NONLINEAR TWO-DIMENSIONAL POTENTIAL PLASMA WAKE WAVES

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Abstract

The condition for potential description of the wake waves, generated by flat or cylindrical driving electron bunch in cold plasma is derived.

The two-dimensional nonlinear equation for potential valid for small values of that is obtained and solved by the separation of variables. Solutions in the form of cnoidal waves, existing behind the moving bunch at small values of vertical coordinate, are obtained. In particular, at some boundary conditions, corresponding to blow-out regime in the underdense plasma, the solution represents by a solitary nonlinear wave.

Approximate solution is also obtained using the method of multiple scales.

The indications are obtained that the dependense of the amplitudes on longitudinal coordinate determines essentially, even in the first approximation, by driving bunch charge distribution. The wake wave amplitude can increase at some conditions along the longitudinal distance from the rear part of the bunch.

1 INTRODUCTION

Analysis of the one dimensional longitudinal, transverse and coupled transverse longitudinal plain nonlinear waves in cold relativistic plasma are given in the review [7] (see therein the references on original works). One dimensional nonlinear longitudinal
waves, generated by the driving bunches with the infinite transverse dimensions, were considered in [?]-[?].

In the present work the two-dimensional nonlinear wake waves, generated by the flat or cylindrical electron bunch, are discussed. The corresponding linear problem was considered in [?],[?] and was found, in correspondence with previous result [?], that the magnetic field in wake wave in linear approximation is equal zero. This result connected with the absence of the energy flow in the wake wave and the absence of the vortices in plasma electron motion in linear approximation.

In the one-dimensional nonlinear treatment [?]-[?] the magnetic field in the wake wave is also zero (by construction), due to the symmetry of the problem relative to transverse displacements.

In two-dimensional wake wave the magnetic field is zero only when vortices connected with the plasma electron motion are zero, which in this case is an additional requirement on the type of the motion of the plasma electrons. The wake waves in this case are potential, i.e., electric field components \( E_z, E_y \) can be expressed as a gradient of one scalar function \( \varphi(y, z) \), and components of the plasma electrons current also can be expressed through one scalar function \( \psi(y, z) \).

The approximate nonlinear equation for potential \( \varphi \) can be obtained, using Maxwell equations and approximate equations of the motion. Equation for potential has an exact solutions with the separated variables for small values of the transverse coordinate.

Among the solutions, which are finite, nonlinear waves by cnoidal nature, at some boundary conditions, associated with the blow-out regime, there exists the solution (on separatirix) in form of the solitary wave.

For arbitrary values of the transverse coordinate the approximate solution for potential, using multiple scales perturbative method is found.

2 VORTEX-FREE WAKE WAVE

Consider the wake wave generated in the cold neutral plasma, with the immobile
ions, by the flat electron bunch, which has horizontal dimensions $2a$ much larger than vertical dimension $2b$, longitudinal dimension is $2d$. The charge density in the bunch is $n_b$, electron plasma density is $n_0$, and we consider both overdense and underdense regimes.

Bunch is moving along $z$-axis with the constant velocity $v_0 < c$ in lab system. All the physical quantities in the question are considered as a function of vertical coordinate $y$ and $\tilde{z} = z - v_0 t$. An electrical field, generated by the bunch $|E_x| \ll E_y \neq 0, E_z \neq 0$ and magnetic field $B_z = 0, |B_y| \ll |B_x| \equiv |B| \neq 0$.

Introduce the dimensionless variables and arguments by

\[
\tilde{E} = \sqrt{4\pi mn_0^2} E' = \frac{\omega_m v_0}{e} \tilde{E}', \\
\tilde{B} = \sqrt{4\pi mn_0^2} B' = \frac{\omega_m v_0}{e} \tilde{B}'.
\]

\[
z', y' = k \tilde{z}, ky, k^2 = \frac{\omega^2}{v_0^2} = \frac{4\pi n_{en}}{mv_0^2}, n'_b = \frac{n_b}{n}, n'_0 = \frac{n_0}{n}
\]

where $n$ is the arbitrary electron density, which is convenient to choose equal $n = n_b$ in the underdense ($n_b > n_0$) case and $n = n_0$ in the overdense ($n_b < n_0$) case.

Following [2], [7] introduce BFTCh-transformation of the variables

\[
V_z = \frac{\beta_{ex} n'}{\beta - \beta_{ez}}, V_y = \frac{\beta_{ey}}{\beta - \beta_{ez}}, \beta = \frac{v_0}{c}, \beta_{ex} = \frac{\tilde{v}_{ex}}{c}
\]

\[
n'_b = \frac{\beta N}{\beta - \beta_{ez}} = N(1 + V_z);
\]

($N \to n'_b, \beta_{ez} \to 0$, when $z, y \to +\infty; v_{ex} = 0$).

The Maxwell equations then can be rewritten in the following form (superscript prime
is omitted in what follows):

\[(a) \quad \frac{\partial B}{\partial y} = \beta NV_z + \beta \frac{\partial E_z}{\partial z} + \beta n_b \]

\[(b) \quad \frac{\partial (B + \beta E_y)}{\partial z} = -\beta NV_y \]

\[(c) \quad \frac{\partial (\beta B + E_y)}{\partial z} = \frac{\partial E_z}{\partial y} \]

\[(d) \quad \frac{\partial E_z}{\partial z} + \frac{\partial E_y}{\partial y} = (n_0 - n_b) - N(1 + V_z) \]

The continuity equation \(\frac{\partial N}{\partial z} = \frac{\partial NV}{\partial y}\) follows from (??,d),(??,a),(??,b). Using (??,a), (??,b),(??,c) we have

\[
\frac{\partial^2 B}{\partial y^2} + (1 - \beta^2) \frac{\partial^2 B}{\partial z^2} = \text{rot}_x(\beta NV) + \text{rot}_x(\beta n_b) \]

which means that the magnetic field is zero in plasma (linear or nonlinear) wake wave only when

\(\text{rot}(\beta NV) = 0\)

i.e. the plasma electrons motion is vortex-free.

In the following, we consider the region of the space occupied by wake wave i.e. \(z < -d\). Maxwell equations (??) for wake waves under condition (??) can be obtained putting in (??) \(B = 0\) and \(n_b = 0\).

\[(a) \quad \frac{\partial E_z}{\partial z} = -NV_z \]

\[(b) \quad \frac{\partial E_y}{\partial z} = -NV_y \]

\[(c) \quad \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y} \]

\[(d) \quad \frac{\partial E_y}{\partial y} = n_0 - N \]
Then from (7) follows that

\[ \vec{E} = -\nabla \phi \]  

i.e. the wake fields under condition (8), as it must be, are potential.

3 THE BASIC EQUATION FOR THE POTENTIAL, EXACT SOLUTION IN SEPARABLE ARGUMENTS

Consider Maxwell equations (4) for wake waves, when \( z < -d \). From (6.a),(6.b) and (6) we have

\[ N = n_0 + \frac{\partial^2 \phi}{\partial y^2} \]  

(10)

\[ \frac{\partial^2 \phi}{\partial z^2} = \left( n_0 + \frac{\partial^2 \phi}{\partial y^2} \right) V_z \]  

(11)

From hydrodynamic equation of the plasma wake wave electrons motion, using (6),(7),(8) it is possible to obtain the relativistic equation of motion for the \( V_z \) component of the generalized velocity:

\[ -\frac{\partial V_z}{\partial z} + V_y \frac{\partial V_z}{\partial y} = -W^{1/2} \left[ E_z \left( 1 + 2V_z + \frac{V_z^2}{\gamma^2} \right) + \beta^2 V_z V_y E_y \right], \]  

(12)

\[ W \equiv 1 + 2V_z + \frac{V_z^2}{\gamma^2} - \beta V_y^2; \]

Neglecting terms with the squares of generalized velocity, compared to the terms with the first power of that, the expression (11) converted to

\[ \frac{\partial V_z}{\partial z} \approx E_z(1 + 3V_z) \]  

(13)

The solution of this equation, using (9), is

\[ V_z = \frac{1}{3}(e^{-3\phi} - 1) \approx -\phi \]  

(14)

with the condition \( \phi = 0 \), when \( V_z = 0 \).

Substituting (9) in (11) we have the basic equation for \( \phi \)

\[ \frac{\partial^2 \phi}{\partial z^2} + \phi \frac{\partial^2 \phi}{\partial y^2} + n_0 \phi = 0 \]  

(15)
Nonlinear term in eq. (28) is proportional to $\frac{\partial^2 \varphi}{\partial y^2}$ and can be large. In the linear approximation solution of eq. (28) lost the $y$-dependence and describes the harmonic oscillation with the plasma frequency on $z$; $y$-dependence of the solution comes from boundary condition at $z = -d$ and coincides with it for all $z < -d$. This is always the case, when wake waves are described as the product of the two functions from separate arguments $y$ and $z$. Such a situation takes place in linear approximation [7],[8],[9].

The eq. (28) permits to search the solution with the separable variables

$$\varphi(y, z) = \varphi_1(y) \varphi_2(z)$$

(16)

$$\frac{\varphi''_2 + n_0 \varphi_2}{\varphi_2^2} = - \varphi''_1 \equiv -k$$

(17)

where $k$ is a separation constant. The equations for $\varphi_1$ and $\varphi_2$ are:

$$\varphi_1'' = k$$

(18)

$$\varphi_2'' + n_0 \varphi_2 + k \varphi_2^2 = 0$$

(19)

Due to the symmetry of the problem the solution of equation (28) must be symmetric on $y$; $\varphi_1'(y = 0) = 0$ due to $E_y = 0$ at $y = 0$. The solution of the linear problem [7],[8] is concentrated in the region of the "trace", falling outside it exponentially. Adopting the same picture of the potential flow for considering case too, the solution of eq. (28) is

$$\varphi_1(y) = \frac{ky^2}{2} + A = k \left( \frac{y^2}{2} + a \right),$$

which is valid for small values of $y$. It means that, the solution of eq. (28) in separable arguments exists only for small values of $y < b$.

The equation (28) is the equation for nonlinear oscillator, with nonlinear part of the force proportional to $\varphi_2^2$ (for mathematical pendulum the first nonlinear term is proportional to $\varphi_2^3$ see e.g. [7]). The general solution of this equation is given in the implicit form by

$$-(z + d) = \pm \int_{\varphi_2}^{\varphi_0} \frac{d\varphi_2}{\sqrt{2[h - F(\varphi_2)]^{1/2}}}.$$
sign $\pm$ corresponds to positive or negative $\frac{d\varphi}{dz}$ subsequently and $h$ is an energy constant, defined by

$$h = \frac{1}{2} \varphi_2^2 + \frac{n_0}{2} \varphi_2^2 + \frac{k}{3} \varphi_2^3$$

(21)

and determined from boundary condition at $z = -d$, $\varphi_2(-d) \equiv \varphi_0$, $\varphi_2'(-d) \equiv \varphi_0'$

The function $F(\varphi_2)$ is

$$F(\varphi_2) = \frac{n_0}{2} \varphi_2^2 + \frac{k}{3} \varphi_2^3$$

(22)

The separatrix, $h_s = \frac{1}{\delta \varphi} n_0^3$, is the tangent to $F(\varphi_2)$ at its maximum point; $F(\varphi_2)$ has three real roots: double root equal zero and one root at $B = \frac{3n_0}{2k}$. The roots of the equation

$$h - F(\varphi_2) = 0,$$

(23)

are $c_i (i = 1, 2, 3)$ The different solutions (23) of the equation (22) defined by the value of the $h$, which in turn, depends on the boundary values $\varphi_0$ and $\varphi_0'$. Finite solutions for $k < 0$ is $c_1 \leq \varphi_2 \leq c_2$, which is existed, when $h \leq h_s$, i.e. $c_s \leq c_1 \leq 0$, where

$c_s = -\frac{mn}{2m}$, and $0 \leq c_2 \leq c_m = \frac{mn}{2m}$,

($c_m$ corresponds to the local maximum of the function $F(\varphi_2)$, equal to $h_s$). The third root of the eq (22) is $c_3$: $c_m \leq c_3 \leq B = \frac{3n_0}{2k}$. For $k > 0$ finite solutions exist, when $c_2 \leq \varphi_2 \leq c_3$, $0 \leq h \leq h_s$; for $k = 1$, $-\frac{3n_0}{2} \leq c_1 \leq -n_0, -n_0 \leq c_2 \leq 0, 0 \leq c_3 \leq \frac{n_0}{2}$.

For the cylindrical bunch with length $2d$ and radius $R_0$ eq. (22) is written in the form

$$\frac{\partial^2 \varphi}{\partial z^2} = \left[ n_0 + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) \right] V_z$$

The approximate equation of motion for $V_z$ has the same form as (22) and $V_z$ approximately is equal to $\varphi$. Basic equation for $\varphi$ is then

$$\frac{\partial^2 \varphi}{\partial z^2} + \varphi \left( \frac{1}{2} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) \right) + n_0 \varphi = 0$$

Solution of this equation in separable arguments $\varphi(r, z) = \varphi_1(r) \varphi_2(z)$ can be obtained by solving the equations:

$$\frac{1}{2} \frac{d}{dr} \left( r \frac{d \varphi_1}{dr} \right) = k$$
\[
\frac{d^2 \varphi_2}{dz^2} + n_0 \varphi_2 + k \varphi_2^2 = 0
\]

The last equations for \( \varphi_2(z) \) coincides with the eq (??) for flat case. Equation for \( \varphi_1(r) \) has the solution finite for small \( r \)

\[
\varphi_1(r) = \frac{k}{2} \left( \frac{r^2}{2} + a \right)
\]

. Hence the cylindrical bunch case is described practically by the same equations as a flat one with the evident changes from \( \varphi_1(y) \) to \( \varphi_1(r) \).

4 BOUNDARY CONDITIONS

The definition \( \varphi = \varphi_1(y) \varphi_2(z) \) permits the transformation \( \varphi_1 = |k| \tilde{\varphi}_1, \varphi_2 = |k|^{-1} \tilde{\varphi}_2 \). Then eqs. (??) and (??) for \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) will have the following forms:

\[
\begin{align*}
\tilde{\varphi}_1'' &= \pm 1, \\
\tilde{\varphi}_2'' &= n_0 \tilde{\varphi}_2 \pm \tilde{\varphi}_2^2 = 0,
\end{align*}
\]  

(24)

which corresponds to value of \( k = \pm 1 \). It means that separation constant \( k \) is arbitrary and can be chosen as \( k = \pm 1 \). In what follows the sign "bar" over \( \varphi_1, \varphi_2 \) is omitted and \( k \) is chosen plus one, which provide meaningful results for our case, when \( z < -d; k = -1 \) is suitable for \( z > 0 \). Then

\[
\varphi = \varphi_1(y) \varphi_2(z) = (a + y^2/2) \varphi_2(z)
\]  

(25)

If at \( z = -d \) the physical quantities are

\[
\begin{align*}
E_z^d &\equiv E_z(y, z = -d) = E_{z0} + E_{zz2}y^2 = -\left( \frac{\partial \varphi}{\partial z} \right)_{z=-d}, \\
V_z^d &\equiv V_z(y, z = -d) = V_{z0} + V_{zz2}y^2 = -\varphi(y, z = -d),
\end{align*}
\]  

(26)

the unknown constants \( \varphi_0, \varphi_0 ', a \) entering in the solution (??, ??, ??, ??) are

\[
\begin{align*}
\varphi_0' &= -2E_{z2}, \\
\varphi_0 &= -2V_{z2}, \\
a &= \frac{2E_{z0}}{E_{z2}} = \frac{V_{z0}}{2V_{z2}}
\end{align*}
\]  

(27)
From (??,a) and (??)

\[ NV_z = \frac{\partial^2 \varphi}{\partial z^2} = \varphi_1(y)\varphi_2''(z) = \varphi_1(y)(-n_0\varphi_2 - \varphi_2') \]  

(28)

and, when \( z=-d \)

\[(NV_z)^d = -(a + y^2/2)(\varphi_0 + n_0)\varphi_0 = V_z^d(\varphi_0 + n_0), \]

i.e.

\[ N^d = \varphi_0 + n_0, \quad N^d = n_e^d(1 + V_z^d) \]  

(29)

and is independent from \( y \).

Consider the case of underdense plasma \( n_b > n_0 \), when all plasma electrons behind the bunch are "blow out" \( n_e^d = 0 \), i.e. \( \varphi_0 = -n_0 \). Following [?] assume that \( E_z^d = 0 \), i.e. \( \varphi_0 = 0 \) according to (??,??). Then constant \( h \) (??) is equal to

\[ h = \frac{n_0}{2} \varphi_0^2 + \frac{1}{3} \varphi_0^3 = \frac{1}{6} n_0^3 = h_s \]

i.e. the solution, corresponding to the "blow out" regime, lies on separatrix.

The constant \( h \) (??), also can be expressed through the roots \( c_i = \alpha_i n_0 \) of the equation (??):

\[ h = \frac{1}{3} c_1 c_2 c_3 = -\frac{\alpha_1 \alpha_2 \alpha_3}{3} n_0^3 \]

(30)

where \(-3/2 \leq \alpha_1 \leq -1, -1 \leq \alpha_2 \leq 0, 0 \leq \alpha_3 \leq 1/2 \) for \( k > 0 \). For the separatrix \( \alpha_1 = \alpha_2 = -1, \alpha_3 = 1/2 \)

\[ h = h_s = \frac{n_0^3}{6} \]

(31)

For the values \( h > h_s \) and \( h < 0 \) as it is evident, the solutions for \( \varphi_2 \) have an infinite values. When \( \varphi_0 < c_2 \) the solution became unphysical, even for \( 0 \leq h < h_s \).

5 FINITE NONLINEAR SOLUTIONS

First consider the case when \( c_2 \leq \varphi_0, \varphi_2 < c_3 \). From general solution (??), using known expressions for the elliptic integrals and elliptic functions [?],[?], we have
\[ \varphi_2(z) = c_3 - (c_3 - c_2) \text{sn}^2 z_1, \] 

where

\[ z_1 \equiv F(\gamma_0, q) + \frac{1}{2} \sqrt{\frac{2(c_3 - c_1)}{3}} (z + d) \] 

\[ \gamma_0 = \arcsin \sqrt{\frac{c_3 - \varphi_0}{c_3 - c_2}}, \] 

\[ q = \sqrt{\frac{c_3 - c_2}{c_3 - c_1}} = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}} \] 

and \( F(\gamma_0, q) \) is the elliptic integral of the first kind, \text{sn}z_1- elliptic function.

Using (32),(33),(34) and (35) it is possible to obtain

\[ E_y = -\frac{\partial \varphi}{\partial y} = -y \varphi_2(z) = -yn_0[\alpha_3 - (\alpha_3 - \alpha_1)\text{sn}^2 z_1] \] 

\[ E_z = -\frac{\partial \varphi}{\partial z} = \left( \frac{y^2}{2} + a \right) n_0(\alpha_3 - \alpha_2) \left[ \frac{2}{3} n_0(\alpha_3 - \alpha_1) \right]^{1/2} \text{sn}z_1 \text{ch}z_1 \text{dn}z_1 = \] 

\[ = \frac{1}{2} \left( \frac{y^2}{2} + a \right) n_0(\alpha_3 - \alpha_2) \left[ \frac{2}{3} n_0(\alpha_3 - \alpha_1) \right]^{1/2} \text{sn}^2 z_1 \left[ 1 - \left( \frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1} \right) \text{sn}^4 z_1 \right] \] 

The length \( \lambda_n \) of the nonlinear wave is given by

\[ \frac{\omega}{v_0} \lambda_n = 2 \int_{c_2}^{c_3} \frac{1}{\sqrt{2(h - F(\varphi_2))^{1/2}}}, \] 

\[ \lambda_n = \frac{4v_0}{\omega} \left( \frac{3}{2n_0} \right)^{1/2} \frac{1}{(\alpha_3 + |\alpha_1|)^{1/2}} \times \] 

\[ \times F \left( \frac{\pi}{2}, \left( \frac{\alpha_3 + |\alpha_2|}{\alpha_3 + |\alpha_1|} \right)^{1/2} \right) \] 

In the linear case

\[ h \to 0, \alpha_2 \to 0, \alpha_3 \to 0, \alpha_1 \to -3/2, \] 

and \( \lambda_n \to \lambda_p = \frac{2\pi v}{c_p} \).
From (??) it follows, that \( E_y \) is by the order of magnitude equal to \( E_y \sim y n_0 \), or in the ordinary units

\[
E_y \sim \frac{m \omega_p v \omega_p}{e} y = 4 \pi \epsilon n_p y,
\]
which coincides with the field inside the flat bunch uniformly charged with the plasma electron density \( n_p \).

From (??) the longitudinal component of the electric field is by order of magnitude equal to \( E_z \sim \left( \frac{y^2}{2} + a \right) n_0^{3/2} \), or in ordinary units

\[
E_z \sim \frac{m \omega_p v_0 \omega_p^2}{e} \left( \frac{y^2}{2} + \tilde{a} \right) = 4 \pi \epsilon n_p \omega_p \left( \frac{y^2}{2} + \tilde{a} \right)
\]
where \( \tilde{a} = \frac{2E_0}{E_z} \), \( E_z(y, z = -d) = E_{z0} + E_{z2} y^2 \) in ordinary units, so at some conditions it can be larger than \( E_y \).

In the case \( h = h_s \) the changes due to nonlinearity are more drastic.

As we have seen, this case corresponds to the conditions

\[
n_b > n_0, n_e(z = -d, y = 0) \equiv n_d = 0,
\]
and \( E_{z0} = 0 \), which resembles the blow-out regime in underdense plasma (??). In this case

\[
c_1 = c_2 \rightarrow c_m = -n_0, c_3 \rightarrow c_s = +n_0,
\]

\[
h - F(\varphi_2) \rightarrow \frac{1}{3} (\varphi_2 + n_0)^2 \left( \frac{n_0}{2} - \varphi_2 \right)
\]

From (??) with the minus sign in the front of integral (\( \varphi_2 \) decreases when \( z \) increases from \(-\infty \) up to \(-d \)), using [??] it follows

\[
\varphi_2 = -n_0 + \frac{3}{2} \frac{n_0}{\hbar^2} \frac{\psi}{2}
\]
where

\[
\psi = \theta_0 - \sqrt{n_0(z + d)}, \theta_0 = \ln \left| \frac{\left( \frac{n_0}{2} - \varphi_2 \right)^{1/2} + \left( \frac{3}{2} n_0 \right)^2}{\left( \frac{n_0}{2} - \varphi_0 \right)^{1/2} - \left( \frac{3}{2} n_0 \right)^2} \right|
\]
In the (??) it is necessary to use \( \varphi_0 = -n_0 + \epsilon, \epsilon > 0 \) and pass to limit \( \epsilon \rightarrow 0 \) at fixed \( \psi \). Then at \( \psi = 0, z \rightarrow -\infty \), and at \( \psi \rightarrow -\infty, z \rightarrow -d \).

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The electric field components in the considered case of the blow-out regime are

\[ E_y = \frac{1}{2} y n_0 (3th^2 \psi / 2 - 1) \]  

(41)

\[ E_z = 3 n_0^{3/2} \left( \frac{y^2}{2} + \frac{V_{\psi_0}}{n_0} \right) (1 - th^2 \psi / 2) th \psi / 2 \]  

(42)

The maximum value of the transversal component of the field by the order of magnitude is \( E_{y,\text{max}} \approx y n_0 \).

Longitudinal component \( E_z = 0 \) when \( z \to -d, (\psi \to -\infty) \) and when \( z \to -\infty, (\psi \to 0) \). Its maximum value is at \( \psi_0 = -1, 32 \) and by the order of magnitude is equal

\[ E_{z,\text{max}} \approx n_0^{3/2} \left( \frac{y^2}{2} + \frac{V_{\psi_0}}{n_0} \right). \]

Hence the maximum values of the field coincides with that in case when \( 0 \leq h < h_s \). The difference is in the form of the wave: when \( h < h_s \) the wake wave (??-??) is cnoidal and when \( h = h_s \) (blow out regime) the wave (??-??) is solitary one.

6 APPROXIMATE SOLUTION OBTAINED BY MULTIPLE SCALES METHOD

In order to obtain the solution of the basic eq. (??) valid for larger values of \( y \) it is necessary to solve (??) by some other methods. It seems that the multiple scales approximate method (development of the derivative [??]) is suitable for this propose.

The small parameter in question for considered case is \( \epsilon = \left( \frac{\epsilon_{\text{max}}}{mc^2} \right) \ll 1, (\varphi' \ll 1, \varphi \) here is in ordinary units). According to [??] introduce the different scales variables in \( z \)

\[ z_0 = z, \; z_n = \epsilon^n (z + d), n = 1, 2, 3 \ldots \]

and perform the following developments

\[ \varphi(y, z) = \tilde{\varphi}(y, z_0, z_1, z_2, \ldots) = \]

(43)

\[ = \epsilon \varphi_1(y, z, z_1, z_2, \ldots) + \epsilon^2 \varphi_2(y, z, z_1, z_2, \ldots) + \epsilon^3 \varphi_3(y, z, z_1, z_2, \ldots) \]

\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \epsilon \frac{\partial}{\partial z_1} + \epsilon^2 \frac{\partial}{\partial z_2} + \ldots \]

\[ \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z_0^2} + 2 \epsilon \frac{\partial^2}{\partial z_0 \partial z_1} \epsilon^2 \left( 2 \frac{\partial^2}{\partial z_0 \partial z_2} + \frac{\partial^2}{\partial z_1^2} \right) + \ldots \]
(For dimensionless function and arguments it is necessary to put \( \epsilon = 1 \) in the final results.)

Substitution of the developments (??) in (??) gives the following set of the equations for subsequent approximations:

\[
\frac{\partial^2 \varphi_1}{\partial z_0^2} + n_0 \varphi_1 = 0
\]  
(44)

\[
\frac{\partial^2 \varphi_2}{\partial z_0^2} + n_0 \varphi_2 = -\varphi_1 \frac{\partial^2 \varphi_1}{\partial y^2} - 2 \frac{\partial^2 \varphi_1}{\partial z_0 \partial z_1}
\]  
(45)

\[
\frac{\partial^2 \varphi_3}{\partial z_0^2} + n_0 \varphi_3 = -\varphi_1 \frac{\partial^2 \varphi_2}{\partial y^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial y^2} - 2 \frac{\partial^2 \varphi_2}{\partial z_0 \partial z_1} - \left( \frac{2 \frac{\partial^2 \varphi_1}{\partial z_0 \partial z_1}}{\frac{\partial^2 \varphi_1}{\partial z_0^2} + \frac{\partial^2 \varphi_1}{\partial z_1^2}} \right) \varphi_1
\]  
(46)

The general solution of eq. (??) is

\[
\varphi_1 = a(y_1, z_1, z_2) e^{-i \sqrt{n_0} z_0} + a^*(y_1, z_1, z_2) e^{+i \sqrt{n_0} z_0}
\]  
(47)

and \( y \)-dependence of the solution (??) comes from boundary conditions at \( z_0 = -d_1, z_1 = 0, z_2 = 0 \). If

\[
E_z = -\frac{\partial \varphi_1(y, z_0 = -d, z_1 = 0, z_2 = 0)}{\partial z_0} \equiv g(y)
\]

\[
E_y = -\frac{\partial \varphi_1(y, z_0 = -d, z_1 = 0, z_2 = 0)}{\partial y} \equiv f(y)
\]

then

\[
Rea(y, z_0 = -d, z_1 = 0, z_2 = 0) =
\]

\[= -\frac{1}{2} \int_0^y \left( f(y) \cos n_0 d + \frac{1}{\sqrt{n_0}} g(y) \sin \sqrt{n_0} d \right) dy
\]

\[Ima(y, z_0 = -d, z_1 = 0, z_2 = 0) =
\]

\[= \frac{1}{2} \int_0^y \left( f(y) \sin \sqrt{n_0} d + \frac{1}{\sqrt{n_0}} g(y) \cos \sqrt{n_0} d \right) dy
\]

The second term is right hand side of the eq. (??) is secular due to the solution (??); it can be eliminated if \( \varphi_1 \) is independent on \( z_1 \) i.e.

\[
\frac{\partial \varphi_1}{\partial z_1} = 0
\]  
(49)

It means that \( \varphi_2 \) is also independent on \( z_1 \) and the general solution of the eq. (??), taking into account (??), is

\[
\varphi_2 = B(y, z_2) + A(y, z_2) e^{-2i \sqrt{n_0} z_0} + A^*(y, z_2) e^{2i \sqrt{n_0} z_0} + b(y, z_2) e^{-i \sqrt{n_0} z_0} + b^*(y, z_2) e^{i \sqrt{n_0} z_0}
\]  
(50)
where
\[
B = -\frac{1}{n_0} \left( a \frac{\partial^2 a^*}{\partial y^2} + a^* \frac{\partial^2 a}{\partial y^2} \right), \quad A = -\frac{1}{3n_0} a \frac{\partial^2 a}{\partial y^2}
\]

(51)

Function \( b(y, z_2) \) entering in the solution of the homogeneous part of the eq. (??) can be found from the boundary conditions

\[
\varphi_2(y, z_0 = -d, z_2 = 0) = 0
\]
\[
\frac{\partial \varphi_2(y, z_0 = -d, z_2 = 0)}{\partial z_0} = 0
\]

and has the following value

\[
b = \frac{1}{2} \left( A^* e^{-3i\sqrt{n_0}d} - 3A e^{i\sqrt{n_0}d} + B \right)
\]

(52)

The \( z_2 \)-dependense of the function \( a(y, z_2) \) (and subsequently the \( z_2 \)-dependense of \( A, B, b \)) comes out from the consideration of the eq. (??) for the third approximation. Eq. (??), due to the independense of \( \varphi_1, \varphi_2 \) on \( z_2 \) (??), has the form:

\[
\frac{\partial^2 \varphi_3}{\partial z^2} + n_0 \varphi_3 = -2 \frac{\partial^2 \varphi_1}{\partial z_0 \partial z_2} - \varphi_1 \frac{\partial^2 \varphi_2}{\partial y^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial y^2}
\]

(53)

Using solutions (??, ??) for \( \varphi_1, \varphi_2 \) it is evident that right hand side of the eq. (??) has the secular terms, proportional to \( e^{\pm i\sqrt{n_0}z_2} \). The conditions for their elimination are

\[
\frac{\partial a}{\partial z_2} - \frac{1}{n_0} \frac{\partial^2 a}{\partial y^2} \left( a^* \frac{\partial a^*}{\partial y^2} + a \frac{\partial^2 a}{\partial y^2} \right) + \frac{1}{3n_0} a^* \frac{\partial^3 a}{\partial y^3} \frac{\partial^2 a}{\partial y^2} - \\
- \frac{1}{n_0} \frac{\partial^3 a}{\partial y^3} \left( a^* \frac{\partial^2 a^*}{\partial y^2} + a \frac{\partial^2 a}{\partial y^2} \right) + \frac{1}{3n_0} a^* \frac{\partial^3 a}{\partial y^3} \left( a \frac{\partial a}{\partial y^2} \right)
\]

(54)

and subsequent conjugate expression.

The eq. (??) determines the dependense of \( a(y, z_2) \) from \( z_2 \). Eq. (??) is complicated enough, it is a system of the first order differential equations for \( Re a(y, z_2) \) and \( Im a(y, z_2) \). The \( y \)-dependense of \( a(y, z_2 = 0) \) is given by boundary conditions.

The eq. (??) simplifies under the assumption that \( a(y, z_2) \) is

\[
a(y, z_2) = Y(y) Z(z_2)
\]

(55)
where $Z(z_2 = 0) = 1$, and $Y(y)$ is known from the boundary conditions (57). Under (57) eq. (57) takes the form

$$2i\sqrt{\frac{1}{n_0}} \frac{dz}{dz_2} = \psi(y)|Z|^2Z$$

(56)

where $\psi(y)$ is

$$\psi(y) = \frac{8}{3n_0}Y''' + \frac{2}{3n_0}Y''Y'' - \frac{1}{2}YY'''$$

$$+ \frac{2}{n_0}(Y''Y'' + Y''' + \frac{1}{2}YY''')$$

(57)

The function $\psi(y)$ is slowly varied on $y$, when $|y| \leq b$ and is zero, when $|y| > b$. It is reasonable to average the eq. (57) on $y$. Strictly speaking the need of such kind of procedure indicates that assumption (57) is not in full appropriate, but practically can work for slowly varying $\psi(y)$.

After averaging eq. (57) and its conjugate one gives the following system for $x \equiv Re Z$ and $y \equiv Im Z$

$$2\sqrt{\frac{1}{n_0}} \frac{dx}{dz_2} = Im\bar{\psi}|Z|^2x + Re\bar{\psi}|Z|^2y$$

(58)

$$-2\sqrt{\frac{1}{n_0}} \frac{dy}{dz_2} = Re\bar{\psi}|Z|^2x - Im\bar{\psi}|Z|^2y$$

(59)

where

$$\bar{\psi} = \frac{1}{b} \int_0^b \psi(y)dy.$$

From (57-57) follows

$$\frac{d|Z|^2}{dz_2} = \frac{2Im\bar{\psi}}{\sqrt{\frac{1}{n_0}}}|Z|^4.$$  

(60)

The eq. (57) has the solution

$$|Z|^2 = \frac{1}{c - \frac{2Im\bar{\psi}}{\sqrt{\frac{1}{n_0}}}} \geq 0,$$

(61)

with the arbitrary constant $c > \frac{2Im\bar{\psi}}{\sqrt{\frac{1}{n_0}}}$. In the simplest case, when $Im\bar{\psi} = 0$, $|Z|^2 = c^{-1}$ and the system (57-57) has the solution

$$x = c^{-1/2} \cos \left[ \frac{Re\bar{\psi}}{2\sqrt{\frac{1}{n_0}}z_2 + \theta_0} \right]$$

(62)

$$y = c^{-1/2} \sin \left[ \frac{Re\bar{\psi}}{2\sqrt{\frac{1}{n_0}}z_2 + \theta_0} \right]$$
The constants $c^{-1/2} = 1, \theta_0 = 0$, due to the boundary condition $Z(z_2 = 0) = 1$. When $Im\tilde{\psi} \neq 0, x = |Z|\cos \theta, y = -|Z|\sin \theta$, where $\theta$ is the solution of equation

$$\frac{d\theta}{dz_2} = -\frac{|Z|^2}{2\sqrt{n_0}}Re\tilde{\psi} \tag{63}$$

at boundary condition $Z(z_2 = 0) = 1, (\theta = 0)$.

$$\theta = \frac{1}{4 Im\tilde{\psi}} \ln(1 - \frac{2 Im\tilde{\psi}}{\sqrt{n_0} z_2}) \tag{64}$$

and $a(y, z_2)$ for the first approximation (??) is $a(y, z_2) = Y(y)Z(z_2) = Y(y)|Z|^2\theta$, where $|Z|$ is given by (??), $\theta$ by (??) and $Y(y)$ can be found from boundary condition (??). When $Im\tilde{\psi} \to 0$ solution (??),(??) turns to the solution (??).

The solution (??) is valid for $|z_2| = c^2|z'| = (\frac{mc^2}{e\varphi_{max}})^2 |z'| \leq 1$ i.e. for $|z'| = k_p|z| \leq (\frac{mc^2}{e\varphi_{max}})^2$. In the considered domain $z_2 < 0, (z < -d)$ and the solutions (??),(??) have different behaviour, when $Im\tilde{\psi} > 0$ and $Im\tilde{\psi} < 0$. When $Im\tilde{\psi} > 0$ from (??) it is seen that $|Z|^2$ (and consequently the amplitude $a(y, z_2)$) decreases when $|z_2|$ increases; when $Im\tilde{\psi} < 0, |Z|^2$ and amplitude $a(y, z_2)$ increases, when $|z_2| = c^2|z|$ increases up to allowed value:

$$|z|_{\text{max}} = \min\left\{ \left( \frac{mc^2}{e\varphi_{max}} \right)^2, \left( \frac{mc^2}{e\varphi_{max}} \right)^2 \sqrt{n_0} \frac{2 Im\tilde{\psi}}{4 Im\tilde{\psi}} \right\}$$

An outlined approximate procedure, based on multiple scales method for solving basic nonlinear equation (??) for potential wake waves, shows that in the lowest (first, second and third) approximations solution of eq. (??) is represented by set of harmonics $\sim e^{i\varphi_{max}mz}(m = 0, 1, 2, 3 \ldots)$ with amplitudes $(a, B, A, b \text{ eqs.}(??-??), (??),(??-??))$ slowly increasing or decreasing with $|z|$. The $z$ dependence of amplitudes is rather combersome, but it is clear, that some singularity can appear at certain boundary conditions when $Im\tilde{\psi} < 0$ and $|z_2| \sim \frac{\sqrt{n_0}}{2 Im\tilde{\psi}} \leq \left( \frac{mc^2}{e\varphi_{max}} \right)^2$, which is inside of the region of applicability of the adopted procedure.

May be it is an indication of some new quality of the considered nonlinear potential wake wave, and, if it is so, it needs an additional consideration. In any case the outlined consideration indicates that boundary conditions at the rear end of the driving
bunch, which can be changed by appropriate choice of the bunch transverse and longitudinal charge distributions, can essentially affect the kind of the nonlinear wave amplitude dependence on the longitudinal coordinate z, even in the first approximation.

It seems, that subsequent experimental investigation of the dependence of the wake wave amplitude on the driving bunch transverse and longitudinal charge distributions could be useful.

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