Wave Function Renormalization at Finite Temperature

S. Esposito, G. Mangano, G. Miele and O. Pisanti

Dipartimento di Scienze Fisiche, Università di Napoli "Federico II",
and
INFN, Sezione di Napoli, Mostra D’Oltremare Pad. 20, I-80125 Napoli, Italy

We present a derivation of the medium dependent wave function renormalization for a spinor field in presence of a thermal bath. We show that, as already pointed out in literature, projector operators are not multiplicatively renormalized and the effect involves a non trivial spinor dependence, which disappears in the zero temperature covariant limit. The results, which differ from what already found in literature, are then applied to the decay of a massive scalar boson into two fermions and to the $\beta$-decay and crossed related processes relevant for primordial nucleosynthesis.

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I. INTRODUCTION

The large amount of new and precise data on the structure of the universe, which will be provided by the new generation of experiments [1], represents a new exciting source of information for particle physics too. In particular, new measurements on the primordial abundance of light elements will probably provide a severe arena where to test new models for fundamental interactions. In this respect, the theoretical predictions on Big Bang Nucleosynthesis (BBN), which are considered one of the great successes of the hot Big Bang theory, must be refined in order to reach the same level of precision of the new experimental data.

One of the necessary improvement in the BBN algorithm is to take into account that all the nucleosynthesis reactions occur in a plasma of photons, electron-positron pairs and neutrinos whose temperature
varies, in the relevant epoch for BBN, in the range 0.1 \div 10 \text{ MeV} (nucleons and nuclei due to their large mass with respect to temperature may be safely considered at } T = 0). This observation led several authors [2,3] to add finite temperature contributions to weak transitions like \( n + \nu_e \leftrightarrow p + e^- \) and crossed phenomena, due to thermal radiative electromagnetic corrections. The size of these effects on relevant cosmological observables, like the ratio \( n/p \), which is the key parameter for the evaluation of the abundances of light nuclei, as \( D, \ ^3 \text{He or } ^4 \text{He} \), has been estimated to be not larger than few percent [2,3]. However, since the new experimental data will presumably be sensible to corrections of this order of magnitude, the study of the thermal effects is certainly physically well-motivated.

Although there is a general agreement in the literature on how to compute finite temperature effects on phase-space, vertex, mass corrections and photon emission/absorption [2]-[12], the finite temperature wave function renormalization still remains an open problem. Since Refs [2,3], where unfortunately the problem was not addressed in the proper way, several approaches have been proposed in literature [5]-[13]. They all agree on the idea of using finite temperature Dirac spinors to obtain the corresponding effective projection operator. The final results, however, striking differ [14].

In this paper we approach this problem in a different but straightforward way. Our results, obtained in section II, differ from what has been obtained in [5] for an additional term which is due to a peculiar property of the heat bath. It has a different spinorial structure and spoils the multiplicative character of renormalization of wave-function. The presence of such a term was first recognized in [13], but we disagree on its explicit form. We then perform a comparative analysis of the several approaches and, in section III, discuss some physical implications, like the change on the decay process of a heavy scalar boson into lepton-antilepton pairs, \( H \rightarrow l^+l^- \), and on \( \beta \)-decay \( n \rightarrow p + e^- + \bar{\nu}_e \) and crossed related processes. Finally, in section IV we give our conclusions.

II. FINITE TEMPERATURE PROJECTION OPERATORS

The ambiguity in the wave function renormalization at finite temperature is basically due to the lack of Lorentz covariance. The thermal bath introduces a preferred frame, as the one in which the time-like four vector generalizing the non-relativistic temperature parameter takes the form \((\beta, \bar{0})\). In this rest frame, the equilibrium particle distribution are the usual Bose-Einstein or Fermi-Dirac expressions, with \( \beta \) the inverse temperature. The very definition of a particle state, which corresponds in zero temperature quantum field theory to an irreducible representation of the Poincaré group, becomes tricky and different
points of view may naturally lead to quite different results for the medium dependent wave-function renormalization, basically due to different choices for the renormalized fields. As noticed in [13], however, the non-local, momentum dependent effects of the medium imply that such a field, as a local field, could even be not defined.

We will take, as in [13], quite a different point of view, identifying the particle states as corresponding to the energy poles in the propagator. In the covariant case, for a spinor field, the propagator will always develop poles in the Lorentz invariant $\tilde{p}$, or equivalently in $p^2$, the effect of the interactions being the shift of the pole and a change in the corresponding residue. For a particle propagating in a thermal bath, the interactions with the surrounding plasma spoil, in general, this property, and the location of the pole cannot be represented by an invariant statement. The mass shift would in fact acquire a momentum (and frame) dependence, which is to say that particles propagating with different speeds in the bath rest frame will acquire, in general, different inertia due to the interactions with the plasma (spatial dispersion).

Notice that, unlike the zero temperature case, the effect of the wave function renormalization, being momentum dependent as well, cannot be absorbed via the introduction of a local counterterm in the bare lagrangian density. It therefore represents a genuine physical effect of the medium on the propagating particles.

It is not the aim of this paper to investigate the fundamental aspects sketched above, but rather to discuss a prescription for evaluating the wave-function renormalization effects. They cannot, nevertheless, be hidden under the carpet, and they render our point of view, at least with our present understanding, only a reasonable way of dealing with this problem.

As already mentioned, our starting point will be to identify the particle states as corresponding to the energy poles in the field propagator $G(E, \vec{p})$, which can be perturbatively evaluated taking into account the effects of interactions with the surrounding medium. Actually, only the poles which are perturbatively close to the free particle ones will be considered, and not, for example, the hole branches described in [15]. The particle state will be therefore characterized by a momentum $\vec{p}$ and a new dispersion relation $E = E(\vec{p})$ given by the new position of the pole. The wave function renormalization, which is in general momentum dependent, can then be read off by evaluating the residue of $G$ at the pole. This prescription has been already considered by Sawyer [13], but obtaining quite different results from ours. We will comment on this later. The relevant quantity which appears in the expressions of scattering cross sections and decay rates are the projection operators on positive (+), or negative (−), energy states.
\[ \Lambda^{\pm} \]. The fact that they modify with respect to the free field case is of course due to the wave function renormalization, and is the way this effect shows up in interaction processes. In what follows, we will report our results directly in terms of these quantities. Hereafter all calculations are performed in the medium rest frame.

In order to develop a general approach to the finite temperature wave function renormalization, let us first consider the simple and well-known zero temperature case, where the problem can be easily solved using covariance, and looking for the pole shift in \( \hat{p} \). If \( \Sigma = \Sigma_1 \hat{p} + \Sigma_0 \) is the self-energy produced by radiative corrections at one-loop, the propagator for a particle of four-momentum \( p^\mu \equiv (E, \vec{p}) \) and bare rest mass \( m_0 \) takes the form

\[
G = (\hat{p} - m_0 - \Sigma)^{-1} \approx \frac{(\hat{p} - \Sigma_1 \hat{p} + m_0 + \Sigma_0)}{(p^2 - 2\Sigma_1 p^2 - m_0^2 - 2m_0 \Sigma_0)} ,
\]

where \( \Sigma_i = \Sigma_i(p^2) \) with \( i = 0, 1 \) are two scalar functions. Note that in the denominator of (2.1) we have neglected second order terms in \( \Sigma_i \). Using standard manipulations one obtains the following expressions for the mass shift \( \delta m \) and the wave function renormalization factor \( Z_2 \), respectively,

\[
\delta m = \Sigma_{m_0} = m_0 \Sigma_1 + m_0 \Sigma_0 + 2m_0 \frac{d \Sigma_1}{dp^2} + 2m_0 \frac{d \Sigma_0}{dp^2} ,
\]

\[
Z_2 = 1 + \frac{d \Sigma}{dp} \bigg|_{p=m_0} = 1 + \Sigma_1 + 2m_0 \frac{d \Sigma_1}{dp^2} + 2m_0 \frac{d \Sigma_0}{dp^2} .
\]

These results can be equally obtained by expanding the propagator around the free positive energy pole \( E = \omega_p \equiv \sqrt{\vec{p}^2 + m_0^2} \) (for the negative energy pole one proceeds in the same way), and looking for the perturbed value \( \omega_p^R \). The residue at this pole will provide the wave function renormalization factor and give again (2.3). Expanding \( \Sigma_i \) up to first order in \( E - \omega_p \) we get in fact

\[
(p^2 - 2\Sigma_1 p^2 - m_0^2 - 2m_0 \Sigma_0)^{-1} \approx \frac{1}{(E - \omega_p)(E - \omega_p^R)} \left( \Sigma_1 + \Sigma_1^R (E - \omega_p^R) + \frac{m_0^2}{E - \omega_p^R} \Sigma_0 \right) + \frac{m_0^2}{E - \omega_p} \Sigma_0 ,
\]

where \( \Sigma_i \) and \( \Sigma_i^R \) denote the \( \Sigma_i \) functions and their derivatives with respect to \( E \) evaluated at \( E = \omega_p \), which is to say, in this covariant case, at \( p^2 = m_0^2 \). Furthermore, \( \omega_p^R \) stands for the shifted energy pole, perturbatively close to \( \omega_p \)

\[
\omega_p^R \approx \omega_p + \omega_p \frac{m_0^2}{\omega_p} \Sigma_0 ,
\]

\[
\omega_p^R \approx \frac{m_0^2}{\omega_p} \Sigma_1 + \frac{m_0}{\omega_p} \Sigma_0 ,
\]

\[\text{Actually the } \Sigma_i \text{ should be expanded around the still unknown value } \omega_p^R, \text{ but the difference is of higher order in perturbation theory.}\]
which is equivalent to (2.2).

By substituting (2.4) in (2.1) one gets for $E \simeq \omega_p^R$

$$G \simeq \frac{(\hat{p} + m_0) \left( 1 + 2 \tilde{\Sigma}_1 + \frac{m_0^2}{\omega_p^R} \tilde{\Sigma}_1^I + \frac{m_0}{\omega_p^R} \tilde{\Sigma}_0^I \right) - \tilde{\Sigma}_1 \hat{p} + \tilde{\Sigma}_0}{2 \omega_p^R (E - \omega_p^R)} + \text{finite terms}$$

$$= \frac{1 + \tilde{\Sigma}_1 + \frac{m_0^2}{\omega_p^R} \tilde{\Sigma}_1^I + \frac{m_0}{\omega_p^R} \tilde{\Sigma}_0^I}{E - \omega_p^R} \frac{\hat{p} + (m_0 + \delta m)}{2 \omega_p^R} + \text{finite terms} \quad ,$$

(2.6)

where now $\hat{p}$ is understood to be $\hat{p} = \omega_p^R - \vec{p} \gamma$ and $\delta m = m_0 \tilde{\Sigma}_1 + \tilde{\Sigma}_0$, which is again (2.4). From this expression we immediately read off the residue at the shifted positive pole $\omega_p^R$ which represents the renormalized projector operator on positive energy states

$$\Lambda_R^+ = \left( 1 + \tilde{\Sigma}_1 + \frac{m_0^2}{\omega_p^R} \tilde{\Sigma}_1^I + \frac{m_0}{\omega_p^R} \tilde{\Sigma}_0^I \right) \frac{\hat{p} + (m_0 + \delta m)}{2 \omega_p^R} \equiv Z_2 \frac{\hat{p} + m_0 + \delta m}{2 \omega_p^R} \quad .$$

(2.7)

Thus, as well known, in the covariant zero temperature case, $\Lambda_R^+$ preserves the same spinor structure of the free field projector and, as expected, the value of $Z_2$ obtained within this approach coincides with the expression (2.3).

The renormalized projector operator for negative energy states (antiparticles), $\Lambda_R^-$, is obtained from (2.7) by simply replacing $(\omega_p^R, \omega_p^-) \rightarrow (- \omega_p^R, - \omega_p^-)$ and $\vec{p} \rightarrow -\vec{p}$ (more simply $p^0 \rightarrow -p^0$). This is true since the Lorentz invariance of $\Sigma_i$ guarantees a quadratic dependence of these quantities on $E$. Thus, $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_i^I$ of (2.7), evaluated at the positive energy pole, are simply connected with the same quantities evaluated at the negative energy pole, which occurs in the expression of $\Lambda_R^-$. As we will see later, the situation can be quite different in presence of a medium.

We now consider the finite temperature case. As before, we first expand the propagator around the shifted positive (negative) energy pole. The residue of $G$ at the pole will automatically give the perturbed positive (negative) energy projector operator. In the medium rest frame, the radiative correction $\Sigma$ of Eq. (2.1) takes the general form

$$\Sigma = A(E, |\vec{p}|) \, E \gamma^0 - B(E, |\vec{p}|) \, \vec{p} \gamma - C(E, |\vec{p}|) \quad ,$$

(2.8)

where $A$, $B$ and $C$ are scalar functions. Note that the covariant limit corresponds to take $A = B$ and all dependence of $A$, $B$ and $C$ on momentum and energy via the scalar quantity $p^0$. Using the above expression for $\Sigma$ we get for the propagator

$$G \simeq \frac{(\hat{p} + m_0 - A E \gamma^0 + B \vec{p} \gamma - C)}{(\vec{p}^2 - 2 A E^2 - m_0^2 + 2 B |\vec{p}|^2 + 2 m_0 C)} \quad .$$

(2.9)
As for Eq. (2.4), also in this case we expand $A$, $B$ and $C$ near $E = \omega_p$ and by using the same notation we have

\[
\left( \hat{p}^2 - 2\hat{A}E^2 - m_0^2 + 2B|\hat{p}|^2 + 2m_0C \right)^{-1} \simeq \\
\frac{1 + 2 \left( \hat{A} + \frac{E}{E + i\omega_p} \hat{A} - \hat{B}^2 \frac{E + i\omega_p}{E + i\omega_p} \hat{C}^i \right)}{(E + i\omega_p)(E - \omega_p^R)} ,
\]

(2.10)

with

\[
\omega_p^R = \omega_p + \omega_A \hat{A} - \frac{|\hat{p}|^2}{\omega_p} \hat{B} - \frac{m_0}{\omega_p} \hat{C} ,
\]

(2.11)

representing the dispersion relation for the finite temperature physical state and the new positive energy pole. This result can be also written as

\[
\omega_p^R = \sqrt{|\hat{p}|^2 + (m_0 + \delta m)^2} \simeq \omega_p + \frac{m_0 \delta m}{\omega_p} ,
\]

(2.12)

where, from (2.11), the mass shift $\delta m$ is given by

\[
\delta m = \frac{\omega_p^2 \hat{A}}{m_0} - \frac{|\hat{p}|^2}{m_0} \hat{B} - \hat{C} .
\]

(2.13)

Thus for $E \simeq \omega_p^R$ the propagator reads

\[
G \simeq \frac{1 + 2 \hat{A} + \omega_p \hat{A} - \hat{B}^2 \frac{E + i\omega_p}{\omega_p} \hat{C}^i}{2\omega_p^R (E - \omega_p^R)} \left( \hat{p} + m_0 - \hat{A}E\gamma^0 + \hat{B} \hat{p}\gamma^i - \hat{C} \right) + \text{finite terms} ,
\]

(2.14)

with $\hat{p} = \omega_p^R \gamma_0 - \hat{p}\gamma^i$. Using the expression of $\delta m$ and rearranging the terms we get for the residue at $\omega_p^R$, i.e. for the renormalized projector on positive energy states

\[
\Lambda_p^R = \left( 1 + \hat{A} + \omega_p \hat{A} - \frac{|\hat{p}|^2}{\omega_p} \hat{B} - \frac{m_0}{\omega_p} \hat{C}^i \right) \frac{(\hat{p} + m_0) + \left( \hat{B} - \hat{A} \right)}{2\omega_p^R} + \left( \hat{B} - \hat{A} \right) \left[ \hat{p}\gamma^i + \frac{|\hat{p}|^2}{m_0} \right] ,
\]

(2.15)

with $m_R = m_0 + \delta m$.

From this expression we read the two effects produced by the interactions with the medium. First of all the zero temperature mass is shifted to the new effective value $m_R$, which is in general momentum dependent. The second one is a genuine contribution due to the wave function renormalization, which consists not only of a non trivial multiplicative factor but also introduces an additional term proportional to $\hat{B} - \hat{A}$ and with a different spinorial structure. This is the peculiar signature of breaking of Poincaré invariance which introduces a spinor dependence in wave function renormalization. Notice that both terms are momentum (and frame) dependent. It is easily seen that at zero temperature, i.e. in the
covariant limit, \( A = B \), the second term vanishes and the first one reproduces the usual multiplicative renormalization factor.

In order to obtain the renormalized projector on negative energy states one should perform an analogous computation but expanding now all the involved quantities around the negative energy pole \( -\omega_p \). In the general case, \( A, B \) and \( C \) do not have definite properties of parity with respect to \( E \) and so \( \Lambda_R^+ \) and \( \Lambda_R^- \) can have quite different expressions which are not simply connected one each other. This is strongly related with the property of charge conjugation invariance of medium and interaction. For example, for \( e^- \) and \( e^+ \) in thermal equilibrium with photons one only expects their chemical potentials to be opposite. However, if the two distributions are equally populated, and thus chemical potentials vanish, the electron-positron plasma is charge conjugation invariant and hence \( A, B \) and \( C \) are even functions of \( E \). This is for example the situation occurring at the time of BBN. Hereafter we will make this assumption, though more intriguing situations could be considered as well. Under the assumption of charge conjugation invariance, \( \Lambda_R^- \) results to be

\[
\Lambda_R^- = \left( 1 + \hat{A} + \omega_p \hat{A} - \frac{\vec{p}^2}{\omega_p} \vec{B} - \frac{m_0}{\omega_p} \vec{C} \right) \frac{(\hat{p} - m_R)}{2\omega_p} + \left( \frac{\vec{B} - \vec{A}}{2\omega_p} \right) \left[ \vec{\gamma} \vec{\gamma} - \frac{\vec{\gamma}^2}{m_0} \right].
\]

The results of (2.15) and (2.16) are, in a sense, half way between the approaches of Refs. [5] and [13]. The simple multiplicative factor, in fact, is exactly the one obtained in [5], but in that approach the additional term is absent. Denoting with \( \Lambda_{DH}^\pm \) the projector obtained in [5] we get

\[
\Lambda_{DH}^\pm = \Lambda_R^\pm - \left( \frac{\vec{B} - \vec{A}}{2\omega_p} \right) \left[ \vec{\gamma} \vec{\gamma} - \frac{\vec{\gamma}^2}{m_0} \right].
\]

Actually the approach followed there is substantially different than ours. They start introducing finite temperature spinors \( \tilde{\psi} \), chosen as the solution of the nonlinear Dirac equation

\[
(\hat{p} - m_0 - \Sigma) \tilde{\psi} = 0,
\]

whose corresponding creation and annihilation operators are assumed to satisfy ordinary, zero temperature, anticommutation relations. Expanding the propagator in terms of these spinors they obtain a wave function renormalization factor which is only multiplicative. We think that the assumption made on the canonical spinor basis to be used is responsible for this feature and represents the essential difference with our approach. As mentioned, we do not make any hypothesis on the renormalized field, if any, to be used, but simply recover the particle content and the corresponding projector operators from the poles of \( G \) and their residues, respectively.
The fact that the renormalized spinors are related to the free Dirac ones via a momentum dependent transformation in spinor space has been first stressed in [13]. In this analysis the projector operators are deduced, as in our approach, looking at the residue of the propagator, and the following result is obtained, in term of our \( \Lambda_R^\pm \)

\[
\Lambda_S^\pm = \Lambda_R^\pm - \frac{\left( \hat{B} - \hat{A} + \frac{m_\nu}{\omega_p} \hat{C} - \frac{m_\nu^2}{\omega_p} \hat{B} \right)}{2\omega_p} \left[ p^\gamma \gamma^\mu \frac{p_\mu}{m_0} \right]. \quad (2.19)
\]

It is still unclear to us the reason for the difference between \( \Lambda_S^\pm \) and our result, since the starting point, in both cases, is the same. It should be stressed, however, that \( \Lambda_S^\pm \) does not reproduce the expected behaviour in the zero temperature, covariant limit. In fact, since we have already shown that \( \Lambda_R^\pm \) gives back in this limit the correct result, \( \Lambda_S^\pm \) will still contain an explicitly noncovariant term, only vanishing if \( \hat{C}' = m_0 \hat{B}' \), which is not guaranteed by any general principle.

**III. APPLICATIONS TO SIMPLE PROCESSES**

In order to show the physical differences of the three different approaches to thermal wave function renormalization, simply summarized by the use of projectors \( \Lambda_R^\pm, \Lambda_{DH}^\pm \) and \( \Lambda_S^\pm \), let us discuss a relevant example like the leptonic Higgs decay \( H \to l^+ l^- \). A similar analysis has been performed in [14]. At tree-level the matrix element for this process reads

\[
\mathcal{A}(H \to l^+ l^-) = -\frac{ig\mu}{2M_W} m(l^-) \nu(l^+) ,
\]

where \( m_l \) stands for the zero temperature lepton mass, and \( l^-, \nu(l^+) \) denote the \( T = 0 \) Dirac spinors for lepton and antilepton, respectively. In a thermal bath of equally populated lepton-antilepton pairs\(^1\), the above decay width acquires an additional contribution, due to the lepton wave function renormalization, which at one loop results to be

\[
\left[ \mathcal{M} \right]^2_T = \sum_{\lambda^+, \lambda^-} \left| \mathcal{A}(H \to l^+(\lambda^+)l^-(\lambda^-)) \right|^2 = \frac{g^2 m_l^2}{4M_W^2} \text{Tr} [\Lambda^+(l^-)\Lambda^-(l^+)]
\]

\[
= \frac{g^2 m_l^2}{4M_W^2} \text{Tr} [(\Lambda_0^+(l^-) + \delta \Lambda^+(l^-)) (\Lambda_0^-(l^+) + \delta \Lambda^-(l^+))]
\]

\[
\simeq \frac{g^2 m_l^2}{4M_W^2} \left\{ \text{Tr} [\Lambda_0^+(l^-)\Lambda_0^-(l^+)] + \text{Tr} [\delta \Lambda^+(l^-)\Lambda_0^-(l^+) + \Lambda_0^+(l^-)\delta \Lambda^-(l^+)] \right\}
\]

\[
= \left[ \mathcal{M} \right]^2_0 + \delta \left[ \mathcal{M} \right]^2 . \quad (3.2)
\]

\(^1\)We assume the Higgs mass much larger than the lepton temperature so that the decaying particle can be considered at \( T = 0 \).
In Eq. (3.2) \( \lambda^\pm \) stand for the lepton polarizations and with \( \Lambda_0^\pm (l^-) \) and \( \delta \Lambda^\pm (l^-) \) we denote the zero temperature lepton projector and its first order thermal correction, respectively. The second term in the r.h.s. of (3.2), \( \delta |\mathcal{M}|^2 \), then represents the thermal contribution to the decay width due to thermal wave function renormalization. The expression for \( \delta \Lambda^\pm (l^-) \) can be extracted from \( \Lambda^\pm \) of Eqs (2.15),(2.16), \( \Lambda_{DH}^\pm \) or \( \Lambda_S^\pm \). Once substituted in \( \delta |\mathcal{M}|^2 \) we get in the c.m. reference frame

\[
\delta |\mathcal{M}|^2_{IR} = \frac{g^2 m_l^2}{M_W^2} \left( 1 - \frac{4 m_l^2}{M_H^2} \right) \left[ 2 \hat{A} - \hat{B} + \frac{M_H}{2} \hat{A} - \frac{M_H}{2} \left( 1 - \frac{4 m_l^2}{M_H^2} \right) \hat{B}' - \frac{2 m_l}{M_H} \right], \tag{3.3}
\]

\[
\delta |\mathcal{M}|^2_{DH} = \frac{g^2 m_l^2}{M_W^2} \left( 1 - \frac{4 m_l^2}{M_H^2} \right) \left[ \hat{A} + \frac{M_H}{2} \hat{A} - \frac{M_H}{2} \left( 1 - \frac{4 m_l^2}{M_H^2} \right) \hat{B}' - \frac{2 m_l}{M_H} \right], \tag{3.4}
\]

\[
\delta |\mathcal{M}|^2_{S} = \frac{g^2 m_l^2}{M_W^2} \left( 1 - \frac{4 m_l^2}{M_H^2} \right) \left[ \hat{A} + \frac{M_H}{2} \left( \hat{A}' - \hat{B}' \right) \right]. \tag{3.5}
\]

We have not included the effect due to the renormalization of the lepton mass, in order to disentangle the two different effects. To consider its contribution it is sufficient to substitute the value of \( m_l \) and \( \omega_p \) with \( m_l^R = m_l + \delta m \) and \( \omega_p^R = \omega_p + \delta \omega \) reported in (2.13) and (2.12) respectively, in the tree level squared amplitude

\[
|\mathcal{M}|^2_0 \Rightarrow \frac{g^2 m_l^2}{2 M_W^2} \left( 1 - \frac{4 (m_l^R)^2}{M_H^2} \right). \tag{3.6}
\]

A similar shift can also be performed in any of the contributions (3.3)-(3.5), since the difference we introduce in this way on the squared amplitude is of higher order in perturbation expansion.

All approaches give the expected behaviour for the squared amplitude\(^4\), which must vanish if \( M_H = 2m_l \). Nevertheless, in the expressions (3.3)-(3.5) the thermal corrections take different forms. It is worth observing that both our result and the one obtained using \( \Lambda_{DH}^S \) coincide in the covariant limit and give the expected \( Z_2 \) factor reported in (2.3). This is not the case if \( \Lambda_S^\pm \) is instead used.

Another example where to compute the additional contribution due to the finite temperature renormalization wave function for electrons is the \( \beta \)-decay \( n \rightarrow p + e^- + \nu_e \) and the crossed related processes. Actually, the need for a consistent way of computing thermal corrections to these processes, relevant for primordial nucleosynthesis, represents one of the main motivation for our present analysis. According to the notation adopted in [3] a simple computation shows that, in the nonrelativistic limit for both neutron and proton, the use of the renormalized projector \( \Lambda_R^\pm \) for the outgoing electron gives the additional contribution to the rate per neutron

\(^4\)In this respect we note that the result quoted in [14] as equation (11) is incorrect.
\[
\Delta \omega_S = \frac{G_F (\beta^2 + 3 \alpha^2)}{2 \pi^3} \int_0^\infty d|\vec{p}| \, |\vec{p}|^2 Q^2 \theta(Q) \left( \hat{A}(\vec{p}) + \omega_p \hat{A}^{\dagger}(\vec{p}) - \frac{m_\nu}{\omega_p} \hat{B}(\vec{p}) - \frac{m_\nu}{\omega_p} \hat{B}^{\dagger}(\vec{p}) \right) \\
\times (1 - F_\nu(Q)(1 - F_e(\omega_p)),
\]

where $\vec{p}$ is electron momentum, $\omega_p$ its energy, $Q = M_n - M_p - \omega_p$ the neutrino energy and $F_{\nu e}$ are the neutrino and electron Fermi-Dirac distributions, respectively. The corresponding corrections due to electron (positron) wave-function renormalization for the other crossed processes are easily obtained replacing the thermal factors and the expression of $Q$ as reported in Table 1 of ref. [3]. Despite of the different forms of $\Lambda^+_R$, $\Lambda^+_S$ and $\Lambda^+_{DH}$, it is interesting to note that unlike the scalar boson decay previously considered, all give the same result (3.7) in the nonrelativistic limit. The reason is that the extra pieces which make different the three expressions (see Eq.s 2.17 and 2.19) give a correction which vanishes after the angular integration of the differential rate is performed. This fact has been already noticed in [14].

IV. CONCLUSIONS

In this paper we have addressed the problem of finite temperature renormalization effects on spinor wave function. This issue has been previously considered by many authors, with different results. The main question seems to be whether this renormalization effect is simply multiplicative, as in ordinary quantum field theory, or rather involves a non trivial spinorial dependence as well. Our starting point has been to look for the residue at the energy poles of the propagator, corrected for thermal interactions with the surrounding medium. These poles give the new energy–momentum dispersion relation and illustrate the particle content of the theory. This point of view bypass the difficulty and ambiguity related with the choice of the renormalized fields to be used at finite temperature, which are due to the non-local and frame dependent character of the medium effects. The projector operators on positive and negative states we have obtained show that, as pointed out in [13], the breaking of Poincaré invariance leads to the appearance of extra contributions with a different spinorial structure. They reduce to ordinary renormalized projectors in the covariant limit.

The physical interest for this problem is mainly connected with the study of thermal corrections to nuclear interactions in the early universe, during the primordial nucleosynthesis epoch. Notably, our results for the case of $\beta$-decay and related processes completely agree, in the non relativistic regime, with the ones which can be obtained using different approaches or results for wave function renormalization,
since the extra contributions mentioned above play no role in this limit. This is not the case for the decay of a scalar boson in fermion–antifermion pair where we have found using our approach, a different result for the corresponding change in the decay rate.