Isolated vacua in supersymmetric Yang–Mills theories

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Abstract
An explicit proof of the existence of nontrivial vacua in the pure supersymmetric Yang–Mills theories with higher orthogonal $SO(N), N \geq 7$ or the $G_2$ gauge group defined on a 3–torus with periodic boundary conditions is given. Extra vacuum states are separated by an energy barrier from the perturbative vacuum $A_i = 0$ and its gauge copies.

It was shown by Witten long time ago [1] that, in a pure $N = 1$ supersymmetric gauge theory with any simple gauge group, the supersymmetry is not broken spontaneously. Placing the theory in a finite spatial box, the number of supersymmetric vacuum states [the Witten index $\text{Tr}(\frac{1}{2})F$] was calculated to be $\text{Tr}(\frac{1}{2})F = r + 1$ where $r$ is the rank of the gauge group. This results conforms with other estimates for $\text{Tr}(\frac{1}{2})F$ for unitary and symplectic groups. It disagrees, however, with the general result 1

$$\text{Tr}(\frac{1}{2})F = T(G)$$  \hfill (1)

($T(G)$ is the Dynkin index of the adjoint representation) for higher orthogonal and exceptional groups. For $SO(N \geq 7)$, $T(G) = N - 2 > r + 1$. Also for exceptional groups $G_2, F_4, E_{6,7,8}$, the index (1) is larger than Witten’s original estimate.

This paradox persisting for more than 15 years has been recently resolved by Witten himself [3]. He has found a flaw in his original arguments and shown that, for $SO(N \geq 7)$, vacuum moduli space is richer than it was thought before so that the total number of quantum vacua is $N - 2$ in accordance with the result (1). This note presents basically a comment to Witten’s recent paper. Its raison d’être is to derive this result in an explicit way and in a form understandable to pedestrians (the paper [3] is full of special mathematical terminology and concepts which makes it difficult to understand for a person not familiar with this language). We also extend the analysis to the $G_2$ gauge group.

Let us first recall briefly Witten’s original reasoning.

1which follows e.g. from the counting of gluino zero modes on the instanton background [1, 2]
• Put our theory on the spatial 3D torus and impose periodic boundary conditions on the gauge fields\(^2\). Choose the gauge \(A_0^a = 0\). A classical vacuum is defined as a gauge field configuration \(A_i^a(x, y, z)\) with zero field strength (a flat connection in mathematical language).

• For any flat periodic connection, we can pick out a particular point in our torus \((0, 0, 0) \equiv (L, 0, 0) \equiv \ldots\) and define a set of holonomies (Wilson loops along non-trivial cycles of the torus)

\[
\begin{align*}
\Omega_1 & = P \exp \left\{ i \int_0^L A_1(x, 0, 0) \, dx \right\} \\
\Omega_2 & = P \exp \left\{ i \int_0^L A_2(0, y, 0) \, dy \right\} \\
\Omega_3 & = P \exp \left\{ i \int_0^L A_3(0, 0, z) \, dz \right\}
\end{align*}
\]

(\(A_i = A_i^a T^a\) where \(T^a\) are the group generators in a given representation). \(\text{Tr}\{\Omega_i\}\) are invariant under periodic gauge transformations.

• A necessary condition for the connection to be flat is that all the holonomies (2) commute \([\Omega_i, \Omega_j] = 0\). We will see shortly that, for a simple connected group with \(\pi_1(G) = 0\), it is also a sufficient condition for a flat periodic connection with given holonomies to exist.

• A sufficient condition for the group matrices to commute is that their logarithms belong to a Cartan subalgebra of the corresponding Lie algebra. For unitary and simplectic groups, this is also a necessary condition. In other words, any set of commuting group matrices \(\Omega_i\) with \([\Omega_i, \Omega_j] = 0\) can be presented in the form

\[
\Omega_i = \exp\{iC_i\}, \quad [C_i, C_j] = 0
\]

A flat connection with the holonomies \(\Omega_i\) is then just \(A_i = C_i/L\). Witten’s original assumption which came out not to be true is that this is also the case for all other groups. Assuming this, Witten constructed an effective Born–Oppenheimer hamiltonian for the slow variables \(A_i^a\). It involves \(3r\) bosonic degrees of freedom (\(r\) is the rank of the group) and their fermionic counterparts. Imposing further the condition of the Weyl symmetry (a remnant of the original gauge symmetry) for the eigenstates of this hamiltonian, one finds \(r + 1\) supersymmetric quantum vacuum states.

Before proceeding further, let us prove a simple

**Theorem:** For any set \(\{\Omega_1, \Omega_2, \Omega_3\}\), \(\Omega_i \in G\) where \(G\) is a simple, connected, and simply connected group, \([\Omega_i, \Omega_j] = 0\) \(\forall i, j\), a periodic flat connection exists such that \(\Omega_i\) are the holonomies (2).

\(^2\)For unitary groups, one can perform the counting also with ‘t Hooft twisted boundary conditions, but for the orthogonal and exceptional groups where the mismatch in the Witten index calculations was observed this method does not work.
Proof: A flat connection (a pure gauge configuration) can be presented in the form
\[ A_i = -i \partial_i U U^{-1} \] with \( U(x, y, z) \in G \). Let us seek for \( U(x, y, z) \) satisfying the following boundary conditions
\[
\begin{align*}
U(x + L, y, z) &= U(x, y, z) \Omega_1 \\
U(x, y + L, z) &= U(x, y, z) \Omega_2 \\
U(x, y, z + L) &= U(x, y, z) \Omega_3
\end{align*}
\] (4)
with constant commuting \( \Omega_i \) (commutativity of \( \Omega_i \) is important for the matrix \( U \) to be uniquely defined). Then \( A_i(x, y, z) \) is periodic. If choosing \( U(0, 0, 0) = 1 \), the matrices \( \Omega_i \) are the holonomies (2). We construct the matrix \( U(x, y, z) \) in several steps.

- At the first step, we define
\[
\begin{align*}
U(x, 0, 0) &= \exp \left\{ i \pi T_1 \frac{x}{L} \right\} \\
U(0, y, 0) &= \exp \left\{ i \pi T_2 \frac{y}{L} \right\} \\
U(0, 0, z) &= \exp \left\{ i \pi T_3 \frac{z}{L} \right\}
\end{align*}
\] (5)
where \( \Omega_i = \exp \{ i \pi T_i \} \) (The choice of \( T_i \) once \( \Omega_i \) are given is not unique, but it is irrelevant. Take some set of the logarithms of holonomies \( \Omega_i \)). Having done this, we can extend the construction over all other edges of the 3-cube so that the boundary conditions (4) are fulfilled. For example, we define
\[
U(L, y, 0) = \exp \left\{ i \pi T_2 \frac{y}{L} \right\} \Omega_1, \quad U(x, L, 0) = \exp \left\{ i \pi T_1 \frac{x}{L} \right\} \Omega_2
\]

- With \( U(x, y, z) \) defined on the edges of the cube in hand, we can continue \( U \) also to the faces of the cube due to the fact that, according to our assumption, \( \pi_1(G) = 0 \) i.e. any loop in the group is contractible. Let us do this first for 3 faces adjacent to the vertex \( (0,0,0) \).

- With \( U(x, y, 0) \), \( U(x, 0, z) \), and \( U(0, y, z) \) in hand, we can find \( U(x, y, z) \) on the other 3 faces of the cube:
\[
U(x, y, L) = U(x, y, 0) \Omega_3, \quad U(x, L, z) = U(x, 0, z) \Omega_2, \quad U(L, y, z) = U(0, y, z) \Omega_1
\]

- With \( U(x, y, z) \) defined on the surface of the cube, we can continue it into the interior using the fact that \( \pi_2(G) = 0 \) for all simple Lie groups.

By construction, \( U(x, y, z) \) satisfies the boundary conditions (4) and hence \( A_i(x, y, z) \) is periodic.

The skeleton construction just outlined is rather common in homotopy theory and can be found also in physical literature (see e.g. [4]). The proof works only for simply connected groups. If \( \pi_1(G) \neq 0 \), it is generally not true, i.e. not for every set of commuting
Ω, a flat periodic connection with the holonomies Ω, exists. The simplest counterexample is the set of three \( SO(3) \) matrices

\[
\Omega_1 = \text{diag}(1, -1, -1); \quad \Omega_2 = \text{diag}(-1, 1, -1); \quad \Omega_3 = \text{diag}(-1, -1, 1)
\]  

(6)

Being diagonal, they obviously commute, but no corresponding periodic flat connection can be constructed. Indeed, suppose we have an \( SO(3) \) connection \( A^a_i(x, y, z) \) with the holonomies (6). The functions \( A^a_i(x, y, z) \) can be thought of also as a connection corresponding to the covering group \( SU(2) \). Nothing prevents us then from constructing the holonomies in the fundamental \( SU(2) \) representation, which must be the liftings of the holonomies (6) to the group \( SU(2) \). One can readily derive

\[
\Omega^1_{SU(2)} = \pm i\sigma_1, \quad \Omega^2_{SU(2)} = \pm i\sigma_2, \quad \Omega^3_{SU(2)} = \pm i\sigma_3
\]  

(7)

But the matrices (7) do not commute anymore, the logarithms of the products \( \Omega^1_{SU(2)} \Omega^2_{SU(2)} [\Omega^1_{SU(2)}]^{-1} [\Omega^2_{SU(2)}]^{-1} \), etc. define nonzero fluxes of the magnetic field along the corresponding directions on the 3-torus. Thus, the connection \( A^a_i(x, y, z) \) cannot be flat.

In reference [3], Witten constructs a set of 3 commuting \( SO(7) \) matrices such that

i) they cannot be presented in the form \( \Omega_i = \exp\{iC_i\} \) with commuting \( C_i \); and

ii) the corresponding holonomies in the covering group \( \text{Spin}(7) \) do commute as well.

As the group \( \text{Spin}(7) \) [in contrast with \( SO(7) \)] is simply connected, the theorem which we have just proven guarantees the existence of the corresponding nontrivial periodic flat connection. A particular convenient choice of 3 commuting \( SO(7) \) matrices is

\[
\Omega_1 = \text{diag}(1, -1, -1, 1, 1, -1, 1, 1, -1) \\
\Omega_2 = \text{diag}(1, -1, 1, 1, -1, 1, -1, 1, -1) \\
\Omega_3 = \text{diag}(-1, 1, 1, -1, 1, -1, -1, 1, -1)
\]  

(8)

This set consists of seven “rows” with \( \pm 1 \) as the elements such that every one of the seven possible combinations like

\[
\begin{pmatrix}
1 \\
1 \\
-1
\end{pmatrix}
\]

involving at least one minus appears just once.

As far as \( SO(7) \) is concerned, many other choices of the set \( \{\Omega_i\} \) differing from (8) by permutations of the rows is possible. They all can be obtained from each other by global \( SO(7) \) rotations. We have chosen a particular order of the rows anticipating the further \( G_2 \) applications.

Each \( \Omega_i \) can be represented as an exponential of an \( SO(7) \) generator. This representation is far from being unique. For example,

\[
\Omega_1 = \exp\{i\pi[T_{34} - T_{27}]\} = \exp\{i\pi[T_{23} - T_{47}]\} = \ldots \\
\Omega_2 = \exp\{i\pi[T_{56} + T_{27}]\} = \exp\{i\pi[T_{25} + T_{67}]\} = \ldots \\
\Omega_3 = \exp\{i\pi[T_{16} - T_{47}]\} = \exp\{i\pi[T_{14} + T_{57}]\} = \ldots
\]  

(9)
where $T_{ij}$ are the generators of the rotations in the $(ij)$ plane. The point is that one cannot choose the “logarithms” of $\Omega_i$ so that all of them commute. Suppose one could, then write $\Omega_i = \exp\{i\pi S_i\}$. [$S_i, S_j = 0 \forall i, j$ implies $[S_i, \Omega_j] = 0 \forall i, j$. But, as one can easily check, a matrix that commutes with all three $\Omega_j$ given has to be diagonal. The generators of $SO(7)$ however are antisymmetric, so no generator of $SO(7)$ commutes with all $\Omega_i$. This proves that the assumption $[S_i, S_j] = 0$ is wrong.

The new vacuum is isolated. This can be seen as follows: Try to perturb the $\Omega_i$

$$\Omega_i' = \Omega_i (1 + i\alpha_i^a T^a)$$

(10)

and require that all $\Omega_i'$ still commute. This implies the conditions

$$\alpha_i^a [\Omega_i^{\prime}, T^a, \Omega_j] = \alpha_j^a [\Omega_j^{\prime}, T^a, \Omega_i]$$

To solve these, we note that:

- With the given $\Omega^i$ and the standard basis $T^a$ of the $so(7)$ Lie algebra, either $[\Omega^i, T^a] = 0$ or $\{\Omega^i, T^a\} = 0$
- $\alpha_i^a = 0$ if $[T^a, \Omega_i] = 0$ (since $[T^a, \Omega_j] \neq 0$ for some $i \neq j$)
- $\alpha_i^a = \alpha_j^a$ if both $[T^a, \Omega_i] \neq 0$ and $[T^a, \Omega_j] \neq 0$

From these observations it follows that we can rewrite (10) as

$$\Omega_i' = \Omega_i + i\beta^a [\Omega_i, T^a]$$

(11)

with $\beta^a$ independent of $i$. This is a global group rotation, and not a nontrivial deformation. The new vacuum does not admit deformations, and hence is isolated.

$SO(7)$ has a real 8–component spinor representation. Spinors are transformed under rotations. A set of all the corresponding $8 \times 8$ matrices is called the Spin(7) group. Two Spin(7) matrices differing by a sign correspond to one and the same $SO(7)$ matrix. Thereby a covering $Spin(7) \xrightarrow{2} SO(7)$ exists much analogous to the familiar covering $SU(2) \xrightarrow{2} SO(3)$. Witten proves that the Spin(7) holonomies corresponding to the $SO(7)$ holonomies (8) commute using a powerful mathematical machinery involving notions like Stieffel–Whitney class etc. We will show this explicitly.

The generators of Spin(7) are

$$T^{s}_{ij} = \frac{1}{4} [\Gamma_i, \Gamma_j]$$

(12)

where $\Gamma_i$ are 7–dimensional $\Gamma$–matrices satisfying the Clifford algebra $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2 \delta_{ij}$. One particular choice for the $\Gamma$–matrices is

$$\Gamma_1 = i \sigma^2 \otimes \sigma^2 \otimes \sigma^2; \quad \Gamma_2 = -i \otimes \sigma^1 \otimes \sigma^2; \quad \Gamma_3 = -i \otimes \sigma^3 \otimes \sigma^2; \quad \Gamma_4 = i \sigma^1 \otimes \sigma^2 \otimes 1; \quad \Gamma_5 = -i \sigma^3 \otimes \sigma^2 \otimes 1; \quad \Gamma_6 = -i \sigma^2 \otimes 1 \otimes \sigma^1; \quad \Gamma_7 = -i \sigma^2 \otimes 1 \otimes \sigma^3$$

(13)

It is not difficult now to construct explicitly the generators of Spin(7) and exponentiate them as in Eq.(9). It does not matter which particular representation for $\Omega_i$ is chosen, it
affects only the overall sign. The set of the Spin(7) holonomies corresponding to the set (8) of the SO(7) holonomies is

$$\Omega^\text{spin}_1 = \pm \sigma^3 \otimes 1 \otimes \sigma^3; \quad \Omega^\text{spin}_2 = \pm \sigma^3 \otimes \sigma^3 \otimes 1; \quad \Omega^\text{spin}_3 = \pm \sigma^3 \otimes 1 \otimes 1$$  \hspace{1cm} (14)

It is easy to see that $[\Omega^\text{spin}_i, \Omega^\text{spin}_j] = 0$. As $\pi_1[\text{Spin}(7)] = 0$, non-trivial periodic flat connections with the holonomies (14, 8) exist.

A similar statement for the $G_2$ gauge group can be obtained free of charge. The $G_2$ group can be defined as a subgroup of $SO(7)$ leaving invariant the combination $f_{ijk} Q^i P^j R^k$ where $Q^i, P^j, R^k$ are 3 arbitrary 7–vectors and $f_{ijk}$ is a certain antisymmetric tensor. One particular convention for $f_{ijk}$ is

$$f_{165} = f_{341} = f_{523} = f_{271} = f_{673} = f_{475} = f_{246} = 1$$  \hspace{1cm} (15)

and all other non-zero components are recovered using antisymmetry. It is easy to see now that the matrices (8) do belong to the $G_2$ subgroup of $SO(7)$. Another way to see the same is to define $G_2$ as a subgroup of Spin(7) leaving a particular spinor invariant. The matrices (14) leave invariant the spinor $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and hence belong to $G_2$ (and, incidentally, $f_{ijk} = \eta^T \Gamma_i \Gamma_j \Gamma_k \eta$). As $G_2$ is simply connected, $\pi_1(G_2) = 0$ \footnote{This can be seen from the fact that the unique compact simply-connected group with Lie-algebra $G_2$ has a trivial center (see e.g. [5])}. One can apply our general theorem immediately and make sure thereby that a non-trivial periodic flat $G_2$ connection exists. This extra vacuum state together with $r_{G_2} + 1 = 3$ “old” states associated with the constant gauge potentials belonging to the Cartan subalgebra makes the total vacuum state counting in accordance with the result $\text{Tr}(-1)^F = T(G_2) = 4$.

For $SO(7)$ and $G_2$, the new corner in the moduli space of classical vacua presents just a single point. The same is true for $SO(8)$: up to a global gauge transformation, any set of commuting $SO(8)$ matrices whose logarithms do not commute can be presented in the form $\Omega^i_{SO(8)} = \text{diag}(\Omega^i_{SO(7)}, 1)$ with $\Omega^i_{SO(7)}$ given by Eq.(8). Consider still higher orthogonal groups. Starting from $SO(9)$, an additional freedom appears associated with Cartan rotations in extra dimensions; any set $\Omega^i_{SO(N)} = \text{diag}(\Omega^i_{SO(7)}, \omega^i_{SO(N-7)})$ with logarithms of $\omega^i_{SO(N-7)}$ belonging to the Cartan subalgebra of $SO(N-7)$ gives rise to a nontrivial $SO(N)$ connection. The extra component of the moduli space is not an isolated point anymore, but presents a manifold. Its dimension is $3r_{SO(N-7)}$. There are $r_{SO(N-7)} + 1$ eigenstates of the corresponding Born–Oppenheimer hamiltonian. All together we have $(r_{SO(N)} + 1) + (r_{SO(N-7)} + 1) = N - 2$ vacuum states [3] in accordance with the counting (1).

There is some subtlety for $SO(9)$, where the continuous unbroken symmetry group is $SO(2)$, which is abelian. The index for $SO(2)$-theory is $\text{Tr}(-1)^F = 0$, which seems to lead to a wrong answer for $SO(9)$-theory. This is resolved as follows. Apart from the continuous $SO(2)$, there are also some discrete symmetries unbroken. An example of such a discrete symmetry is represented by the matrix $\text{diag}(1, 1, 1, 1, 1, -1, -1, -1)$. This matrix commutes with the holonomies $\text{diag}(\Omega^i_{SO(7)}, 1, 1)$, and acts as $\text{diag}(1, -1)$ in the unbroken $SO(2)$-subgroup. It is a gauge symmetry, so we have to demand invariance under this
symmetry. In this way, the unbroken $SO(2)$ is enhanced to $O(2)$, and we need $\text{Tr}(-1)^F$ for $O(2)$-theory, not $SO(2)$. To calculate the index we can simply repeat the analysis for $SO(2)$ from [1], with the requirement of invariance under the extra symmetry $\text{diag}(1, -1)$. One finds that, of the four states mentioned in [1], the two states with one fermion are not invariant under the extra symmetry, while the two bosonic states (two fermions or none) are invariant. In this way we find $\text{Tr}(-1)^F = 2$ for $O(2)$-theory, in contrast to the zero result of $SO(2)$-theory. Hence for $SO(9)$ one finds $(r_{SO(9)} + 1) + 2 = 7$, the right number.

In an analogous way one finds that for the higher orthogonal groups, the unbroken symmetry group is actually $O(N - 7)$ (for the Spin groups it is $\text{Pin}(N - 7)$, the double cover of $O(N - 7)$). However, the extra symmetry does not affect the analysis in this case, and the previous results stay valid.

At first sight, starting from $N = 14$, a new corner in the moduli space associated with the matrices $\text{diag}(\Omega^{SO(7)}_1, \Omega^{SO(7)}_2)$ might appear. This is not so, however. One can write explicitly

$$\Omega^{SO(14)}_1 = \text{diag}(1, -1, -1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, 1) = \exp\{i\pi[T_{2,9} + T_{3,10} + T_{4,11} + T_{7,14}]\} \quad \text{(16)}$$

and similarly for $\Omega_2, \Omega_3$. $\log \Omega^{SO(14)}_i$ defined according to the prescription (16) commute and belong to the Cartan subalgebra of $SO(14)$. So we obtain nothing new. Also for still higher groups nothing new happens. After possibly rotating away non-trivial $SO(14)$-blocks, as in the above, all the information about the holonomies will be contained in an $SO(p)$- subgroup (with $p < 14$). Hence for $SO(N)$, there are never more than two components in the moduli space, no matter how large $N$ gets.

It is an interesting and still unresolved problem how to find a similar explicit construction revealing extra $T(G) - r_G - 1$ vacuum states for four other exceptional groups. The corresponding numbers are listed in the Table.

<table>
<thead>
<tr>
<th>group G</th>
<th>$F_4$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
</tr>
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<tbody>
<tr>
<td>$r + 1$</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>$T(G)$</td>
<td>9</td>
<td>12</td>
<td>18</td>
<td>30</td>
</tr>
<tr>
<td>mismatch</td>
<td>4</td>
<td>5</td>
<td>10</td>
<td>21</td>
</tr>
</tbody>
</table>

**Table**: Vacuum counting for higher exceptional groups.

For $E_7$, $T(G) > 2(r + 1)$ while, for $E_8$, $T(G) > 3(r + 1)$. Assuming that each disconnected component might contribute not more than $r + 1$ states in the total counting, it suggests the presence of at least three disconnected components for $E_7$ and at least four components for $E_8$.

The presence of extra disconnected components in the vacuum moduli space implies the existence of classical solutions to the Yang–Mills equations of motion for the field living on $T^3 \otimes R_\tau$ which interpolate between trivial vacua with constant commuting $A_i$ at $\tau = -\infty$ and the non–trivial one at $\tau = \infty$. Indeed, one can start with the configuration

$$A_i^\text{trial}(\tau, \vec{x}) = \frac{1 + \tanh(\mu \tau)}{2} A_i^\text{isol. vac.}(\vec{x}) \quad \text{(17)}$$
and then deform it with the boundary conditions at $\tau = \pm \infty$ fixed so that the action be minimized. A solution thus obtained should have a finite action $\sim 1/g^2$. Probably, it can be found only numerically in a way similar to how the toron–like Euclidean solutions for the $SU(2)$ gauge group with 't Hooft twisted boundary conditions were earlier found in Ref.[6].

These new Euclidean solutions have nothing to do with conventional instantons: the latter interpolate between trivial vacua and the vacua with nonzero integer Chern–Simons number but with the same trivial holonomies. An interesting question is, what is the Chern–Simons number of our isolated vacuum.

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References


