Parametric resonance of neutrino oscillations and passage of solar and atmospheric neutrinos through the earth

E. Kh. Akhmedov *

The Abdus Salam International Centre for Theoretical Physics, I-34100 Trieste, Italy

Abstract

We present an exact analytic solution to the neutrino evolution equation in matter with periodic step-function density profile and discuss in detail the parametric resonance of neutrino oscillations that can occur in such a system. Solar and atmospheric neutrinos traversing the earth pass through layers of alternating density and can therefore experience parametric resonance of their oscillations. Atmospheric neutrinos can undergo parametrically enhanced oscillations in the earth when their trajectories deviate from the vertical by about $26^\circ - 32^\circ$. Solar neutrinos traversing the earth can experience a strong parametric resonance of their oscillations in a wide range of zenith angles. If the small mixing angle MSW effect is the solution of the solar neutrino problem, the oscillations of solar neutrinos crossing the core of the earth must undergo strong parametric resonance; this phenomenon should facilitate significantly the observation of the day-night effect in oscillations of solar neutrinos. If observed, the enhanced day-night effect for core crossing neutrinos would therefore confirm both the MSW solution of the solar neutrino problem and the parametric resonance of neutrino oscillations.

*On leave from National Research Centre Kurchatov Institute, Moscow 123182, Russia. E-mail: akhmedov@sissa.it
1 Introduction

It is well known that the resonantly enhanced neutrino oscillations in matter, the Mikheyev-
Smirnov-Wolfenstein (MSW) effect [1, 2], have a simple analogue in classical mechanics:
oscillations of two weakly coupled pendulums with slowly changing frequencies [3, 4]. One
then naturally wonders if there are any other resonance phenomena in mechanics which
might have analogues in neutrino physics.

One such phenomenon is parametric resonance. The parametric resonance can occur
in dynamical systems whose parameters vary periodically with time. A textbook example
is a pendulum with vertically oscillating point of support [5, 6]. Under certain conditions
topmost, normally unstable, equilibrium point becomes stable. The pendulum can oscillate
around this point in the upside-down position. Another example, familiar to everybody,
is a swing, which is just a pendulum with periodically changing effective length. It is the
parametric resonance that makes it possible to swing on a swing.

What would be an analogue of the parametric resonance for neutrino systems? Since
matter affects neutrino oscillations, periodically varying conditions can be achieved if a beam
of oscillating neutrinos propagates through a medium with periodically modulated density.
For certain relations between the period of density modulation and oscillation length, the
parametric resonance occurs and the oscillations get enhanced. The probability of neutrino
transition from one flavor state to another may become close to unity even for small mixing
angle.

This phenomenon is very different from the MSW effect. Indeed, at the MSW resonance
the neutrino mixing in matter becomes maximal ($\theta_m = \pi/4$) even if the vacuum mixing angle
$\theta_{vac}$ is small. This leads to large-amplitude neutrino oscillations in a matter of constant
density equal (or almost equal) to the resonance density, or to a strong flavor conversion in
the case of matter density slowly varying along the neutrino path and passing through the
resonance value. The situation is quite different in the case of the parametric resonance.
The mixing angle in matter (the oscillation amplitude) does not in general become large. What happens is an amplification of the transition probability because of specific phase relationships. Thus, in the case of the parametric resonance it is the *phase* of oscillations (rather than their amplitude) that undergoes important modification.

The parametric resonance of neutrino oscillations has a very simple physical interpretation: the average value of the transition probability, around which the oscillations occur, drifts. This may lead to large probabilities of flavor transitions. We shall discuss this interpretation in the last section of this paper.

The possibility of the parametric resonance of neutrino oscillations was suggested independently in [7] and [8]. In ref. [7] an approximate solution for sinusoidal matter density profile was found. In [8] an exact analytic solution for the periodic step-function density profile was obtained. However, in ref. [8] the results were presented only for a simplified case of small matter effects on the oscillations length and mixing angle, $l_m \approx l_{\text{vac}}, \theta_m \approx \theta_{\text{vac}}$. Parametric effects in neutrino oscillations were further studied in [9] where combined action of parametric and MSW resonances and possible consequences for solar and supernova neutrinos were discussed.

Recently, there has been an increasing interest in parametric resonance of neutrino oscillations. It was pointed out [10] that atmospheric neutrinos traversing the earth travel through layers of alternating density and can therefore undergo parametrically enhanced oscillations. The same was also shown to be true for solar neutrinos passing through the earth [11]. Interestingly, this situation very closely corresponds to a periodic step-function density profile studied in [8]. We therefore believe that it would be useful to present the exact analytic solution for this case in full, without assuming smallness of matter effects. We also study the evolution of oscillating neutrinos in the earth and derive an exact analytic expression for the neutrino evolution matrix in the constant-density-layers model of the earth structure. Although in this case neutrinos travel only over “one and a half” periods of density modulation, parametric resonance effects are possible and can be very large in
This case, too. We discuss these effects and their possible manifestations for solar and atmospheric neutrinos.

This paper is organized as follows. The evolution equation for a system of oscillating neutrinos in matter is reviewed and the solution for the periodic step-function density profile is derived in Sec. 2. Various limiting cases are considered and the parametric resonance of neutrino oscillations is studied in Sec. 3. Parametric enhancement of oscillations of solar and atmospheric neutrinos in the matter of the earth is considered in Sec. 4. The results are discussed and a simple physical interpretation of the parametric resonance in neutrino oscillations is given Sec. 5.

2 Evolution equation and its solution

Consider a system of two relativistic neutrino species with mixing. The evolution of a such a system in matter is described in the weak eigenstate basis by the Schrödinger equation [2]

\[ i \frac{\partial}{\partial t} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} -A(t) & B \\ B & A(t) \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \equiv H(t) \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \]

(1)

Here \( \nu_{a,b}(t) \) are the probability amplitudes of finding neutrinos of the corresponding flavor \( a, b \) at a time \( t \) (in particular, one of these two species can be a sterile neutrino \( \nu_s \)). The parameters \( A \) and \( B \) are

\[ A(t) = \frac{\Delta m^2}{4E} \cos 2\theta_{\text{vac}} - \frac{G_F}{\sqrt{2}} N(t) , \quad B = \frac{\Delta m^2}{4E} \sin 2\theta_{\text{vac}} . \]

(2)

Here \( G_F \) is the Fermi constant, \( E \) is neutrino energy, \( \Delta m^2 = m_2^2 - m_1^2 \), where \( m_{1,2} \) are the neutrino mass eigenvalues, and \( \theta_{\text{vac}} \) is the mixing angle in vacuum. The effective density \( N(t) \) depends on the type of the neutrinos taking part in the oscillations:

\[ N = \begin{cases} 
N_e & \text{for } \nu_e \leftrightarrow \nu_{\mu,\tau} \\
0 & \text{for } \nu_\mu \leftrightarrow \nu_\tau \\
N_e - N_\alpha/2 & \text{for } \nu_e \leftrightarrow \nu_s \\
-N_\alpha/2 & \text{for } \nu_{\mu,\tau} \leftrightarrow \nu_s .
\end{cases} \]

(3)
Here $N_e$ and $N_n$ are the electron and neutron number densities, respectively. For transitions between antineutrinos one should substitute $-N$ for $N$ in eq. (2). If overall matter density and/or chemical composition varies along the neutrino path, the effective density $N$ depends on the neutrino coordinate $t$. The instantaneous oscillation length $l_m(t)$ and mixing angle $\theta_m(t)$ in matter are given by

$$l_m(t) = \frac{\pi}{\omega(t)}, \quad \sin 2\theta_m(t) = \frac{B}{\omega(t)}, \quad \omega(t) \equiv \sqrt{B^2 + A(t)^2}.$$  \hspace{1cm} (4)

The MSW resonance corresponds to $A(t_{res}) = 0$, $\sin 2\theta_m(t_{res}) = 1$.

We now consider a special case when the effective density $N(t)$ (and therefore $A(t)$) is a periodic step function:

$$N(t) = \begin{cases} N_1, & 0 \leq t < T_1 \\ N_2, & T_1 \leq t < T_1 + T_2 \end{cases}$$

$$N(t + T) = N(t), \quad T = T_1 + T_2.$$  \hspace{1cm} (5)

Here $N_1$ and $N_2$ are constants. We shall call this the “castle wall” density profile. The function $A(t)$ is expressed by a similar formula with constants $A_1$ and $A_2$. Thus, the Hamiltonian $H(t)$ is also a periodic function of time with the period $T$: $H(t + T) = H(t)$.

Let us denote

$$\delta = \frac{\Delta m^2}{4E}, \quad V_i = \frac{G_F}{\sqrt{2}} N_i, \quad (i = 1, 2).$$  \hspace{1cm} (6)

In this notation

$$A_i = \cos 2\theta_{vac} \delta - V_i, \quad B = \sin 2\theta_{vac} \delta, \quad \omega_i = \sqrt{(\cos 2\theta_{vac} \delta - V_i)^2 + (\sin 2\theta_{vac} \delta)^2}.$$  \hspace{1cm} (7)

Any instant of time in the evolution of the neutrino system belongs to one of the two kinds of the time intervals:

$$\begin{align*}
(1) : & \quad 0 + nT \leq t < T_1 + nT \\
(2) : & \quad T_1 + nT \leq t < T_1 + T_2 + nT, \quad n = 0, 1, 2, ...
\end{align*}$$  \hspace{1cm} (8)
In either of these time intervals the Hamiltonian $H$ is a constant matrix which we denote $H_1$ and $H_2$, respectively. Let us define the evolution matrices for the intervals of time $(0, T_1)$ and $(T_1, T_1 + T_2)$:

$$U_1 = \exp(-iH_1T_1), \quad U_2 = \exp(-iH_2T_2)$$

The evolution matrix for the period is then

$$U_T = U_2U_1.$$  \hspace{1cm} (10)

It is convenient to rewrite the Hamiltonian $H$ given by eq. (1) using Pauli’s $\sigma$ matrices:

$$H(t) = B\sigma_1 - A(t)\sigma_3,$$  \hspace{1cm} (11)

Let us introduce the unit vectors

$$n_1 = \frac{1}{\omega_1}(B, 0, -A_1) = (\sin 2\theta_1, 0, -\cos 2\theta_1),$$
$$n_2 = \frac{1}{\omega_2}(B, 0, -A_2) = (\sin 2\theta_2, 0, -\cos 2\theta_2),$$  \hspace{1cm} (12)

where $\theta_{1,2}$ are the mixing angles in matter at densities $N_1$ and $N_2$: $\theta_1 = \theta_m(N_1), \theta_2 = \theta_m(N_2)$. Then one can write

$$H_i = \omega_i(\sigma n_i).$$  \hspace{1cm} (13)

Notice that eigenvalues of $H_i$ are $\pm\omega_i$. Using eqs. (9) and (10) one arrives at the following expression for the matrix of evolution over the period:

$$U_T = Y - i\sigma X = \exp[-i(\sigma \dot{X})\Phi].$$  \hspace{1cm} (14)

Here

$$Y = c_1c_2 - (n_1n_2)s_1s_2,$$  \hspace{1cm} (15)
$$X = s_1c_2 n_1 + s_2c_1 n_2 - s_1s_2 (n_1 \times n_2),$$  \hspace{1cm} (16)
$$\Phi = \arccos Y = \arcsin X, \quad \dot{X} = \frac{X}{X};$$  \hspace{1cm} (17)

and we have used the notation

$$s_i = \sin \phi_i, \quad c_i = \cos \phi_i, \quad \phi_i = \omega_i T_i \quad (i = 1, 2).$$  \hspace{1cm} (18)
Notice that $Y^2 + X^2 = 1$ as a consequence of unitarity of $U_T$. From eq. (12) one has

$$\mathbf{n}_1 \mathbf{n}_2 = \frac{1}{\omega_1 \omega_2} (B^2 + A_1 A_2) = \cos(2\theta_1 - 2\theta_2),$$  \hspace{1cm} (19)

$$(\mathbf{n}_1 \times \mathbf{n}_2) = \frac{1}{\omega_1 \omega_2} (0, B(A_2 - A_1), 0) = (0, \sin(2\theta_1 - 2\theta_2), 0).$$ \hspace{1cm} (20)

The vector $\mathbf{X}$ can be written in components as

$$\mathbf{X} = \left( B \left( \frac{s_1 c_2}{\omega_1} + \frac{s_2 c_1}{\omega_2} \right), \frac{B s_1 s_2}{\omega_1 \omega_2} (A_1 - A_2), -\left( \frac{A_1}{\omega_1} s_1 c_2 + \frac{A_2}{\omega_2} s_2 c_1 \right) \right).$$ \hspace{1cm} (21)

The evolution matrix for $n$ periods ($n=1, 2,...$) can be obtained by raising $U_T$ to the $n$th power:

$$U_{nT} \equiv U(t = nT, 0) = \exp[-i(\sigma \hat{X})n\Phi].$$ \hspace{1cm} (22)

Eqs. (14)-(22) give the exact solution of the evolution equation for any instant of time that is an integer multiple of the period $T$. In order to obtain the solution for $nT < t < (n+1)T$ one has to evolve the solution at $t = nT$ by applying the evolution matrix

$$U_1(t, nT) = \exp[-iH_1 \cdot (t - nT)] \hspace{1cm} (23)$$

for $nT < t < nT + T_1$ or

$$U_2(t, nT + T_1)U_1 = \exp[-iH_2 \cdot (t - nT - T_1)] \exp[-iH_1 T_1] \hspace{1cm} (24)$$

for $nT + T_1 \leq t < (n+1)T$, with $H_{1,2}$ given by eq. (13).

In the limit $A_2 = A_1$ eqs. (9)-(24) reproduce the well known results for neutrino oscillations in a medium of uniform density, as they should.

3 Parametric resonance

Assume that the initial neutrino state at $t = 0$ is $\nu_a$. The probability of finding $\nu_b$ at a time $t > 0$ (transition probability) is then $P(\nu_a \rightarrow \nu_b, t) = |U_{21}(t)|^2$ where $U(t)$ is the evolution
matrix. We will concentrate now on neutrino transition probabilities for \( t = nT \). Having found it, one can apply the procedure described in the end of the previous section to find \( P(\nu_a \to \nu_b, \ t) \) for an arbitrary \( t \).

From eq. (22) one finds
\[
P(\nu_a \to \nu_b, \ t = nT) = \frac{X_i^2 + X_j^2}{X_i^2 + X_j^2 + X_k^2} \sin^2(n\Phi) = \frac{X_i^2 + X_j^2}{X_i^2 + X_j^2 + X_k^2} \sin^2\left(\frac{\Phi t}{T}\right).
\]

(25)

This expression resembles the neutrino oscillation probability in a matter of constant density. However, the pre-sine factor and the argument of the sine in (25) are different from the amplitude and phase of neutrino oscillations in matter of either of the two densities, \( N_1 \) or \( N_2 \). In particular, the pre-sine factor need not be small even when both \( \sin^2 2\theta_1 \) and \( \sin^2 2\theta_2 \) are small. This is a consequence of the parametric enhancement of neutrino oscillations. The parametric resonance occurs when the pre-sine factor in (25) becomes equal to unity. The resonance condition is therefore
\[
X_3^2 = \left(\frac{A_1}{\omega_1} s_1 c_2 + \frac{A_2}{\omega_2} s_2 c_1\right)^2 = 0.
\]

(26)

For given values of \( N_1, N_2, T_1 \) and \( T_2 \) this equation determines the resonance energy. At the resonance, \( X^2 \) takes the value
\[
(X^2)_{\text{res}} = (X_1^2 + X_2^2)_{\text{res}} \approx \frac{B^2 s_1^2 (A_1 - A_2)^2}{\omega_1^2 \omega_2^2} \left[ 1 + c_2^2 B^2 A_2 \right].
\]

(27)

Eqs. (25)-(27) give a general description of the parametric resonance of neutrino oscillations for the “castle wall” density profile of eq. (5). We shall now analyze the resonance in several particular cases.

### 3.1 Large \( |A_i| \)

Assume first that the oscillation frequencies \( \omega_{1,2} \) are dominated by the \( A_{1,2} \) terms, i.e.
\[
A_i^2 \gg B^2, \quad \omega_i \approx |A_i|.
\]

(28)
This condition is satisfied in the following cases:

(i) small vacuum mixing angle and the system is far from the MSW resonance, or

(ii) \((G_F/\sqrt{2})N_{1,2} \gg \Delta m^2/4E\). The condition (28) is satisfied in this case irrespective of the value of \(\theta_{\text{vac}}\).

Eq. (28) ensures smallness of the oscillation amplitudes in matter with effective densities \(N_1\) and \(N_2\). We shall consider two distinct possibilities now.

(A) Same sign \(A_1\) and \(A_2\).

This corresponds to \(N_1, N_2 < N_{MSW}\) or \(N_1, N_2 > N_{MSW}\) where \(N_{MSW} = \cos 2\theta_{\text{vac}} \delta(\sqrt{2}/G_F)\) is the MSW resonance density. In this case \(X_3^2 \simeq \sin^2(\phi_1 + \phi_2)\), and the parametric resonance condition (26) becomes

\[
\phi_1 + \phi_2 = \bar{\omega} T = k\pi, \quad k = 1, 2, ... \tag{29}
\]

Here \(\bar{\omega}\) is the mean oscillation frequency,

\[
\bar{\omega} = \omega_1 \frac{T_1}{T} + \omega_2 \frac{T_2}{T}. \tag{30}
\]

One can also write the resonance condition (29) as

\[
\Omega = \frac{2\bar{\omega}}{k}, \quad \Omega \equiv \frac{2\pi}{T}. \tag{31}
\]

This is nothing but the well known parametric resonance condition in the case of small-amplitude oscillations [5, 6]. Eq. (29) can also be written as

\[
\bar{l}_m \equiv \frac{\pi}{\bar{\omega}} = \frac{T}{k}, \tag{32}
\]

i.e. the parametric resonance occurs when the mean oscillation length in matter \(\bar{l}_m\) equals the period of the density modulation divided by an integer.

(B) Opposite sign \(A_1\) and \(A_2\).

This corresponds to \(N_1 < N_{MSW} < N_2\) or \(N_2 < N_{MSW} < N_1\). In this case one gets \(X_3^2 \simeq \sin^2(\phi_1 - \phi_2)\), and the resonance condition is

\[
\phi_1 - \phi_2 = k'\pi, \quad k' = 0, \pm 1, ... \tag{33}
\]
or
\[
\left( \frac{\omega_0 \Delta T}{T} + \frac{\Delta \omega}{2} \right) = \frac{k' \Omega}{2} .
\] (34)

Here we have used
\[
\phi_1 - \phi_2 = \omega_0 \Delta T + \frac{1}{2} \Delta \omega T , \quad \omega_0 \equiv \frac{\omega_1 + \omega_2}{2} , \quad \Delta T \equiv T_1 - T_2 , \quad \Delta \omega \equiv \omega_1 - \omega_2 .
\] (35)

For both cases (A) and (B), at the resonance one has
\[
(X^2)_{\text{res}} = (X_1^2 + X_2^2)_{\text{res}} = \frac{B^2 s_1^2}{\omega_1^2 \omega_2^2} (A_1 - A_2)^2 = s_1^2 \sin^2(2\theta_1 - 2\theta_2) .
\] (36)

This follows from eqs. (27) and (28). In the vicinity of the resonance \( \sin^2(\phi_1 \pm \phi_2) \simeq [ (\phi_1 \pm \phi_2) - k\pi ]^2 \), and the transition probability can be written as
\[
P(\nu_a \rightarrow \nu_b , \ t = nT) \simeq \frac{D_k^2}{D_k^2 + (\omega_e - k\Omega/2)^2} \sin^2 \left( \sqrt{D_k^2 + (\omega_e - k\Omega/2)^2} \ t \right) .
\] (37)

Here
\[
D_k^2 = \frac{1}{T^2} s_1^2 \sin^2(2\theta_1 - 2\theta_2) \simeq \frac{\omega_e^2}{k^2 \pi^2} s_1^2 \sin^2(2\theta_1 - 2\theta_2) ,
\] (38)
\[
\omega_e = \begin{cases} 
\bar{\omega} = (\omega_1 T_1 + \omega_2 T_2)/T , & k = 1, 2, \ldots \text{ same sign } A_1 \text{ and } A_2 \\
\omega_0 \Delta T/T + \Delta \omega/2 , & k = 0, \pm 1, \ldots \text{ opposite sign } A_1 \text{ and } A_2 
\end{cases} ,
\] (39)

and we have used the fact that in the case under consideration \( X \ll 1 \), so that \( \Phi \simeq X \).

The conditions (29) and (33) determine the sum or the difference of the phases \( \phi_1 \) and \( \phi_2 \) in the cases (A) and (B) respectively, but not the values of the phases themselves. These values can be fixed through the following consideration. The parametric resonance condition ensures that the pre-sine factor in eqs. (25) and (37) is equal to unity. This, however, is not sufficient for the transition probability to be appreciable. For this one should also require that the argument of the sine be not too small. At the resonance, this argument is proportional to \( D_k \), so the fastest growth of the transition probability is realized when \( |D_k| \) reaches its maximum. As follows from (38), for fixed values of mixing angles in matter \( \theta_{1,2} \) this amounts to \( s_1 = \pm 1 \), i.e.
\[
\phi_1 = \pi/2 + k'\pi .
\] (40)
Thus, the optimal conditions for having a sizeable transition probability are realized when there is a parametric resonance and in addition $D_k^2$ reaches its maximum. This can be summarized as

$$
\phi_1 = \frac{\pi}{2} + k\pi, \quad \phi_2 = \frac{\pi}{2} + k'\pi, \quad k, k' = 0, 1, 2, ... \tag{41}
$$

which follows from eqs. (29), (33) and (40). The conditions (41) apply to both cases (A) and (B).

The parameter $D_k$ also determines the width of the resonance. The pre-sine factor in (37) becomes equal to 1/2 when the detuning $(\omega_e - k\Omega/2) = \pm D_k$. If the parametric resonance condition is considered as a condition on neutrino energy, $D_k$ determines the energy width of the resonance. It is easy to find the energy width at half height $\Delta E$. The condition (28) is satisfied provided that either $\sin^2 \theta_{\text{vac}} \ll 1$ (barring the proximity to the MSW resonance), or $V_i \gg \delta$. In the first case, as well as in the second case for not too large vacuum mixing angles ($\cos 2\theta_{\text{vac}} \gtrsim \delta/2V_i \ll 1$) one finds

$$
\frac{\Delta E}{E_0} \simeq \frac{2D_k}{\cos 2\theta_{\text{vac}} \delta_0} = \frac{|2s_1 \sin(2\theta_1 - 2\theta_2)|}{\cos 2\theta_{\text{vac}} \delta_0 T}. \tag{42}
$$

Here $E_0$ is the resonance energy and $\delta_0 = \delta(E_0)$. In the case $V_i \gg \delta_0$ and vacuum mixing close to maximal the result is

$$
\frac{\Delta E}{E_0} \simeq \frac{2D_k}{\delta_0} \frac{T V_1 V_2}{\delta_0(T_1 V_1 + T_2 V_2)}, \quad \cos 2\theta_{\text{vac}} \lesssim \frac{\delta_0}{2V_i} \ll 1. \tag{43}
$$

The resonance widths $\Delta E/E_0$ in eqs. (42) and (43) are maximal when $s_1 = \pm 1$, i.e. eq. (40) is satisfied. The widths rapidly decrease with increasing resonance order $k$. For the principal resonance, $k = 1$ ($k = 0$ in the case of the opposite sign $A_1$ and $A_2$),

$$
\frac{\delta E}{E_0} \sim |\sin(2\theta_1 - 2\theta_2)|, \quad \delta_0 \gtrsim V_i; \quad \frac{\delta E}{E_0} \sim |\sin(2\theta_1 - 2\theta_2)| \frac{V_i}{\delta_0}, \quad \delta_0 \ll V_i. \tag{44}
$$

Consider now the transition probabilities in more detail. If the condition (40) is satisfied, $X_{\text{res}} = |D_k|_{\text{max}} = |\sin(2\theta_1 - 2\theta_2)|$, $\Phi_{\text{res}} = 2(\theta_1 - \theta_2)$, and the transition probability over one period of density modulation $T$ is

$$
P(\nu_a \rightarrow \nu_b, T) = \sin^2(2\theta_1 - 2\theta_2). \tag{45}
$$
The smallness of neutrino mixing in matter, ensured by eq. (28), implies that \( \theta_1 \) and \( \theta_2 \) are close to 0 or \( \pi/2 \), depending on the signs of \( A_1 \) and \( A_2 \). We first consider the case (A), when \( A_1 \) and \( A_2 \) are of the same sign. For \( A_1, A_2 > 0 \) (\( \theta_{1,2} \) close to 0) the probability (45) is smaller than the maximal transition probability in matter of constant density equal to either \( N_1 \) or \( N_2 \), namely, \( \sin^2 2\theta_1 \) or \( \sin^2 2\theta_2 \). When \( A_1, A_2 < 0 \) (\( \theta_{1,2} \) close to \( \pi/2 \)) the probability (45) is always smaller than the largest of \( \sin^2 2\theta_1 \) and \( \sin^2 2\theta_2 \). Thus, in this case the transition probability over the period cannot exceed the maximal probability in the case of the matter of constant density. However, the parametric resonance does lead to an important gain. In a medium of constant density the transition probability cannot exceed \( \sin^2 2\theta_m \), no matter how long the distance that neutrinos travel. On the contrary, in the matter with “castle wall” density profile, if the optimal parametric resonance conditions (41) are satisfied, the transition probability can become large provided neutrinos travel large enough distance. For \( t = nT \) the transition probability takes the value

\[
P(\nu_a \rightarrow \nu_b, \ t = nT) = \sin^2[2n(\theta_1 - \theta_2)].
\]  

This probability can become quite sizeable even for small \( \theta_1 \) and \( \theta_2 \) provided that neutrinos have traveled sufficiently large distance. The number of periods neutrinos have to pass in order to experience a complete (or almost complete) conversion is

\[
n \simeq \frac{\pi}{4(\theta_1 - \theta_2)}.
\]  

However, the transition probability can be appreciable even for smaller values of \( n \).

Consider now the transition probabilities in the case (B), opposite sign \( A_1 \) and \( A_2 \). Assume for definiteness \( A_1 > 0, A_2 < 0 \). Since at the MSW resonance \( A(t) = 0 \), in this case \( N_1 < N_{MSW} < N_2 \). Small mixing in matter [eq. (28)] then implies that \( \theta_1 \) is close to 0 and \( \theta_2 \) is close to \( \pi/2 \). The transition probability over one period and \( n \) periods in the “optimal parametric resonance” are again given by eqs. (45) and (46) respectively. However in this case, for \( \theta_2 > \pi/4 + \theta_1/2 \) (which is always satisfied for small mixing in matter), one has \( \sin^2(2\theta_2 - 2\theta_1) > \sin^2 2\theta_1, \sin^2 2\theta_2 \). This means that even for the time interval equal
to one period of matter density modulation the transition probability exceeds the maximal probabilities of oscillations in matter of constant densities \(N_1\) and \(N_2\). The effect increases with increasing mixing in matter. This is a very important case; as we shall see in Sec. 4, this parametric enhancement is further magnified in the case of neutrinos traveling over “one and a half” periods of density modulation, which has important implications for neutrinos traversing the earth.

### 3.2 Small frequency variations

We will assume here that the frequency variations are small, \(\omega_1 \approx \omega_2\). This is the case when either relative variation of density is small, \(|N_1 - N_2| \ll N_1\), or matter effects are small, \(V_{1,2} \ll \cos 2\theta_{\text{vac}} \delta\). However, we will not be assuming the condition (28) in this subsection.

Let us introduce \(A_0 = (A_1 + A_2)/2\) and write

\[
A_1 = A_0 + a, \quad A_2 = A_0 - a, \quad |a| \ll |A_0|.
\]

(48)

Then

\[
\omega_1 \approx \omega_0 \left(1 + \frac{A_0}{\omega_0^2} a \right), \quad \omega_2 \approx \omega_0 \left(1 - \frac{A_0}{\omega_0^2} a \right), \quad \omega_0 = (\omega_1 + \omega_2)/2 \approx \sqrt{B^2 + A_0^2}.
\]

(49)

Notice that the mean oscillation frequency

\[
\bar{\omega} = \frac{1}{T}(\phi_1 + \phi_2) = \omega_0(1 + \mathcal{O}(\Delta\omega/\omega_0)).
\]

(50)

The resonance condition (26) in this case coincides with eq. (29), but eq. (40) does not have to satisfied.

In the vicinity of the parametric resonance \(\bar{\omega} \approx k\Omega/2\) one can write

\[
X_3^2 \approx \frac{A_0^2}{\omega_0^2} \sin^2(\phi_1 + \phi_2) \approx \frac{A_0^2}{\omega_0^2} T^2 \left(\bar{\omega} - \frac{k\Omega}{2}\right)^2,
\]

(51)

so that the probability of the oscillations becomes

\[
P(\nu_a \to \nu_b, \ t = nT) \approx \frac{D_k^2}{D_k^2 + (\bar{\omega} - k\Omega/2)^2} \sin^2 \left(\frac{A_0}{\omega_0} \sqrt{D_k^2 + (\bar{\omega} - k\Omega/2)^2} \ t\right).
\]

(52)
The parameter $D^2_k$ here is defined through

$$D^2_k \equiv \frac{\omega_1^2}{A_0^2 T^2} (X_1^2 + X_2^2)_{\text{res}}. \quad (53)$$

It coincides with the one given in eq. (38) in the limit $A_i^2 \gg B^2$. Consider now the case $T_1 = T_2 = T/2$. We have two distinct situations.

(i) Odd $k$ (this includes the principal resonance, $k = 1$). In this case $\phi_1 - \phi_2 \simeq 0$, and

$$X_2^2_{\text{res}} = \frac{B^2 (A_2 - A_1)^2}{\omega_1^2 \omega_2} = \sin^2(2\theta_1 - 2\theta_2), \quad (X_1^2)^{\text{res}} \ll (X_2^2)^{\text{res}}, \quad (54)$$

$$D^2_k = \frac{B^2 (A_1 - A_2)^2}{A_0^2 \pi^2 k^2}, \quad (55)$$

(ii) Even $k$. In this case the main terms in $X_2$ cancel, $X_2 = \mathcal{O}(a^3/A_0^3)$, and $X_{\text{res}}$ is dominated by $(X_1)^{\text{res}}$ which is of the order of $(a^2/A_0^2)$. One finds

$$D_k = \frac{1}{4} \frac{B (A_1 - A_2)^2}{A_0^2}. \quad (56)$$

For the principal resonance ($k = 1$) the resonance width at half-height is

$$\frac{\Delta E}{E_0} = \frac{2}{\pi} \frac{|A_1 - A_2|}{A_1 + A_2} \sin 2\theta_{\text{vac}} \frac{1}{\sqrt{1 - |V_0/\omega_0(E_0)|^2 \sin^2 2\theta_{\text{vac}}}}, \quad (57)$$

where $V_0 \equiv (V_1 + V_2)/2$. The distance that neutrinos should travel in order for the oscillation probability to become equal or close to one is $t = nT$ with

$$n \simeq \frac{\pi}{2} \frac{\omega_0}{|D_k| T} \frac{1}{|D_k| T}. \quad (58)$$

In the limit $V_{1,2} \ll \delta$ the results of ref. [8] are recovered.

4 Evolution of oscillating neutrinos in the earth

The earth consists of two main structures – the mantle, with density ranging from 2.7 g/cm$^3$ at the surface to 5.5 g/cm$^3$ at the bottom, and the core, with density ranging from 9.9 to
12.5 g/cm$^3$ (see, e.g., [12]). The electron number fraction $Y_e$ is very close to 1/2 both in the mantle and in the core. The radius of the earth $R = 6371$ km. At the border of the core, which has the radius $r = 3486$ km, there is a sharp jump of matter density from 5.5 to 9.9 g/cm$^3$, i.e. by about a factor of two. This jump is a very important feature of the matter density distribution in the earth. The matter densities within the mantle and within the core vary little compared to the density jump on the border of the core and mantle; therefore to a very good approximation one can consider the earth as consisting of mantle and core of constant densities equal to the corresponding average densities, $\bar{\rho}_m \simeq 4.5$ g/cm$^3$ and $\bar{\rho}_c \simeq 11.5$ g/cm$^3$. Comparison of neutrino transition probabilities calculated with such a simplified matter density profile with those calculated with actual density profile of the Stacey model [12] shows a very good agreement [10]. More recent models of earth give very similar profiles of the matter density distribution in the earth (for a discussion see, e.g., [13]). We therefore adopt such a “constant-density-layers” model of the earth density distribution$^1$.

Interestingly, this model of matter density profile exactly corresponds to the periodic step-function (“castle wall”) density profile studied in ref. [8] and in sections 2 and 3 of this paper. The peculiarity of the earth’s density profile is that neutrinos traversing the earth do not pass through many periods of density modulation; at most, they can only pass through, loosely speaking, “one and a half” periods (this would be exactly one and a half periods if the distances neutrinos travel in the mantle and core were equal). However, as we have seen in the previous section, even in the case of one period of density modulation the parametric resonance can lead to a very specific amplification of the probability of neutrino oscillations; especially for the case of opposite sign $A_1$ and $A_2$ the parametric resonance conditions ensure the largest increase of the transition probability over one period. As we shall see, this effect is further enhanced when neutrinos travel over “one and a half” periods of density modulation.

$^1$The same approximation was previously used in [14, 15]. A different analytic approach to propagation of oscillating neutrinos in the earth was developed in [16].
4.1 Evolution over “one and a half” periods

Consider now the evolution matrix for neutrinos traversing the earth. We will be interested in the situation when neutrinos traverse the mantle, core and then again mantle; this happens for the neutrino trajectories with the zenith angle $180^\circ \pm 33.17^\circ$ (nadir angle $\leq 33.17^\circ$). The evolution matrix in this case is

$$U_3 = U_1 U_2 U_1 = U_1 U_T = Z - i\sigma W,$$  \hspace{1cm} (59)

where

$$Z = 2c_1 Y - c_2,$$  \hspace{1cm} (60)

$$W = 2s_1 Y n_1 + s_2 n_2,$$  \hspace{1cm} (61)

and $Y$ has been defined in (15). The vector $W$ can be written in components as

$$W = \left( 2s_1 Y \frac{B}{\omega_1} + s_2 \frac{B}{\omega_2}, 0, - \left( 2s_1 Y \frac{A_1}{\omega_1} + s_2 \frac{A_2}{\omega_2} \right) \right).$$  \hspace{1cm} (62)

The transition probability in this case is simply

$$P(\nu_{a} \rightarrow \nu_{b}) = W_1^2.$$  \hspace{1cm} (63)

Eqs. (59)-(63) give the neutrino transition probability in the earth in the constant-density-layers approximation in general case, i.e. for arbitrary values of oscillation parameters and neutrino energies. We shall now study the conditions for the parametric enhancement of the oscillations. For fixed values of $N_1$, $N_2$, $\Delta m^2/4E$ and $\theta_{\text{vac}}$ these conditions give the optimal values of the phases $\phi_1$ and $\phi_2$, i.e. of the distances $T_1$ and $T_2$. It is straightforward to find extrema of $W_1^2$. There are extrema of three types.

---

The transition probability (63) applies when the initial neutrino state at $t = 0$ consists only of $\nu_a$. In some situations, such as for solar neutrinos, the neutrino state arriving at the surface of the earth is a coherent superposition of $\nu_a$ and $\nu_b$. This case can be easily treated using the evolution matrix (59). For example, for the probability $P_{2e} \equiv P(\nu_2 \rightarrow \nu_e)$ relevant for oscillations of solar neutrinos in the earth [17] one finds $P_{2e} = \sin^2 \theta_{\text{vac}} + W_1^2 \cos 2\theta_{\text{vac}} + W_1 W_3 \sin 2\theta_{\text{vac}}$. Different (but equivalent) expressions for $P_{2e}$ were derived in [14] and [11]. For small $\theta_{\text{vac}}$ the probability $P_{2e}$ practically coincides with (63).
1) Extrema at $c_1 = c_2 = 0$, i.e. $\phi_1 = \pi/2 + k\pi$, $\phi_2 = \pi/2 + k'\pi$. These are exactly the parametric resonance conditions (41) found in Sec. 3. Thus, in the case of “one and a half” periods, as well as in the case of $n$ periods, the oscillation probability reaches an extremum when the parametric resonance conditions are satisfied. The extremum is a maximum provided that either

$$f_1 \equiv n_1 n_2 - \left( \omega_1/\omega_2 - \sqrt{\omega_1^2/\omega_2^2 + 8} \right)/4 < 0,$$

(64)

or

$$f_2 \equiv n_1 n_2 - \left( \omega_1/\omega_2 + \sqrt{\omega_1^2/\omega_2^2 + 8} \right)/4 > 0,$$

(65)

otherwise it is a saddle point. The functions $f_1$ and $f_2$ are plotted in Fig. 1 along with $A_1$ and $A_2$ for $\sin^2 2\theta_{\text{vac}} = 0.01$. For small $\theta_{\text{vac}}$ the conditions (64) and (65) are satisfied for $\delta < V_2$ (except in a small region of $\delta$ in the vicinity of the point where $A_1 = 0$). We shall discuss these conditions in more detail below.

At the resonance the transition probability (63) takes the value

$$P(\nu_a \rightarrow \nu_b) = \sin^2 2(\theta_2 - 2\theta_1) \quad (c_1 = c_2 = 0).$$

(66)

The lowest order resonance corresponds to $\phi_1 = \phi_2 = \pi/2$. The parametric resonance of this type is illustrated by Fig. 2. The resonance width is rather large; the transition probability decreases by a factor of two for

$$\Delta \phi_{2,2} = |\phi_{1,2} - \pi/2| \simeq \frac{\pi}{4}.$$

(67)

Thus, the resonance enhancement of neutrino oscillations can occur even for quite sizeable detuning of the phases $\phi_{1,2}$.

2) Extremum at $s_1 = c_2 = 0$. It is a maximum provided $A_2 > 0$ and a saddle point otherwise. The transition probability is

$$P(\nu_a \rightarrow \nu_b) = \sin^2 2\theta_2 \quad (s_1 = c_2 = 0).$$

(68)
This case corresponds to zero transition probability in the mantle and maximal possible transition probability in the core. The extremum of this type is illustrated by Fig. 3. It is not a parametric resonance.

(3) Maximum at \( c_1^2 = s_1^2 = 1/2, \ s_2 = 0, \ A_1 = 0 \). The latter condition means that the MSW resonance occurs at the density \( N_1 \). The transition probability is equal to unity. This case is illustrated by Fig. 4; it requires very special conditions and we shall not discuss it anymore.

The behavior of the neutrino transition probability for \( \nu_e \leftrightarrow \nu_{\mu,\tau} \) oscillations as a function of the phases \( \phi_1 \) and \( \phi_2 \) for various values of \( \delta \) is shown in Figs. 2-8. For large \( \delta \) for which \( A_2 > 0 \) the maximum of the type \( (2) \) occurs (Fig. 3). In this domain of \( \delta \) the conditions \( (64) \) and \( (65) \) are not satisfied (see Fig. 1), so the extremum at \( c_1 = c_2 = 0 \) is a saddle point rather than maximum. Eq. \( (64) \) is fulfilled in the range \( 9.7 \times 10^{-14} \text{eV} < \delta < 2.12 \times 10^{-13} \text{eV} \), which results in a clear parametric peak at \( \phi_1 = \phi_2 = \pi/2 \) (Fig. 2). For \( \delta \) in the range \( 7.8 \times 10^{-14} \text{eV} < \delta < 9.7 \times 10^{-14} \text{eV} \) the function \( f_1 \) is positive while \( f_2 \) is negative, so the extremum at \( \phi_1 = \phi_2 = \pi/4 \) is again a saddle point; however, the value of the transition probability at this point is close to the maximal one (Figs. 5 and 6) except in the very close vicinity of the value \( \delta = \delta_1 = 8.6 \times 10^{-14} \text{eV} \) which corresponds to the MSW resonance at \( N = N_1 \). The transition probability for \( \delta = \delta_1 \) is shown in Fig. 4. The maxima of the transition probability are located at \( c_1^2 = s_1^2 = 1/2, \ s_2 = 0 \), as discussed in the case \( (3) \) above. For \( \delta < 7.8 \times 10^{-14} \text{eV} \) the function \( f_2 \) becomes positive and the extremum at \( \phi_1 = \phi_2 = \pi/4 \) again becomes a maximum (Fig. 7). Notice, however, that with decreasing \( \delta \) the height of the resonance peak rapidly decreases. The shape of the transition probability changes especially quickly in the region \( 7.5 \times 10^{-14} \text{eV} \lesssim \delta \lesssim 1 \times 10^{-13} \text{eV} \), i.e. close to \( \delta_1 \). For comparison, we also plotted in Fig. 8 the transition probability for \( N_2 = N_1 \); since matter density is constant, no parametric resonance occurs in this case. The probability of \( \nu_{e,\mu,\tau} \leftrightarrow \nu_s \) oscillations exhibits a similar pattern but for \( \delta \) scaled down by a factor of two.

As follows from (66), whether or not the parametric resonance can lead to a significant
increase of the transition probability over the time equal to “one and a half” periods of density modulation depends on the values of the mixing angles in matter $\theta_1$ and $\theta_2$. The expression (66) for the transition probability at the resonance has been found in [10] and used there to interpret a specific enhancement of the probability of atmospheric neutrino oscillations in the $\nu_{\mu} \leftrightarrow \nu_s$ channel in terms of the parametric resonance. However, in [10] only the case $\delta \ll V_1, V_2$ was considered. It is easy to see that, at least for not too large mixing in matter, effect is in fact the largest for $\delta$ in the interval $V_1 < \cos 2\theta_{\text{vac}} \delta < V_2$ (opposite sign $A_1$ and $A_2$), or at least close to this interval. In this range of $\delta$ one has $\theta_1 = \theta_{\text{mantle}} < \pi/4$, $\theta_2 = \theta_{\text{core}} > \pi/4$, and the expression $\theta_2 - 2\theta_1$ is closer to $\pi/4$ (which gives maximal transition probability) than either of $\theta_1$ and $\theta_2$ provided that $\theta_2 > \pi/4 + \theta_1$, $\theta_2 > 3\theta_1$ (small mixing in matter). Typically, the enhancement effect is even stronger in this situation than in the case of oscillations over one period of density modulation. Indeed, in that case the relevant combination of the mixing angles was $\theta_2 - \theta_1$ [see (45)]; for $\theta_2 > \pi/4 + 3\theta_1/2$ the combination of the mixing angles $\theta_2 - 2\theta_1$ is closer to $\pi/4$ than $\theta_2 - \theta_1$.

4.2 Parametric enhancement of neutrino oscillations in the earth

In the previous subsection we discussed the behavior of the transition probability as a function of the phases $\phi_1$ and $\phi_2$ treating these phases as free parameters. For neutrino oscillations in the earth, however, the phases $\phi_1$ and $\phi_2$ are not arbitrary: they depend on neutrino energy and on the lengths of neutrino paths in the mantle and core. These lengths vary with the zenith angle of the neutrino source, but their changes are correlated and they cannot take arbitrary values. It is therefore not obvious whether there are any values of the zenith angles and neutrino parameters for which the parametric resonance conditions can be satisfied. As we shall see, the answer to this question is positive.

Consider neutrinos coming to the detector from the lower hemisphere and traversing the mantle, core and then again mantle. As we have mentioned, this corresponds to zenith angles of the neutrino source $\Theta = 180^\circ \pm 33.17^\circ$. The distance $T_1$ that neutrinos travel in
the mantle (each layer) and the distance $T_2$ that they travel in the core are related to the zenith angle $\Theta$, earth’s radius $R$ and core radius $r$ by

$$
T_1 = R \left( -\cos \Theta - \sqrt{r^2/R^2 - \sin^2 \Theta} \right), \quad T_2 = 2R \sqrt{r^2/R^2 - \sin^2 \Theta}.
$$

(69)

Assuming the average matter densities in the mantle and core 4.5 and 11.5 $g/cm^2$ respectively, one finds for the $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations involving only active neutrinos the following values of $V_1$ and $V_2$:

$$
V_1 = 8.58 \times 10^{-14} \text{ eV}, \quad V_2 = 2.19 \times 10^{-13} \text{ eV}.
$$

(70)

For transitions involving sterile neutrinos $\nu_e \leftrightarrow \nu_s$ and $\nu_{\mu,\tau} \leftrightarrow \nu_s$, these parameters are a factor of two smaller.

Consider the lowest-order parametric resonance,

$$
\omega_1 T_1 = \frac{\pi}{2}, \quad \omega_2 T_2 = \frac{\pi}{2}.
$$

(71)

Given a value of $\delta$, these conditions yield magnitudes of $T_1$ and $T_2$ and so fix the zenith angle for which they are satisfied (provided, of course, that such an angle exists). The problem therefore reduces to finding values of $\delta$ and $\theta_{\text{vac}}$ for which eqs. (71) have solutions. A nontrivial point here is that both conditions in eq. (71) have to be satisfied at the same value of $\Theta$. Notice that these conditions constrain the allowed values of the vacuum mixing angle. Assuming that the resonance condition can be satisfied for all neutrino trajectories that cross the core (including the vertical ones) one can derive from the second equality in (71) the following upper limit

$$
\sin^2 2\theta_{\text{vac}} \leq \frac{\pi^2}{4(T_2)^2_{\text{max}} V_2^2} \simeq 0.04.
$$

(72)

If one excludes the zenith angles close to 180$^\circ$, the constraint becomes less stringent. For example, for $\sin^2 \Theta \geq 0.12$ one obtains $\sin^2 2\theta_{\text{vac}} \leq 0.07$.

\[3\text{An analogous upper bound following from the first equality is less restrictive.}\]
The analysis shows that the parametric resonance conditions for neutrino oscillations in the earth are indeed satisfied in a number of cases; those include $\nu_e \leftrightarrow \nu_{\mu,\tau}$ as well as $\nu_{e,\mu,\tau} \leftrightarrow \nu_s$ oscillations. A systematic study of all the cases fully covering the space of parameters $\delta$, $\sin^2 2\theta_{\text{vac}}$ and $\Theta$ is beyond the scope of the present paper; here we will discuss only a few cases of particular interest. In what follows, we shall discuss the neutrino trajectories in the earth in terms of the nadir angle $\Theta_n = 180^\circ - \Theta$.

(i) $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations, $\sin^2 2\theta_{\text{vac}} \ll 1$ (this case is especially relevant for oscillations of solar neutrinos in the earth). There is a broad range of nadir angles for which the parametric resonance conditions (71) are approximately satisfied: $0 \leq \Theta_n \lesssim 25^\circ$. The maximum of transition probability corresponds to $\delta \simeq (1.7 - 1.9) \times 10^{-13}$ eV, depending on $\Theta_n$. Fig. 9 shows the transition probability (63) as a function of $\delta$ for $\sin^2 2\theta_{\text{vac}} = 0.01$ and $\Theta_n = 11.5^\circ$. There is a parametric resonance peak at $\delta \simeq 1.9 \times 10^{-13}$ eV. A “shoulder” on the right slope of the peak is due to the MSW resonance in the core. The transition probability at the peak exceeds the biggest of $\sin^2 2\theta_1$ and $\sin^2 2\theta_2$ (which are the maximal oscillation probabilities in matter of constant densities equal to $N_1$ and $N_2$) by about a factor of 3. The values of $\phi_1 - \pi/2$ and $\phi_2 - \pi/2$ for this case are shown in Fig. 10. One can see that these functions vanish at very close values of $\delta$, i.e there is a range of $\delta$ (around the resonance value $\delta_0 \simeq 1.9 \times 10^{-13}$ eV) where both conditions in eq. (71) are satisfied to a very good accuracy. This pattern does not change in the whole interval $0 \leq \Theta_n \lesssim 25^\circ$: with varying $\Theta_n$ the lowest intersection point of the curves $\phi_1 - \pi/2$ and $\phi_2 - \pi/2$ moves slightly along the $\delta$ axis but remains close to it. For comparison, we have also plotted in Fig. 9 the transition probability for $V_2 = V_1$ which exhibits no parametric resonance.

In fact, a strong parametric enhancement of the transition probability persists up to the nadir angle $\Theta_n = 32.9^\circ$ (after which the resonance peak merges with the MSW resonance at $N = N_1$). The corresponding resonance values of $\delta$ are $\delta_0 \simeq (1.1 - 1.5) \times 10^{-13}$ eV. In this region of the nadir angles, the value of the transition probability at the peak is typically about a factor of 1.5 - 2 smaller than it is for $0 \leq \Theta_n \lesssim 25^\circ$. This is related to the fact
that the conditions (71) are only approximately satisfied: the detuning of $\phi_1$ and $\phi_2$ varies between $-0.3$ and $-0.9$ in this regime. However, since in this case the resonance energy is farther away from the energies corresponding to the MSW resonance in the mantle and core, the relative enhancement of the transition probability is even stronger: for example, for $\Theta_n = 30^\circ$ the transition probability at the peak exceeds the biggest of $\sin^2 2\theta_1$, $\sin^2 2\theta_2$ by a factor of 5.

$(ii)$ $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations, $\sin^2 2\theta_{\text{vac}} = \mathcal{O}(1)$. This case may be relevant for the large mixing angle MSW solution of the solar neutrino problem and is also of interest for atmospheric neutrino oscillations. For $\sin^2 2\theta_{\text{vac}} \simeq 0.6 - 0.8$ there is a parametric resonance in the narrow range of the nadir angles $\Theta_n \simeq 31.1^\circ - 32.7^\circ$. The corresponding resonance values of $\delta$ are $\delta_0 \simeq (6 - 7) \times 10^{-14}$ eV (see Figs. 11 and 12). In Fig. 11 a narrower high peak at $\delta \simeq 3.1 \times 10^{-13}$ eV is due to the MSW effect in the core.

$(iii)$ $\nu_{e,\mu,\tau} \leftrightarrow \nu_s$ oscillations, $\sin^2 2\theta_{\text{vac}} \ll 1$. At small nadir angles, $\Theta_n \lesssim 18^\circ$, the conditions (71) are satisfied for $\delta \simeq (1.5 - 1.7) \times 10^{-13}$ eV. However, since $\delta > V_2^2 / \cos 2\theta_{\text{vac}}$, eqs. (64) and (65) are not fulfilled; therefore this solution corresponds to a saddle point rather than to a maximum. There is no parametric enhancement for this range of $\delta$.

For a wide range of the nadir angles, $0 \leq \Theta_n \leq 32.6^\circ$, there is a maximum of transition probability with $\delta_0$ varying from $1 \times 10^{-13}$ eV (at $\Theta_n = 0$) to $6.8 \times 10^{-14}$ eV (at $\Theta_n = 32.6^\circ$). The corresponding values of $\phi_1$ and $\phi_2$, though, are $\phi_1 \simeq \phi_2 \simeq \pi/4$ rather than $\pi/2$. Nevertheless, this peak can be interpreted as a parametric resonance, at least for $\delta_0$ not very close to the ends of the allowed interval. The point is that the width of the parametric resonance in the $(\phi_1, \phi_2)$ plane is rather large, and detuning $\sim \pi/4$ just decreases the transition probability by about a factor of two (see eq. (67) and Fig. 2). Also, the approximate equality of $\phi_1$ and $\phi_2$ indicates the parametric origin of the peak. On the other hand, close to the end points of the allowed intervals of $\delta_0$, i.e. $\delta_0 \simeq 6.8 \times 10^{-14}$ eV and $\delta_0 \simeq 1 \times 10^{-13}$ eV, the peak values of the transition probabilities nearly coincide with either $\sin^2 2\theta_1$ or $\sin^2 2\theta_2$. In the vicinity of these points an interplay of the parametric and MSW
resonances takes place.

The parametric resonance for $\Theta_n \simeq 30.9^\circ$ ($\delta_0 = 7.8 \times 10^{-14}$ eV) is illustrated by Figs. 13 and 14. The transition probability at the peak exceeds $\sin^2 2\theta_1 \simeq \sin^2 2\theta_2$ by about a factor of 2.7.

(iv) $\nu_{e,\mu,\tau} \leftrightarrow \nu_s$ oscillations, $\sin^2 2\theta_{\text{vac}} = 1$. There is a parametric resonance peak for $\Theta_n = 25.8^\circ - 32.4^\circ$ and the resonance values $\delta_0 = (5 - 7) \times 10^{-14}$ eV. This is the parametric resonance found in [10].

5 Discussion

Neutrino oscillations can be parametrically enhanced in a medium with periodically varying density. The periodic step-function (“castle wall”) density profile is a very simple example; it allows for an exact solution of the neutrino evolution equation and therefore is very illuminating. In addition, this example is of practical importance since to a good approximation the density profile of the earth can be considered as a step-like function with nearly constant densities of the mantle and core.

The parametric resonance realizes very special conditions for neutrino oscillations, leading to a possibility of a striking increase of the transition probabilities. Typically, neutrinos have to travel over many periods of density modulation in order for the parametric enhancement to manifest itself. However, under certain conditions even for neutrino evolution during one period the parametric effects can be quite sizeable. Of course, in this case the neutrinos are not exposed to a periodic potential, and so one may wonder how the parametric enhancement of neutrino oscillations can occur. This can be explained as follows. The parametric resonance implies that the changes of the oscillation phase and matter density profile with the coordinate along the neutrino path are correlated in a very special way. This “synchronization” allows the transition probability to overbuild after every half-period of density modulation (see below); if the parameters $A_1$ and $A_2$ defined in eq. (7) are of
opposite sign, and in addition the oscillation amplitudes at the densities $N_1$ and $N_2$ are not too small, a considerable increase of the transition probability is possible even for one period. The parametric enhancement for one and a half periods of density modulation is then even larger.

It is instructive to examine under what conditions a complete neutrino flavor conversion over one period of density modulation is possible. Though rather contrived, this case clarifies the essence of the parametric resonance of neutrino oscillations. From eq. (45) we find that a complete conversion over period $T$ would require

$$2(\theta_1 - \theta_2) = \pm \pi/2$$  \hspace{1cm} (73)

in addition to the conditions (71). It is easy to see that, when eq. (73) is satisfied, the transition probability (45) is equal to the sum of the amplitudes of the oscillations in matter of constant densities $N_1$ and $N_2$, i.e.

$$\sin^2(2\theta_1 - 2\theta_2) = \sin^2 2\theta_1 + \sin^2 2\theta_2.$$  \hspace{1cm} (74)

Since at the resonance $\phi_1 = \phi_2 = \pi/2$, we have the following pattern of neutrino oscillations in this case. During the first part of the period, $0 \leq t \leq T_1$, usual oscillations in matter of constant density $N_1$ occur. At the time $t = T_1$ maximal possible in this case transition probability is achieved, which is $\sin^2 2\theta_1$. If the density remained constant, the transition probability $P$ would start decreasing after that and would return to zero at $t = 2T_1$. However, the density changes to $N_2$ and, as a consequence of the parametric resonance, the transition probability continues growing instead of starting decreasing. From eqs. (74) and (73) it follows that in the time interval $(T_1, T_1 + T_2)$ the transition probability $P$ undergoes one more half-period increase (with the amplitude $\sin^2 2\theta_2$), but starting from the initial value $\sin^2 2\theta_1$ and not from zero. Thus, in this idealized case the parametric resonance places one half-wave piece of the transition probability on the top of the other: the probability never decreases until it reaches the maximal value equal to unity. If the condition (71) is only approximately satisfied, there is some decrease of the transition probability after
the first “half-period”, but $P$ does not reach zero, and the decrease is followed by another increase. What happens is essentially that the average value around which the transition probability oscillates starts drifting towards larger values. In this way $P$ can become quite large even if the amplitude of the oscillations around the average value is small. This is illustrated by Fig. 15, where the dependence of the transition probability on the coordinate along the neutrino trajectory is shown.

Neutrino oscillations in the earth can undergo a strong parametric enhancement. Neutrinos coming to the detector from the lower hemisphere from a source with the zenith angle in the interval $\Theta = 180^\circ \pm 33.17^\circ$ traverse the earth’s mantle, core and then again mantle and so pass “one and a half” periods of density modulation. The possibility of parametric enhancement of atmospheric neutrino oscillations in the earth was pointed out in [10] where the case $\delta = \Delta m^2/4E \ll V_1, V_2$ and the oscillations in the $\nu_\mu \leftrightarrow \nu_s$ channel were discussed. It was shown that the parametric resonance can modify the zenith angle distribution of the atmospheric neutrino events. As follows from our consideration, the parametric resonance can occur for oscillations of atmospheric neutrinos in both $\nu_\mu \leftrightarrow \nu_e$ and $\nu_\mu \leftrightarrow \nu_s$ channels for large enough nadir angles, $\Theta_n \simeq 26^\circ - 32^\circ$.

The energy width of the parametric resonance $\Delta E/E_0$ is typically $\sim 2 - 3$. Since the atmospheric neutrino anomaly has been observed in a rather wide range of neutrino energies (sub-GeV and multi-GeV), it seems unlikely that the parametric resonance can alter considerably the gross features of the atmospheric neutrino oscillations, such as the allowed values of neutrino parameters $\Delta m^2$ and $\sin^2 2\theta_{\text{vac}}$. In any case, numerical analyses of the atmospheric neutrino data should have automatically taken the parametric enhancement into account. Still, we believe that it may be worth re-analyzing the data paying more attention to the parametric resonance and looking for its possible manifestations. It should be also stressed that the parametric resonance can occur in oscillations of atmospheric neutrinos even in those channels that are not responsible for the atmospheric neutrino anomaly; such effects are potentially observable and of considerable interest.
The parametric resonance may be very important for oscillations of the solar neutrinos in the earth. For small $\sin^2 2\theta_{\text{vac}}$, in a wide range of zenith angles almost completely covering the earth’s core, the $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations exhibit a strong parametric resonance with the peak at $\delta \simeq (1.7 - 1.9) \times 10^{-13}$ eV. For typical energy of $^8$B solar neutrinos $E \simeq 8$ MeV these values of $\delta$ give $\Delta m^2 \simeq (5.5 - 6.1) \times 10^{-6}$ eV$^2$. Amazingly, this almost exactly corresponds to the center of the allowed interval of $\Delta m^2$ for the so called “small mixing angle” MSW solution of the solar neutrino problem (for recent discussions see, e.g., [18, 13, 19]). Strong parametric enhancement of the probability of the oscillations of solar neutrinos in the earth, by up to a factor of 7, occurs in the whole allowed region of $\Delta m^2$ and $\sin^2 2\theta_{\text{vac}}$. The small mixing angle MSW solution gives the best fit of the available solar neutrino data and so is the most likely solution of the solar neutrino problem. The parametric resonance can also occur in the $\nu_e \leftrightarrow \nu_s$ oscillations in the earth, but the effect is smaller in this case. The resonance values $\delta \simeq (6.8 - 10) \times 10^{-14}$ eV correspond to $\Delta m^2 \simeq (2.2 - 3.2) \times 10^{-6}$ eV$^2$ (for $E = 8$ MeV). These values are close to the lower bound of the allowed interval of $\Delta m^2$ for the small mixing angle solution in this channel of oscillations. For large mixing angle the parametric resonance is also possible, but only in a rather limited interval of zenith angles of neutrino trajectories. In addition, the resonance values of $\Delta m^2$ in this case are below the allowed interval.

In both cases of $\nu_e \leftrightarrow \nu_{\mu,\tau}$ and $\nu_e \leftrightarrow \nu_s$ oscillations the parametric effects are strongest at neutrino energies which are between the energies corresponding to the MSW resonances in the core and in the mantle of the earth. This is illustrated by Fig. 16; one can see that the parametric peak at $\delta = 1.5 \times 10^{-13}$ eV is between the maxima of $\sin^2 2\theta_1$ and $\sin^2 2\theta_2$ which are due to the MSW resonances in the mantle and in the core respectively. In other words, the MSW resonance density for neutrinos with $\delta \simeq 1.5 \times 10^{-13}$ eV is about $7.8 \, g/cm^3$; this density is between the core density and the mantle density. The value of the transition probability at the parametric peak exceeds the amplitudes of neutrino oscillations in the mantle $\sin^2 2\theta_1$ and in the core $\sin^2 2\theta_2$ by more than a factor of six.
The enhancement of the transition probability in matter in the case when the minimal and maximal densities satisfy $N_1 < N_{\text{MSW}} < N_2$ would be of no surprise if the periodic density profile was, e.g., of the sinusoidal type. Then somewhere the matter density would be equal to the MSW resonance density, and one would expect an increase of the transition probability [9]. However, in the case of the "castle wall" density distribution the MSW resonance condition is not satisfied anywhere because the density profile is discontinuous: the only allowed values of density are $N_1$ and $N_2$. It is quite remarkable that in this case, too, there is a sizeable enhancement of the transition probability compared to the case of matter of constant density. This effect cannot be explained by the usual MSW-type increase of the oscillation amplitude; it is related to a specific phase relationships which are due to the parametric resonance.

One of the most promising ways of testing the MSW solution of the solar neutrino problem is to look for the day-night effect due to regeneration of solar $\nu_e$'s in the matter of the earth during the night [17, 20, 21]. For the most probable small mixing angle solution, however, this effect is expected to be rather small – at the level of $(1-3)\%$. The parametric resonance must increase significantly the day-night effect for neutrinos crossing the earth’s core. Unfortunately, due to their geographical location, the existing solar neutrino detectors have a relatively low time during which solar neutrinos pass through the core of the earth to reach the detector every calendar year. The Super-Kamiokande detector has a largest fractional core coverage time equal to 7\%. In [22] it was suggested to build a new detector close to the equator in order to increase the sensitivity to the earth regeneration effect; this would also maximize the parametric resonance effects in oscillations of solar neutrinos in the earth.

The parametric enhancement of oscillations of solar neutrinos in the earth for neutrinos that travel significant distances in the earth’s core was discovered numerically (but not recognized as the parametric resonance) in [21, 15, 22, 18]. The authors of [18] found that the day-night effect in the oscillations of solar neutrinos in the earth can be enhanced by
up to a factor of six when solar neutrinos cross the earth’s core. These results were quite surprising as the largest effect resulted for neutrinos of energies which correspond to the MSW resonance at the densities of about \((6 - 8) \text{g/cm}^3\), the values lying between the core and mantle densities. For neutrinos of those energies the MSW resonance is inoperative. The possibility of explaining these findings in terms of the parametric resonance was pointed out in [11], where also a different interpretation was suggested which the author of [11] seems to prefer. Our results support the parametric–resonance interpretation.

It is remarkable that the parametric resonance conditions can be fulfilled for neutrino oscillations in the earth. An important observation here is [22, 10, 23] that the potentials \(V_1\) and \(V_2\) corresponding to the matter densities in the mantle and core, inverse radius of the earth \(R^{-1}\), and typical values of \(\delta \equiv \Delta m^2/4E\) of interest for solar (and atmospheric) neutrinos, are all of the same order of magnitude – \((3 \times 10^{-14} - 3 \times 10^{-13}) \text{eV}\). It is this surprising coincidence that makes appreciable earth effects on oscillations of solar and atmospheric neutrinos possible. Even more remarkable fact is that the very specific conditions on the phases of neutrino oscillations in the mantle and core of the earth, \(\phi_1 \simeq \phi_2 \simeq \pi/2\), which lead to the parametric resonance of neutrino oscillations, are also satisfied in many cases of interest. Those include atmospheric neutrinos, and, most importantly, solar neutrinos.

For small vacuum mixing angle, the parametric resonance occurs for the values of \(\delta = \Delta m^2/4E\) which exactly correspond to the allowed values for the small mixing angle MSW effect in the sun. If the small mixing angle MSW effect is the solution of the solar neutrino problem, the oscillations of solar neutrinos in the earth therefore must undergo strong parametric resonance. The parametric resonance provides a natural interpretation for the increase of the night-time regeneration effect in the case of core-crossing neutrinos found numerically in [21, 15, 22, 18]. By selecting only those neutrinos, one can improve the prospects of observing the day-night effect (though at the expense of reduced statistics).

\[\text{Strictly speaking, the density profile of the earth is not discontinues; the densities (6 – 8) g/cm}^3\text{ can be achieved on the border of the mantle and core where the density jumps from 5.5 to 9.9 g/cm}^3\text{. However, the jump occurs in a very thin layer and so the MSW resonance does not develop.}\]
If observed, the enhanced day-night effect for core crossing neutrinos would therefore confirm both the MSW solution of the solar neutrino problem and the parametric resonance of neutrino oscillations.

The author is grateful to Q.Y. Liu and A.Yu. Smirnov for useful discussions and to S.T. Petcov for informing him about the results of the paper [11] prior to its publication.

References


Figure captions

Fig. 1. Functions $f_1(\delta)$ (upper solid curve) and $f_2(\delta)$ (lower solid curve) [eqs. (64) and (65)]. Also shown are $A_1(\delta) \times 10^{13}$ (dashed line) and $A_2(\delta) \times 10^{13}$ (short-dashed line), where $A_{1,2}$ are defined in eq. (7). $\sin^2 2\theta_{\text{vac}} = 0.01, \Theta_n = 11.5^\circ$.

Fig. 2. Transition probability (63) for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations over “one and a half periods” of density modulation ($t = 2T_1 + T_2$) vs oscillation phases $\phi_1$ and $\phi_2$ acquired during the time intervals $T_1$ and $T_2$. $\sin^2 2\theta_{\text{vac}} = 0.01, \delta = 1.2 \times 10^{-13}$ eV.

Fig. 3. Same as Fig. 2, but for $\delta = 2.5 \times 10^{-13}$ eV.

Fig. 4. Same as Fig. 2, but for $\delta = 8.6 \times 10^{-14}$ eV.

Fig. 5. Transition probability (63) for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth for core-crossing neutrinos vs $\delta$ (solid curve). Also shown is the probability in the case $V_2 = V_1 = 8.58 \times 10^{-14}$ eV (dashed curve). $\sin^2 2\theta_{\text{vac}} = 0.01, \Theta_n = 11.5^\circ$.

Fig. 6. Phases $\phi_1 - \pi/2$ (dashed curve) and $\phi_2 - \pi/2$ (solid curve) vs $\delta$ for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth. $\sin^2 2\theta_{\text{vac}} = 0.01, \Theta_n = 11.5^\circ$.

Fig. 7. Transition probability (63) for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth for core-crossing neutrinos vs $\delta$. $\sin^2 2\theta_{\text{vac}} = 0.6, \Theta_n = 32.2^\circ$.

Fig. 8. Phases $\phi_1 - \pi/2$ (solid curve) and $\phi_2 - \pi/2$ (dashed curve) vs $\delta$ for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth. $\sin^2 2\theta_{\text{vac}} = 0.6, \Theta_n = 32.2^\circ$.

Fig. 9. Transition probability (63) for $\nu_{\mu,\tau} \leftrightarrow \nu_s$ oscillations in the earth for core-crossing neutrinos vs $\delta$. $\sin^2 2\theta_{\text{vac}} = 0.01, \Theta_n = 30.9^\circ$.

Fig. 10. Phases $\phi_1 - \pi/4$ (solid curve) and $\phi_2 - \pi/4$ (dashed curve) vs $\delta$ for $\nu_{\mu,\tau} \leftrightarrow \nu_s$ oscillations in the earth. $\sin^2 2\theta_{\text{vac}} = 0.01, \Theta_n = 30.9^\circ$.

Fig. 11. Transition probability (63) for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth as a function of the distance $t$ (measured in units of the earth’s radius $R$) along the neutrino trajectory. $\delta = 1.7 \times 10^{-13}$ eV, $\sin^2 2\theta_{\text{vac}} = 0.01, \Theta_n = 11.5^\circ$. 31
Fig. 12. Transition probability (63) for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth. (solid curve). Also shown are $\sin^2 2\theta_1$ (dashed curve) and $\sin^2 2\theta_2$ (short-dashed curve). $\sin^2 2\theta_{\text{vac}} = 0.01$, $\Theta_n = 28.6^\circ$. 
Fig. 1.

Fig. 2.
Fig. 3.

Fig. 4.

34
Fig. 5.

Fig. 6.
Fig. 7.

Fig. 8.

36
Fig. 9.

Fig. 10.
Fig. 11.

Fig. 12.