][\chi_0^2 + e^{-2\phi_0}]}}. \hspace{1cm} (22)$$

However, we see that the string tension around the source is only balanced (required for the stability of the configuration) if we consider the following simple possibility. We take the source to be the end point of a F-string extending in the same two dimensional plane making an angle $\theta_2$ with the D-string such that

$$\cos \theta_2 = \frac{\chi_0}{\sqrt{\chi_0^2 + e^{-2\phi_0}}}. \hspace{1cm} (23)$$
This is thus the three string junction of [11]. However, we would like to stress here that we found it out by studying the dual gauge field $\tilde{A}$, which is a simple consequence of working with a $SL(2, Z)$ covariant action. Moreover, since we know the transformation properties of $F$ and $J$ under $SL(2, Z)$, we can obtain a generic junction configuration simply by applying $SL(2, Z)$ matrix on (20) and the source term.

In a similar manner, we can study the effect of $J = \begin{pmatrix} n \\ 0 \end{pmatrix}$ source on the D-string. The resultant $(n, 1)$ string makes an angle $\theta_1$ with the D-string given by

$$\cos\theta_1 = \frac{e^{-2\phi_0} + (\chi_0 - n)\chi_0}{\sqrt{(n - \chi_0)^2 + e^{-2\phi_0} + e^{-2\phi_0}}}.$$  \hspace{1cm} (24)

Instead of (22). Furthermore, if we consider the source to be the end point of a $(n, 0)$ string, the tension around the junction is only balanced when $(n, 0)$ string makes an angle $\theta_2$ given by (23) with the D-string. It is interesting to note that in the large $n$ limit, various angles around the junction are independent of $n$ and determined completely by the background values of $\chi$ and $\phi$. The angle is given by

$$\cos\theta_1 = \frac{\chi_0}{\sqrt{\chi_0^2 + e^{-2\phi_0}}}.$$  \hspace{1cm} (25)

It is precisely this limit when the world-sheet electric field reaches its critical value [13]. If we further add open strings to the $(n, 1)$ string, the angle is hardly changed. This indicates that in the large $n$ limit, the open string that ends on the $(n, 1)$ string becomes tensionless. From (25) it also follows that if the R-R field is set to zero, the $(n, 1)$ string is completely bent making an angle $\frac{\pi}{2}$ with the original D-string and is independent of the string coupling $e^{\phi_0}$.

From the above discussion, therefore, question naturally arises as to what exactly happens to the open string that end on $(n, 1)$ string as we increase $n$? Using the boundary state formalism [14], it is possible to identify certain set of operators in the corresponding conformal field theory which generates deformations along the direction of the $(n, 1)$ string. It turns out that, in the large $n$ limit, these deformations correspond to pure gauge deformations and hence decouple from the theory. A detail discussion of this issue will be presented elsewhere.

Finally, we would like to make the following comments. In type IIB string theory, besides strings, we have other branes as solitons. Among them, the three brane is self-dual
under $SL(2, Z)$. At the level of world-volume theory, it has been analyzed in [4]. Furthermore, in [15], a detail analysis of self-duality properties was carried out for four dimensional Maxwell electrodynamics. Generalizing their result for the three-brane world volume theory is straight forward. A preliminary analysis shows that external electro-magnetic sources can be introduced on the world-volume maintaining self-duality properties of the three-brane. In the context of type IIB string theory, it has the following interpretation. The end point of a $(p, q)$ string acts as source of electro-magnetic field when it ends on a three-brane. Thus it is possible to have string and three-brane junction preserving the self-dual properties of the three-brane (for a detail discussion of such junction configurations, see for example [16]). Recently, in [17], a manifestly $SL(2, Z)$ invariant three-brane world-volume action has been proposed by again introducing auxiliary gauge field. It will thus be of interest to understand the world-volume physics by coupling it with $SL(2, Z)$ invariant source. We hope to report on it in the future.

Acknowledgements: I would like to thank César Gómez, Patrick Meessen and especially Tomás Ortín for useful discussions. The work is supported by Ministerio de Educación y Cultura of Spain and also through the grant CICYT-AEN 97-1678.
References