Large-angle fluctuations in the cosmic X-ray background are investigated by a new formalism with a simple model of the X-ray sources. Our method is formulated from the Boltzmann equation and a simple extension of the work by Lahav et al. to be applicable to a hyperbolic (open) universe. The low multipole fluctuations due to the source clustering are analyzed in various cosmological models in both numerical and analytic way. The fluctuations strongly depend on the X-ray sources evolution model, as pointed out previously. It turns out that the nearby \( z < 0.1 \) sources are the dominant contributors to the large-angle fluctuations. If these nearby sources are removed in an observed X-ray map, the dipole (low multipole) moment of the fluctuation drastically decreases. In this case the Compton-Getting effect of an observer’s motion can be a dominant contribution to the dipole fluctuation. This feature of fluctuation, relating to the matter power spectrum, is discussed.

**Keywords** cosmology: theory — cosmic X-ray background — large-scale structure
there are many uncertainties. The evolution of the X-ray sources are not completely understood. A simple (power-law) source evolution model is assumed in their paper (Lahav et al. 1997; Treyer et al. 1998). Furthermore only a simple flat cosmological model is assumed. The difference in the cosmological model might yield the large difference in amplitude of the fluctuation. For example, if the X-ray sources at the high-$z$ universe are the dominant contributors to the angular fluctuations, the cosmological parameters, e.g., the curvature of the universe and the cosmological constant $\Lambda$, are significant factors. It is therefore worth examining how the fluctuations depend on the cosmological parameters. We also develop a useful formalism which is applicable to a hyperbolic (open) universe in order to extend the work by Lahav et al. to various cosmological models, which is described in Section 2. Our formalism is based on the Boltzmann approach, which is familiar in analysis of the cosmic microwave anisotropies (e.g., Hu & Sugiyama 1995a;1995b). A simple model for the source distribution is introduced. In Section 3 we solve the Boltzmann equation and obtain the expression for the root mean square of multipole moments of the fluctuations. In Section 4 the fluctuations are analyzed in various cosmological models. Section 5 is devoted to summary and discussions.

Throughout this paper we will use the units $c = \hbar = k_B = 1$; however, we occasionally use the Planck constant $h_P (= 2\pi \hbar)$ to make clear the meaning of equations.

II. FORMALISM

The perturbation of the Friedmann-Robertson-Walker (FRW) space-time in the Newtonian gauge is written as

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)\gamma_{ij}dx^idx^j,$$

(2.1)

where $\Psi$ is the perturbed gravitational potential, $\Phi$ is the curvature perturbation, $\gamma_{ij}$ is the three-metric on a space of constant negative curvature, and $a$ is the scale factor normalized to be unity at present. The scale factor $a$ is given by the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_0}{a^3} + \Omega_K + \Omega_\Lambda a^2\right),$$

(2.2)

where $H_0$ is the Hubble parameter with $H_0 = 100 h$ km/s/Mpc, $\Omega_0$ is the density parameter, $\Omega_\Lambda (= \Lambda/3H_0^2)$ is defined by the cosmological constant $\Lambda$, and $\Omega_K (= 1 - \Omega_0 - \Omega_\Lambda)$ describes the spatial curvature of the universe. For the spatially flat models, $\Omega_K$ is equal to zero. In equation (2.2) the dot denotes $\eta$ differentiation, where $\eta$ is the conformal time defined by $a d\eta = dt$.

The propagation of photon is affected by the expansion and the metric perturbations. The Boltzmann equation in the perturbed FRW universe is derived in Appendix A. The explicit form is expressed as

$$\frac{\partial f}{\partial \eta} + (1 - \Phi + \Psi)\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial q} \left(\frac{\dot{a}}{a} + \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Psi}{\partial x^i} \gamma^i\right) + \frac{\partial f}{\partial \gamma^i} \frac{d\gamma^i}{d\eta} = aS[\eta, x^i, q, \gamma^i],$$

(2.3)

where $q$ is the photon energy in the locally orthonormal frame, and $\gamma^i$ is the spatial vector normalized as $\gamma_{ij}\gamma^j = 1$ in this frame.

We now discuss the source term $S$ in the right-hand side in equation (2.3). The X-ray sources may be the discrete point sources. As we are interested in the much larger scale, we can average the distribution of the X-ray sources over a certain region and regard the source function as a smoothed field with spatially varying emissivity. We write the comoving volume emissivity per frequency at the time $\eta$ as $j_\nu(\eta, x)$ and separate it into two parts as

$$j_\nu(\eta, x) = \bar{j}_\nu(\eta) + \delta j_\nu(\eta, x),$$

(2.4)

where the bar represents a spatially averaged quantity, and $\delta j_\nu$ is the spatially inhomogeneous part of $j_\nu$. In this paper, we consider the cosmological models dominated by CDM. The emissivity may be
changed due to the presence of the inhomogeneity of the CDM. We expect that the value \( j_\nu \) should increase as the CDM density fluctuation \( \delta_c \) \((= (\rho_c - \bar{\rho}_c)/\bar{\rho}_c)\) becomes large. Therefore, the emissivity in the inhomogeneous universe is expressed as

\[
j_\nu(\eta, x) = \tilde{j}_\nu(\eta)(1 + b_X \delta_c)
\]

where \( b_X \) is the bias factor to relate the large-scale distribution of the CDM with that of the X-ray sources. * In this paper we assume

\[
\tilde{j}_\nu(\eta) = j_0(\nu)E(z),
\]

with \( E(z) = (1+z)^p \) for redshift \( z_{\text{min}} \leq z \leq z_{\text{max}} \), and \( E(z) = 0 \) for other period. Note that \( p \) is constant and \( p = 0 \) is the case of no evolution of the X-ray sources. We also assume the power-law form of the energy spectrum, \( j_0(\nu) \propto \nu^{-\alpha} \) with \( \alpha = 0.4 \). Here we set \( \alpha \) to be the observed CXB spectral index in \((3-20) \) keV range (Comastri et al. 1995), however, a reasonable change in \( \alpha \) will not affect our conclusions at a significant level.

The photon number emitted during the proper time \( t \sim t + dt \), in the frequency range \( \nu \sim \nu + d\nu \), in the solid angle \( d\Omega \), from the volume \( x \sim x + dx \), is written as \(^\dagger\)

\[
S = \frac{1}{g3\pi^2} \frac{1}{a^3} \tilde{j}_\nu \frac{d\Omega}{4\pi} \nu^2 d^3x dt
\]

where we assumed that this volume element has the peculiar velocity \( V_X \), and note that the Doppler effect is taken into account.

From the definition of the source term \( S \) in the Boltzmann equation, the photon number emitted during \( t \sim t + dt \), in the momentum range \( q \sim q + dq \), from the volume \( x \sim x + dx \), is \( Sgd^3q q dq dt / hp^3 \), where \( g(= 2) \) is the photon statistical weight. Using the relation \( g(= |q|) = hp\nu \), the photon number emitted during \( t \sim t + dt \), in the frequency range \( \nu \sim \nu + d\nu \), from the volume \( x \sim x + dx \), in the solid angle \( d\Omega \), is written as \( gS\nu^2 d\Omega d\Omega d^3x dt \). Equating this expression and equation (2.7), we find

\[
S = \frac{1}{g3\pi^2} \frac{1}{a^3} \tilde{j}_\nu \frac{d\Omega}{4\pi} \nu^2 (1 + (3 + \alpha)\gamma_i V_X^2)
\]

where we used the relation \( hp = 2\pi \) in our unit. The peculiar velocity of the X-ray sources \( V_X \) does not necessarily agree with that of the CDM, \( V_c \). We introduce another bias factor \( b_V \) for velocity, where \( V_c \) is the CDM velocity field.

We eventually get the source term as

\[
S = \frac{1}{g3\pi^2} \frac{1}{a^3} \tilde{j}_\nu \frac{d\Omega}{4\pi} \nu^2 (1 + b_X \delta_c(\eta, x) + (3 + \alpha)b_V \gamma_i V_c^i(\eta, x))
\]

* It may be unrealistic to assume the epoch-independent biasing. The epoch-dependent biasing is assumed in the recent paper by Treyer et al. (1998) by setting \( b_X(z) = b_X(0) + z[b_X(0) - 1] \). As we describe in section 4 and Appendix D, however, the nearby sources \((z \lesssim 0.1)\) are the dominant contributors to the large-angle fluctuations \((l \lesssim 10)\). In this case the amplitude of large-angle fluctuations will roughly scale in proportion to \( b_X(0) \) unless \( b_X(0) \) becomes very large.

\(^\dagger\)The fluctuation of the metric might have to be taken into account when formulating of the source term. As we will see in the next section (also Lahav et al. 1997), however, the contribution from the metric fluctuations to the large-angle fluctuations in the CXB is small. Therefore we have omitted the metric fluctuations in the source term.
III. SOLUTION OF THE BOLTZMANN EQUATION

In order to solve the above Boltzmann equation (3), we use the linear perturbative expansion method with respect to the inhomogeneity (such as metric perturbations). If the universe is completely homogeneous, the distribution function \( f^{(0)} \) and the source term \( S^{(0)} \) can be described by \( \eta \) and \( q \). With the inhomogeneity, they are modified as

\[
\begin{align*}
  f &= f^{(0)}(\eta, q) + f^{(1)}(\eta, x^i, q, \gamma^i), \\
  S &= S^{(0)}(\eta, q) + S^{(1)}(\eta, x^i, q, \gamma^i),
\end{align*}
\] (3.1)

with,

\[
\begin{align*}
  S^{(0)}(\eta, q) &= \frac{1}{g8\pi^2} \frac{1}{a^3} \frac{j_\nu}{\nu^3}, \\
  S^{(1)}(\eta, x^i, q, \gamma^i) &= S^{(0)}(\eta, q) \left( b_X \delta_e(\eta, x^i) + (3 + \alpha) b_V V_i(\eta, x^i) \right). 
\end{align*}
\] (3.3)

The Boltzmann equations of the zeroth and first order are given by

\[
\begin{align*}
  \frac{df^{(0)}}{d\eta} - q \frac{df^{(0)}}{dq} \frac{\dot{a}}{a} &= aS^{(0)}, \\
  \frac{df^{(1)}}{d\eta} + \gamma^i \frac{df^{(1)}}{dx^i} - q \frac{df^{(1)}}{dq} \frac{\dot{a}}{a} - q \frac{df^{(0)}}{dq} \left( \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Psi}{\partial x^i} \gamma^i \right) + \frac{df^{(1)}}{d\gamma^i} \frac{d\gamma^i}{d\eta} &= aS^{(1)}. 
\end{align*}
\] (3.5)

Note that when neglecting the metric perturbations, equations (3.5) and (3.6) are equivalent to the original Boltzmann equation, and give the exact solution.

Introducing the comoving momentum by \( Q = aq \), the above equations reduce to

\[
\begin{align*}
  \frac{df^{(0)}}{d\eta} &= aS^{(0)}(\eta, Q/a), \\
  \frac{df^{(1)}}{d\eta} + \gamma^i \frac{df^{(1)}}{dx^i} - Q \frac{df^{(0)}}{dQ} \left( \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Psi}{\partial x^i} \gamma^i \right) + \frac{df^{(1)}}{d\gamma^i} \frac{d\gamma^i}{d\eta} &= aS^{(1)}(\eta, x^i, Q/a, \gamma^i),
\end{align*}
\] (3.7)

where \( f^{(0)} \) and \( f^{(1)} \) must be regarded as the functions \( f^{(0)}(\eta, Q) \) and \( f^{(1)}(\eta, x^i, Q, \gamma^i) \), respectively.

Let us first consider the zeroth order equation (3.7), which is formally integrated as

\[
\begin{align*}
  f^{(0)}(\eta, Q) &= \int_0^\eta d\eta' a' S^{(0)}(\eta', Q/a') \\
  &= \frac{\pi}{g} Q^{-3} \int_0^\eta d\eta' a' \tilde{j}_{Q/h, a'}(\eta'), \\
  &= \frac{\pi}{g} Q^{-3} \int_0^\eta d\eta' a' \tilde{j}_{Q/h, a'}(\eta'), \\
  &= \frac{\pi}{g} Q^{-3} \int_0^\eta d\eta' a' \tilde{j}_{Q/h, a'}(\eta'),
\end{align*}
\] (3.9)

where we used \( S^{(0)}(Q/a) = (\pi/gQ^3) \tilde{j}_{Q/h, a}(\eta) \), and defined \( a' = a(\eta') \). Since we have assumed the power-law form, \( \tilde{j}_\nu(\eta) = j_0(\nu)(1 + z)^\nu \), for \( z_{\text{min}} \leq z \leq z_{\text{max}} \), (see equation (2.6)), the specific intensity, \( I_{\nu}^{(0)} = gh \nu^3 f^{(0)} \) (e.g., Shu 1991) is written as

\[
I_{\nu}^{(0)}(\eta_0) = \frac{j_0(\nu)}{4\pi H_0} \int_{z_{\text{min}}}^{z_{\text{max}}} dz (1 + z)^{\nu - 2 - \alpha} \sqrt{\Omega_0(1 + z) + \Omega_K + \Omega_\Lambda/(1 + z)^2},
\] (3.10)
where we used the relation between the conformal time and the redshift (2.2).

We next consider equation (3.8). It is convenient to introduce a new variable \( \Theta(\eta,x^i,Q,\gamma^i) \) as

\[
\Theta(\eta,x^i,Q,\gamma^i) = \frac{f(\eta)}{f(0)} = \frac{\Delta I_\nu}{I_\nu^{(0)}},
\]

where we used the general relation \( I_\nu = gh_\nu n^3 f \) for the second equality. By using \( \Theta(\eta,x^i,Q,\gamma^i) \), equation (3.8) can be written as

\[
\frac{\partial \Theta}{\partial \eta} + \xi(\eta) \Theta + \frac{\partial \Theta}{\partial x^i} \gamma^i + (\alpha + 3) \left( \frac{\partial \Psi}{\partial x^i} \gamma^i + \frac{\partial \Phi}{\partial \eta} \right) + \frac{\partial \Theta}{\partial \gamma^i} \gamma^i = \xi(\eta) \left( b_X \delta_c(x) + b_V (3 + \alpha) \gamma_i V_c^i(x) \right),
\]

where we used the relation \( Q \partial f^{(0)} / \partial Q = (\alpha + 3) f^{(0)} \) and equation (3.7), and \( \xi \) is defined by \( \xi(\eta) = (\partial f^{(0)} / \partial Q) / (\partial f^{(0)} / \partial Q) = f^{(0)} / f^{(0)} = \dot{\epsilon}(\eta) / \epsilon(\eta) \) with

\[
\dot{\epsilon}(\eta) = \int_0^\eta d\eta' a^i \dot{\epsilon}(\eta') a^{i+1} = \int_0^\eta d\eta' E(\eta') a^{i+1} \alpha.
\]

Note that the variable \( Q \) is separated in equation (3.12) and that \( \Theta \) can be regard as the function \( \Theta(\eta,x,\gamma^i) \). This is the result of the simple assumption of the power-law energy spectrum (2.6).

Equation (3.12) can be written as

\[
\frac{d}{d\eta} \left[ \epsilon \left( \Theta(\eta, x(\eta), \gamma(\eta)) + (\alpha + 3) \Psi(\eta, x(\eta)) \right) \right] = 2(\alpha + 3) \epsilon^{1/2} \frac{\partial}{\partial \eta} \left( \epsilon^{1/2} \Psi(\eta, x(\eta)) \right) + \dot{\epsilon} \left( b_X \delta_c(\eta, x) + b_V (3 + \alpha) \gamma_i V_c^i \right),
\]

where we used \( \Phi = -\Psi \) from the linear perturbation theory. This can be easily integrated as

\[
\Theta(\eta, x(\eta), \gamma) + (\alpha + 3) \Psi(\eta, x(\eta)) = \frac{1}{\dot{\epsilon}(\eta)} \int_0^\eta d\eta \left[ 2(\alpha + 3) \epsilon^{1/2} \frac{\partial}{\partial \eta} \left( \epsilon^{1/2} \Psi(\eta, x) \right) + \dot{\epsilon} \left( b_X \delta_c(\eta, x) + b_V (3 + \alpha) \gamma_i V_c^i \right) \right],
\]

where we used \( \dot{\epsilon}(0) = 0 \).

We rewrite this solution by using the multipole expansion method. The formalism for the CMB anisotropies is useful (e.g., Hu & Sugiyama 1995b). Since \( \Theta(\eta, x, \gamma^i) \) follows the linear equation, we can expand it as a sum of modes labeled by \((k,l)\),

\[
\Theta(\eta, x, \gamma^i) = \sum_k \sum_l \Theta_l(\eta, k) G_l(x, \gamma^i),
\]

where \( G_l \) is a certain mode function. The explicit form is written in Appendix B. The solution for \( \Theta_l(\eta_0, k) \) can be found from equation (3.15):

\[
\Theta_l(\eta_0, k) = -(\alpha + 3) \Psi(\eta_0, k) + \frac{1}{\dot{\epsilon}(\eta)} \int_0^\eta d\eta \left[ 2(\alpha + 3) \epsilon^{1/2} \frac{\partial}{\partial \eta} \left( \epsilon^{1/2} \Psi(\eta, k) \right) X_0^0(\Delta \eta) + \dot{\epsilon} b_X \delta_c(\eta, k) X_0^0(\Delta \eta) - \dot{\epsilon} b_V (3 + \alpha) V_c(\eta, k) X_0^0(\Delta \eta) \right],
\]

and
we can express the ensemble average of the two point angular correlation function as
\[\frac{\Theta_l(\eta_0, k)}{2l + 1} = \frac{1}{\varepsilon(\eta_0)} \int_0^{\eta_0} d\eta \left[ 2(\alpha + 3)\varepsilon^{1/2} \frac{\partial}{\partial \eta} (\varepsilon^{1/2} \Psi(\eta, k)) X^l_\omega(\Delta \eta) + \dot{\varepsilon} b_V (3 + \alpha) V_c(\eta, k) \left( \frac{l}{2l + 1} X^l_{\omega - 1}(\Delta \eta) - \frac{l + 1}{2l + 1} \left( 1 - \frac{l(l + 2) K^2}{k^2} \right) X^{l+1}_{\omega}(\Delta \eta) \right) \right], \tag{3.18}\]

with \(\Delta \eta = \sqrt{-K} (\eta_0 - \eta)\) and
\[X^l_\omega(\chi) = \left( \frac{\pi (\omega^2 + 1)^{1/2}}{2 \sinh \chi} \right)^{1/2} P_{l-\omega-1/2}^{-1/2}(\cosh \chi), \tag{3.19}\]

where \(P_l^\nu(x)\) is the Legendre function, and \(\omega\) is defined as \(\omega^2 = k^2 / (-K) - 1\) with the eigen-value \(k^2\) of scalar harmonics and the spatial curvature parameter \(K = -H_0^2 (1 - \Omega_0 - \Omega_\Lambda)\). We can check that equations (3.17) and (3.18) satisfy the perturbation equations for the multipole moments derived in Appendix B.

In the limit of flat universe, i.e. \(K \to 0\), the radial function \(X^l_\omega(\chi)\) reduces to the spherical Bessel function, and equation (3.18) reduces to
\[\frac{\Theta_l(\eta_0, k)}{2l + 1} = \frac{1}{\varepsilon(\eta_0)} \int_0^{\eta_0} d\eta \left[ 2(\alpha + 3)\varepsilon^{1/2} \frac{\partial}{\partial \eta} (\varepsilon^{1/2} \Psi(\eta, k)) j_i(k r_c) + \dot{\varepsilon} b_V (3 + \alpha) V_c(\eta, k) j_i(k r_c) \right] \left( \frac{l}{2l + 1} j_{l-1}(k r_c) - \frac{l + 1}{2l + 1} j_{l+1}(k r_c) \right), \tag{3.20}\]

where \(r_c\) is the path of a photon, defined as \(r_c = \eta_0 - \eta\).

As we are interested in the statistics of the angular fluctuations at a point of an observer \(x_0 = x(\eta_0)\), we consider the two point angular correlation function \(\langle \Theta(\eta_0, x_0, \gamma^i) \Theta(\eta_0, x_0, \gamma^j) \rangle\). Expanding the fluctuation \(\Theta\) at the point \(x_0\) by the spherical harmonics,
\[\Theta(\eta_0, x_0, \gamma^i) = \sum_{lm} a_{lm} Y_{lm}(\Omega_\gamma), \tag{3.21}\]

we can express the ensemble average of the two point angular correlation function as
\[\langle \Theta(\eta_0, x_0, \gamma^i) \Theta(\eta_0, x_0, \gamma^j) \rangle = \frac{1}{4\pi} \sum_{l=0}^\infty (2l + 1) C_l P_l(\cos \theta), \tag{3.22}\]

where we defined \(C_l = \langle a_{lm}^2 \rangle\), and \(\cos \theta = \gamma^i \cdot \gamma^j\). Here the angular power spectrum \(C_l\) is given by (e.g., Hu & Sugiyama 1995b)
\[C_l = \frac{2}{\pi} \int_0^\infty d\tilde{k} \tilde{k}^2 M_l \left( \frac{\Theta_l(\eta_0, k)}{2l + 1} \right)^2, \tag{3.23}\]

where \(M_l = (\tilde{k}^2 - K) \cdots (\tilde{k}^2 - l^2 K) / (\tilde{k}^2 - K)^l\) and \(\tilde{k}^2 = k^2 + K\).

As we see in equations (3.23) with (3.18) or (3.20), the angular power spectrum \(C_l\) consists of three terms, which are due to the source clustering \(\delta_c\), the bulk motion of the sources \(V_c\), and the gravitational potential fluctuations \(\Psi\). As we will see in the below, the term proportional to \(\delta_c\) in the right-hand side in equations (3.18) and (3.20) are the dominant contributors to the fluctuations in the CXB (Lahav et al 1997). Note that the effect of the motion of an observer, i.e. Compton-Getting effect (hereafter C-G effect) is not taken into account in this solution. We integrate the above equation numerically. The results are described in the next section.
IV. FLUCTUATIONS

In this section we present our results. We here consider three cosmological models, a standard CDM model (SCDM), a CDM model with a cosmological constant $\Lambda$ ($\Lambda$CDM), and an open CDM model (OCDM). As for the initial density power spectrum, we use the Harrison-Zeldovich power spectrum for SCDM and $\Lambda$CDM with the COBE normalization (Bunn & White 1997). In the OCDM model, we use the spectrum predicted in a simple open inflation model (Yamamoto & Bunn 1996; White & Silk 1996). The spectrum is almost the same as the Harrison-Zeldovich one at the scale of the large-scale structure, so that our choice of the spectrum does not essentially alter the results. In Appendix C an useful formula for the evolution of the CDM perturbation is summarized.

We have quite a lot of freedom in the parametrization of the X-ray background model. The parameters are classified into two kinds. One is the cosmological parameters and the other is the model parameters of the X-ray sources. We present how the results depend on the two kinds of parameters.

A. dependence on the cosmological models

Before discussing the dependence on the cosmological parameters, we mention the physical contents of the fluctuations. Figure 1 shows the angular power spectrum $C_l$ due to the source clustering, bulk motion, and gravitational potential fluctuations. It is apparent that the source clustering term is the dominant contributor to the large-angle fluctuations. We have checked that the other terms due to the gravitational potential and the bulk motion become important only when $z_{\text{min}} > 0.1$. We therefore take only the source clustering into account hereafter, focusing on the case $z_{\text{min}} \lesssim 0.1$.

Now let us discuss the dependence of the fluctuations on the cosmological parameters. Fig. 2 shows the dependence of the dipole anisotropy $C_{1/2}^{1/2}$ on $\Omega_0$. As we are not interested in the spectrum shape of the angular fluctuations here, we focus on the amplitude of $C_l=1$. As expected, $\Lambda$CDM and OCDM agree with SCDM in the limit of $\Omega_0 \to 1$. It is shown that the dipole anisotropy is not sensitive to the cosmological parameters except for the case of extreme low value of $\Omega_0$ (Lahav et al. 1997). As we see in Appendix D, this feature of the dipole anisotropy is attributed to the difference of the shape of the power spectrum. The other point to note is that the amplitude of the fluctuations strongly depends on the source parameter $z_{\text{min}}$.

B. dependence on the X-ray sources models

We have three parameters, $z_{\text{min}}$, $z_{\text{max}}$ and $p$, as the X-ray sources model. The evolution of the X-ray sources is governed by $p$ and $z_{\text{max}}$. From now on, we set the cosmological parameters ($\Omega_0 = 1$, $h = 0.5$) for SCDM, ($\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$) for $\Lambda$CDM, and ($\Omega_0 = 0.3$, $\Omega_\Lambda = 0$, $h = 0.7$) for OCDM. In Appendix D, we described an analytic calculation for the angular power spectrum $C_l$. The analytic calculation is limited only to the case of $\Omega_0 = 1$. However, this is instructive to understand the behaviour of the angular power spectrum. Fig. 3 shows the dependence of $C_{1/2}^{1/2}$ on the evolution parameter $p$. The dipole anisotropy drastically drops with $p$. The fluctuation is roughly determined by dividing the anisotropic X-ray flux $\Delta I_\nu$ by the isotropic one $I_{\nu}^{(0)}$. As shown in Appendix D, the dependence on $p$ comes from the isotropic component $I_{\nu}^{(0)}$ of the CXB. The increase of $p$, i.e., the increase of the redshift evolution of sources leads to the relative increase of $I_{\nu}^{(0)}$ compared with $\Delta I_\nu$. Thus the fluctuation decreases with $p$. Fig. 4 shows the dependence of the dipole anisotropy on $z_{\text{max}}$. The amplitude of the fluctuation also decreases with $z_{\text{max}}$. The reason is the same as $p$, but the dependence on $z_{\text{max}}$ is weaker than that on $p$. These features are common to the higher multipoles (Lahav et al. 1997).
Finally, we focus on $z_{\text{min}}$ which was fixed on zero in the paper by Lahav et al. (1997). If the nearby bright X-ray sources were removed from the observational data, it would not necessarily be fixed at $z_{\text{min}} = 0$. Fig. 5 shows the dependence of the dipole anisotropy on $z_{\text{min}}$. It is apparent that the dipole amplitude is sensitive to this parameter especially for $z_{\text{min}} \lesssim 0.1$. The amplitude of the dipole moment rapidly drops with $z_{\text{min}}$ for $z_{\text{min}} \lesssim 0.1$. In contrast, it becomes almost constant for $z_{\text{min}} \gtrsim 0.1$. This strong dependence on $z_{\text{min}}$ is related with the shape of the matter power spectrum $P(k)$ as discussed in Appendix D. The choice of $z_{\text{min}}$ changes the shape of the angular power spectrum of the fluctuations (see Fig. 1). This means that the sources distributed relatively close to the our Galaxy are the dominant contributors to the fluctuations in the CXB for the low multipoles. The peak of the matter power spectrum is located at $\lambda \sim 100h^{-1}\text{Mpc}$, which corresponds to $z \sim 0.03$. The shape of the matter power spectrum is the reason why the sources at the low redshift $z \lesssim 0.1$ are the dominant contributors to the large-angle fluctuations in the CXB.

V. SUMMARY AND DISCUSSIONS

In this paper, we have investigated the large-angle fluctuations in the CXB due to the X-ray source clustering. We have developed the formalism to describe the CXB fluctuations using Boltzmann equation under a simple model of the X-ray sources. Our formalism is a simple extension of that by Lahav et al. (1997) to be applicable to an universe with hyperbolic geometry. The dependence of the fluctuations on the model parameters has been examined in various cosmological models. The fluctuation does not strongly depend on the cosmological parameters. It is quite sensitive to the parameters for the X-ray sources (Lahav et al. 1997). The fluctuation is determined by the ratio of anisotropic X-ray flux to the isotropic one, i.e., $\sqrt{C_l} \sim \Delta I_l/I_l^{(0)}$. $\Delta I_l$ is essentially determined by the nearby sources at low redshift for the low multipole moment, while $I_l^{(0)}$ is done by the high-$z$ sources. The large redshift evolution of the X-ray emissivity (e.g., $p = 3$, and $z_{\text{max}}$ is large) makes the flux from the far sources large and, as a result, the amplitude of fluctuation becomes small.

We have also pointed out the importance of the source distribution parameter $z_{\text{min}}$. The low multipole anisotropies are sensitive to it. This parameter is closely related to the reconstruction of the X-ray map by removing nearby bright sources. The dipole moment with $z_{\text{min}} = 0.1$ is smaller by order of magnitude compared with the case $z_{\text{min}} = 0$. If the sources sufficiently close to the our Galaxy $z \sim 0.1$ are removed, it may be expected that the effect of the peculiar motion of the observer (Compton-Getting (C-G) effect) is dominant in the CXB dipole.

Let us roughly compare the effects which contribute to the dipole anisotropies except for the source clustering effect. The observer’s peculiar motion relative to the CMB was measured by using the COBE four-year data (Lineweaver et al. 1996), which gave the peculiar velocity $V_{\text{obs}} = 368.9 \pm 2.5 \text{ km/s}$ in the direction ($l = 264^\circ; b = 48^\circ$). Assuming we have a similar motion relative to the CXB, the expected C-G dipole is estimated as $\sqrt{C_{l=1}^{CG}} \simeq 5.0 \times 10^{-3}$.

On the other hand, it is well known that the shot noise fluctuation arises from the discreteness of the sources. Since this fluctuation is originated from the Poisson fluctuation of the discrete sources, the spectrum is white noise. The shot noise fluctuations in the CXB have been investigated by Lahav et al. (1997). Following their result, the amplitude of fluctuation is estimated $\sqrt{C_{l=1}^{SN}} \simeq 1.2 \times f_{\text{m}}^{1/4}$, where $f_{\text{m}}$ is a flux cut-off of bright removed sources in unit of erg s$^{-1}$ cm$^{-2}$. If the flux cut-off level becomes lower, the amplitude of shot noise decreases. Thus the shot noise fluctuation depends on the flux cut-off in observational data and is important when comparing a theoretical model with the observed X-ray map (Treyer et al. 1998). The flux cut-off was $\approx 3 \times 10^{-11}$ erg s$^{-1}$ cm$^{-2}$ in the HEAO-1, and the amplitude of shot noise is estimated as $\sqrt{C_{l=1}^{SN}} \simeq 2.8 \times 10^{-3}$. Note also that the cut-off level is relevant to the parameter $z_{\text{min}}$ in the sense of bright source removability.

The dipole owing to the source clustering is shown in Fig.5 ($p = 3, z_{\text{max}} = 3$). In the case of $z_{\text{min}} = 0$, the dipole anisotropy due to the clustering effect is comparable to the C-G effect. However, if $z_{\text{min}}$ is
$O(0.1)$, the C-G dipole becomes well above the dipole due to the source clustering. If the flux cut-off becomes lower by the improvement of observation, the shot noise and the clustering effect could be sufficiently smaller than C-G effect to yield a good chance of measuring the C-G dipole in the CXB. This investigation also suggests that the careful treatment is required when comparing the observational map with the theoretical prediction. When the redshifts of the sources can not be determined, the subtraction of nearby X-ray sources from the observed map may contain a delicate problem because the nearby faint sources may contribute to the fluctuations sensitively. These problems are left as future problems.

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APPENDIX A: BOLTZMANN EQUATION

We write the Boltzmann equation for the distribution function \( f(t, x^i, q, \gamma^i) \) of photons as

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial q} \frac{dq}{dt} + \frac{\partial f}{\partial \gamma^i} \frac{d\gamma^i}{dt} = S(t, x^i, q, \gamma^i), \tag{A1}
\]

with the source term \( S \). Here we defined the energy in the locally orthonormal frame as \( q = |q| \), thus \((q, q^i)\) forms the 4-momentum of the photon in this frame. Since the Boltzmann equation is written in terms of the momentum \( q \) measured by an observer in the cosmological rest frame, we must rewrite the terms \( dx^i/dt \) and \( dq/dt \) in terms of \( x^i \) and \( q \) in order to solve this equation. These relations are given from the geodesic equation of a photon. However, the geodesic equation is commonly written in terms of the 4-momentum \( p^\mu \) in the frame (2.1), where \( p^\mu \) is defined by \( p^\mu = dx^\mu /d \lambda \) with the affine parameter \( \lambda \). The 4-momentum \((q, q^i)\) in the rest frame is related to the 4-momentum \( p^\mu \), as follows:

\[
q = (1 + \Psi)p^0, \tag{A2}
\]
\[
q^i = a(1 + \Phi)p^i. \tag{A3}
\]

The spatial vector \( \gamma^i \) is defined as \( \gamma^i = q^i /q \) and \( \gamma^i \) satisfies \( \gamma_{ij}\gamma^{ij} = 1 \).

Equations (A2) and (A3) give the following relations, up to the first order of \( \Psi \) and \( \Phi \):

\[
\frac{dx^i}{dt} = \frac{p^i}{p^0} = \frac{1}{a}(1 + \Psi - \Phi)\gamma^i, \tag{A4}
\]

and

\[
\frac{dq}{dt} = \left( \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x^j} \frac{dx^j}{dt} \right) p^0 + (1 + \Psi)\frac{dp^0}{dt}. \tag{A5}
\]

Meanwhile the geodesic equation in the first order of the perturbation is

\[
\frac{dp^0}{dt} = q \left( -\frac{1}{a} \frac{da}{dt} - \dot{\Psi} - 2 \frac{\partial \Psi}{\partial x^i} \gamma^i - \dot{\Phi} + \frac{1}{a} \frac{da}{dt} \Psi \right), \tag{A6}
\]

Inserting this into (A5), we have

\[
\frac{dq}{dt} = -q \left( \frac{1}{a} \frac{da}{dt} + \psi + \frac{1}{a} \frac{\partial \Psi}{\partial x^i} \gamma^i \right). \tag{A7}
\]

Thus we can write down the left-hand side of the Boltzmann equation (A1) using equations (A4) and (A7). If we employ the conformal time defined by \( ad\eta = dt \) instead of the proper time, it becomes

\[
\frac{\partial f}{\partial \eta} + (1 - \Phi + \Psi) \frac{\partial f}{\partial x^i} \gamma^i - q \frac{\partial f}{\partial q} (\dot{a} a + \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Psi}{\partial \gamma^j} \gamma^j) + \frac{\partial f}{\partial \gamma^i} \frac{d\gamma^i}{d\eta} = aS(t, x^i, q, \gamma^i). \tag{A8}
\]

APPENDIX B: MATHEMATICAL FORMULAE IN AN OPEN UNIVERSE

In this appendix, we summarize the mathematical formulae which are needed in section 3. The results of this section are based on the previous work (e.g., Gouda, Sugiyama, & Sasaki 1991; Hu & Sugiyama 1995b; Wilson 1983).
We first consider the scalar harmonics \( Q_k(x) \) on a hyperbolic universe. It follows \( Q_{k_i} = -k^2 Q_k \), where \(|i|\) denotes the covariant derivative in the hyperbolic space with the line element
\[
\gamma_{ij} dx^i dx^j = \frac{1}{K} (d\chi^2 + \sinh^2 \chi d\Omega^2_2),
\] (B1)
where \( K = -H_0^2 (1 - \Omega_0 - \Omega_\Lambda) \) is the spatial curvature parameter and \( d\Omega^2_2 \) is the line element on an unit sphere. Since we assumed that the emitted photons do not scatter, the trajectory of the photon is the free streaming. Therefore, the photon geodesics are radial, we may use the radial scalar harmonics to describe the free streaming behaviour. It can be expressed as \( Q_k = X'_k(\chi)Y_{lm}(\Omega) \) using the radial function \( X'_k \) in equation (3.19) and the spherical harmonics \( Y_{lm} \).

The function \( G_i \) for the multipole decomposition in equation (3.16) is defined
\[
G_i(x, \gamma^i) = (-k)^{-i} Q_{k|_{i=1}}(x, k) P^{i_{1} \cdots i_{l}}(x, \gamma^i),
\] (B2)
and
\[
P_{(0)} = 1, \quad P_{(1)} = \gamma^i, \quad P_{(2)} = \frac{1}{2} (3\gamma^i \gamma^j - \gamma^{ij}),
\]
\[
P_{(l+1)}^{i_{1}\cdots i_{l+1}} = \frac{2l+1}{l+1} \gamma^{i_{1}} P_{(l)}^{i_{2}\cdots i_{l+1}} - \frac{l}{l+1} \gamma^{i_{1}i_{2}} P_{(l-1)}^{i_{3}\cdots i_{l+1}},
\]
where the parentheses denote symmetrization about the indices. For the special case of the flat universe, it reduces to \( G_i = (-i)^l \exp(i\cdot k) P_l(\kappa \cdot \gamma^i) \), where \( P_l(x) \) is the Legendre polynomial and \( \kappa = k^i / k \).

We derive the evolution equations for the moments \( \Theta_i \) in equation (3.16). For this purpose, we use the fact that scalar perturbation quantities are decomposed as \( \Phi(\eta, x) = \sum_k \Phi(\eta, k) G_0(x) \), etc., and \( \gamma_i V_{\kappa} = \sum_k V_{\kappa}(\eta, k) G_1(x, \gamma^i) \) for the velocity. The evolution equations for the moments are given from equation (3.12), by the Boltzmann hierarchy:
\[
\dot{\Theta}_0 + \xi \Theta_0 = -\frac{1}{3} k \Theta_1 - (\alpha + 3) \dot{\Phi} + \xi \partial_x \delta_c,
\] (B3)
\[
\dot{\Theta}_1 + \xi \Theta_1 = -\frac{2}{5} \left( 1 - 3 \frac{K}{k^2} \right) \Theta_2 + k \Theta_0 + (\alpha + 3) k \Psi + (\alpha + 3) \xi \partial_x V_c,
\] (B4)
\[
\dot{\Theta}_l + \xi \Theta_l = -\frac{l+1}{2l+3} k \left( 1 - l(l+2) \frac{K}{k^2} \right) \Theta_{l+1} + \frac{l}{2l-1} k \Theta_{l-1}, \quad \text{for} \ (l \geq 2),
\] (B5)
where we used
\[
\gamma^i \frac{\partial G_i}{\partial x^i} = k \left( \frac{l}{2l+1} \left( 1 - l(l-1) \frac{K}{k^2} \right) \Theta_{l-1} - \frac{l+1}{2l+1} G_{l+1} \right).
\] (B6)

**APPENDIX C: USEFUL FORMULA FOR EVOLUTION OF CDM PERTURBATIONS**

We here summarize an useful equation for the evolution of the CDM perturbation (Peebles 1980). The CDM perturbation \( \delta_c(\eta, k) \) obeys
\[
\ddot{\delta}_c + \frac{a}{a} \dot{\delta}_c = \frac{3}{2} \frac{\Omega_\Lambda H^2}{a} \delta_c,
\] (C1)
where the expansion rate \( \dot{a} / a \) is given by equation (2.2). Note that this equation is independent of the scale \( k \), since we are considering the matter dominant stage. Using equation (2.2), we rewrite the above equation as
\begin{align}
(a\Omega_0 + a^2\Omega_K + a^4\Omega_\Lambda) \frac{d^2\delta_c}{da^2} + \left(\frac{3}{2}\Omega_0 + 2a\Omega_K + 3a^2\Omega_\Lambda\right) \frac{d\delta_c}{da} - \frac{3}{a} \delta_c &= 0.
\end{align}

The growing mode solution is written as

\begin{equation}
\delta_c = \frac{5\Omega_0}{2} \sqrt{\frac{\Omega_0}{a^3} + \frac{\Omega_K}{a^2} + \frac{\Omega_\Lambda}{a}} \int_0^a da' \left(\frac{a'}{\Omega_0 + a'\Omega_K + a'^2\Omega_\Lambda}\right)^{3/2},
\end{equation}

which is normalized as \(\delta_c = a\) at \(a \ll 1\).

**APPENDIX D: ANALYTIC APPROACH**

It is very instructive to evaluate the fluctuations in the CXB in an analytic way. To perform the analytic calculation, we consider the standard CDM model. In the case of \(\Omega_0 = 1\), we have \(a = (\eta/\eta_0)^2\) with \(\eta_0 = 2/H_0\) from equation (2.2). Here we use the notations \(\eta_{\text{max}}/\eta_0 = (1 + z_{\text{min}})^{-1/2}\) and \(\eta_{\text{min}}/\eta_0 = (1 + z_{\text{max}})^{-1/2}\). Omitting the terms of the source bulk motion and the gravitational potential fluctuations from equation (2.2), we have

\begin{equation}
\frac{\Theta_l(\eta_0, k)}{2l + 1} = \frac{b_X}{\varepsilon(\eta_0)} \int_0^{\eta_0} \eta \delta_c \eta, k \rangle j_i(k(\eta_0 - \eta)).
\end{equation}

From the definition of \(\varepsilon\), equation (3.13), we can write

\begin{equation}
\frac{\Theta_l(\eta_0, k)}{2l + 1} = \frac{b_X \delta_c(\eta_0, k)}{\varepsilon} \int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \eta \left(\eta/\eta_0\right)^{2(2+\alpha-p)} j_i(k(\eta_0 - \eta)),
\end{equation}

with

\begin{equation}
\varepsilon = \int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \eta \left(\eta/\eta_0\right)^{2(1+\alpha-p)}
\end{equation}

\begin{equation}
= \frac{\eta_0}{3 + 2\alpha - 2p} \left[(1 + z_{\text{min}})^{-3/2-\alpha+p} - (1 + z_{\text{max}})^{-3/2-\alpha+p}\right],
\end{equation}

where we used the relation \(\delta_c(\eta_0, k) = \delta_c(\eta_0, k)\alpha(\eta)\). Putting (D2) into (3.23), we have

\begin{equation}
C_1 = \left(\frac{b_X}{\varepsilon}\right)^2 \int_0^\infty dk \ P(k) \left|\int_{x_1}^{x_2} dx \ \left(1 - \frac{x}{k\eta_0}\right)^{2(2+\alpha-p)} j_i(x)\right|^2,
\end{equation}

where \(x_1 = k(\eta_0 - \eta_{\text{max}})\), \(x_2 = k(\eta_0 - \eta_{\text{min}})\), and we wrote \(P(k) = \langle \delta_c(\eta_0, k)^2\rangle\). Since \(k\) integration is effective at \(k\eta_0 \gg 1\), we approximate it as

\begin{equation}
C_1 \approx \left(\frac{b_X}{\varepsilon}\right)^2 \int_0^\infty dk \ P(k) \left|\int_{x_1}^{x_2} dx \ j_i(x)\right|^2.
\end{equation}

This approximation works within \(\sim \) a few \(\times 10\%\) for \(0 \lesssim p \lesssim 3\) and \(z_{\text{min}} \lesssim 0.1\). Of course this transformation is exact when \(p = 2 + \alpha\).

In the case of \(l = 1\), \(x\) integration can be done. We get

\begin{equation}
C_{l=1} \approx \left(\frac{b_X}{\varepsilon}\right)^2 \int_0^\infty dk \ P(k) \left(j_0(x_1) - j_0(x_2)\right)^2.
\end{equation}
Furthermore we neglect the term of \( j_0(x_2) \) assuming that \( k \) integration is effective at \( k \eta_0 \gg 1 \). We get

\[
C_{l=1} \simeq \frac{2}{\pi} \left( \frac{b_X}{c} \right)^2 \int_0^\infty dk \, P(k) \, j_0(x_1)^2 .
\]  

(D7)

Assuming \( z_{\text{min}} \ll 1 \), we have \( \eta_0 - \eta_{\text{max}} \simeq z_{\text{min}}/H_0 \), then equation (D7) can be written as

\[
C_{l=1} \simeq \pi b_X^2 \delta_H^2 \left( \frac{3 + 2\alpha - 2p}{1 - (1 + z_{\text{max}})^{-3/2} - \alpha + p} \right)^2 \int_0^\infty d\hat{k} T(\hat{k})^2 j_0(\hat{k} z_{\text{min}})^2 \, .
\]  

(D8)

where \( \hat{k} = k/H_0 \), \( \delta_H = 1.94 \times 10^{-5} \) (Bunn & White 1997), and the matter transfer function \( T(k) \) is introduced. This equation is meaningful to understand how the dipole depends on the X-ray sources parameters. The dependence of the dipole on \( p \) and \( z_{\text{max}} \) originally comes from the isotropic component of the X-ray background radiations \( \propto \varepsilon(\eta_0) \), whereas \( z_{\text{min}} \) affects only the fluctuation part in \( k \) integration.

Introducing a cut off parameter \( k_{\text{max}} \) in \( k \) integration instead of the transfer function, we approximate equation (D8) as

\[
C_{l=1} \simeq \pi b_X^2 \delta_H^2 \left( \frac{3 + 2\alpha - 2p}{1 - (1 + z_{\text{max}})^{-3/2} - \alpha + p} \right)^2 \int_0^{\hat{k}_{\text{max}}} d\hat{k} j_0(\hat{k} z_{\text{min}})^2 ,
\]  

(D9)

where \( \hat{k}_{\text{max}} = k_{\text{max}}/H_0 \sim O(10^2) \). Since \( j_0(x) \) oscillates at \( x \gg 1 \), \( k \) integration gives different behaviour depending on the value \( \hat{k}_{\text{max}} z_{\text{min}} \) around unity, as follows:

\[
\int_0^{\hat{k}_{\text{max}}} d\hat{k} j_0(\hat{k} z_{\text{min}})^2 \simeq \begin{cases} \hat{k}_{\text{max}}^2/2, & (\hat{k}_{\text{max}} z_{\text{min}} \ll 1), \\ \ln(2\hat{k}_{\text{max}} z_{\text{min}}) + \gamma_E / 2 z_{\text{min}}^2, & (\hat{k}_{\text{max}} z_{\text{min}} \gg 1), \end{cases}
\]

where \( \gamma_E \) is the Euler constant. Essentially, \( k_{\text{max}} \) corresponds to the peak of the power spectrum, i.e. the scale of galaxy large scale structure \( k_{\text{max}}^{-1} \sim 10 \sim 10^2\text{Mpc} \). The value of \( C_{l=1} \) is therefore very sensitive to \( z_{\text{min}} \) around \( O(10^{-2}) \).
FIGURE CAPTIONS

Fig. 1— The angular power spectrum of the fluctuations for various effects. The labels ‘δ’, ‘V’, and ‘Ψ’ denote $\sqrt{C_l}$ due to the source clustering, the peculiar motion of the sources, and the gravitational potential fluctuations, respectively. Here we set $b_X = b_V = 1$. The cosmological parameters are taken as $h = 0.5$ and $\Omega_0 = 1$ for SCDM model, and $h = 0.7$ and $\Omega_0 = 0.3$ for ΛCDM model. The cases $z_{\text{min}} = 0, 0.02, 0.1$ are shown.

Fig. 2— $\Omega_0$-dependence of $\sqrt{C_{l=1}}/b_X$ for ΛCDM and OCDM models. The Hubble parameter $h = 0.7$ is taken here. Each panel shows the case $z_{\text{min}} = 0, 0.02, 0.1$, respectively.

Fig. 3— $p$-dependence of $\sqrt{C_{l=1}}/b_X$. The cosmological parameters are taken as $h = 0.5$ and $\Omega_0 = 1$ for the SCDM model, and as $h = 0.7$ and $\Omega_0 = 0.3$ for the ΛCDM and OCDM models. Each panel shows the cases $z_{\text{min}} = 0, 0.02, 0.1$, respectively.

Fig. 4— $z_{\text{max}}$-dependence of $\sqrt{C_{l=1}}/b_X$. The cosmological parameters are the same as Fig. 3.

Fig. 5— $z_{\text{min}}$-dependence of $\sqrt{C_{l=1}}/b_X$. The cosmological parameters are the same as Fig. 3.
Miura, et al. Fig. 1
$\sqrt{C_1 / b_x}$

$z_{\text{min}} = 0$

$\Lambda$CDM

OCDM

$z_{\text{min}} = 0.02$

$\Lambda$CDM

OCDM

$z_{\text{min}} = 0.1$

$\Lambda$CDM

OCDM

$\Omega_0$

Miura, et al. Fig. 2
$\sqrt{C_1 / b_x}$ vs $p$

$z_{\text{min}} = 0$

$z_{\text{min}} = 0.02$

$z_{\text{min}} = 0.1$

SCDM, $\Lambda$CDM, OCDM

Miura, et al. Fig. 3
Miura et al. Fig.4
Miura, et al. Fig.5