Photon Damping of Waves in Accretion Disks

Eric Agol and Julian Krolik
Physics and Astronomy Department, Johns Hopkins University, Baltimore, MD 21218

ABSTRACT

MHD turbulence is generally believed to have two important functions in accretion disks: it transports angular momentum outward, and the energy in its shortest wavelength modes is dissipated into the heat that the disks radiate. In this paper we examine a pair of mechanisms which may play an important role in regulating the amplitude and spectrum of this turbulence: photon diffusion and viscosity. We demonstrate that in radiation pressure-dominated disks, photon damping of compressive MHD waves is so rapid that it likely dominates all other dissipation mechanisms.

Subject headings: accretion, accretion disks, turbulence, waves, MHD, radiation

1. Introduction

Turbulence is widely thought to be central to the dynamics of accretion disks. A combination of magnetic and Reynolds turbulent stresses may be responsible for the outward transport of angular momentum without which no accretion could occur (Shakura and Sunyaev 1973, Balbus et al. 1994). The energy put into this turbulence is ultimately deposited as heat, and is therefore the energy source for the radiation by which we observe accretion disks. Although much effort has gone into identifying mechanisms which excite turbulence (Balbus and Hawley 1991), far less attention in the literature has been given to how the turbulence dissipates. In most instances, it is simply assumed that nonlinear couplings transfer energy from long wavelengths to short, and that some dissipative mechanism eventually damps very short wavelength motions.

One reason why little thought has been given to the specifics of dissipation is that, as matter drifts inward through an accretion disk, if the disk is in a time-steady state its lost potential energy is transformed into heat and kinetic energy at a rate which is entirely fixed by global properties. If the gravitational potential is dominated by the mass $M$ of the central object, the heating rate per unit area is

$$Q = \frac{3}{4\pi} \frac{GM\dot{M}}{r^3} R_R(r).$$

(1)

$R_R \simeq 1$ at large radii) describes the reduction of the local heating due both to the kinetic energy carried outward with the angular momentum flux, and relativistic effects should the central object be a neutron star or black hole (Novikov and Thorne 1973).
It is a great simplification to calculations of disk equilibria that the heating rate should depend only on global quantities. However, this fact leaves open the question of how exactly the energy lost by the accretion flow is transformed into heat, and there are strong observational consequences that depend on just how this happens. For example, the existence of weakly-radiative disks (Ichimaru 1977; Rees et al. 1982; Narayan and Yi 1995) depends critically on the assumption that most of the heat goes to the ions, not the electrons. There have been other suggestions that a significant part of the heat goes into non-thermal particle distributions (e.g. Ferrari 1984 or Stecker et al. 1991). Alternatively, the energy can be lost in magnetic fields which escape the disk, forming a corona or outflow (Galeev et al. 1979).

Balbus & Hawley (1991) pointed out that MHD fluctuations should be linearly unstable in weakly-magnetized accretion disks. Fully nonlinear simulations (Hawley, Gammie & Balbus 1995; Brandenburg et al. 1995; Stone et al. 1996) have shown that these fluctuations grow until the field energy density approaches the pressure in the disk, and that nonlinear couplings create fluctuations on shorter and shorter wavelengths. Most recent work on how the energy in these fluctuations is dissipated has concentrated on plasma physics effects that work on modes of very short wavelength (e.g. Bisnovatyi-Kogan & Lovelace 1997; Quataert 1997; Blackman 1997; Gruzinov 1997), especially in low-density, high temperature disks.

Although this focus is well-grounded in reality in the context of MHD turbulence in laboratory plasmas, it ignores the fact that accretion disks are often extremely bright, and can contain such high photon densities that radiation dominates the total pressure. In this paper we point out that photon diffusion and viscosity can, in radiation-dominated accretion disks dominate all other mechanisms of dissipation. When that is so, compressive modes whose wavelengths are almost as great as a disk thickness can be rapidly damped. Significant consequences follow for the amplitude of MHD turbulence, the rate at which angular momentum may be transported, and the way in which the energy associated with the turbulence is dissipated into heat.

The structure of this paper is as follows: we first extend (§2) the theory of MHD modes interacting with a background photon gas by substituting a time-dependent radiation transfer solution for the conventional description in terms of a photon viscosity. Our procedure is similar in character to the one adopted to treat photon diffusion damping of perturbations in the early Universe (“Silk damping”: Silk 1968, Hu & Sugiyama 1996). We then apply this improved theory to conventional accretion disk models (§3). In §4 we discuss the impact of photon damping on both advection-dominated accretion disks and disks in which the dissipation is segregated into a corona. Finally, in §5 we summarize our results and discuss their significance.

We close this introduction with some notes of distinction. There were earlier suggestions by Loeb and Laor (1992) and Tsuribe and Umemura (1997) that photon viscosity due to an external radiation field might explain the radial angular momentum transport in some accretion disks. We do not make that claim; in this paper we consider only how photon kinetic effects help regulate the amplitude of the MHD turbulence that is responsible for angular momentum transport. The effects
of photon damping we consider are for a scattering-dominated plasma, and thus the relations we derive are different from those found by Bogdan and Knölker (1989) and Mihalas and Mihalas (1983), who derived the dispersion relation for a radiation field in LTE, ignoring both scattering and radiation viscosity, which we include. Our problem also differs from that treated by Cassen and Woolum (1996), who considered only optically thick spiral density waves that lose angular momentum through radiation. Our equations are very similar to those of Jedamzik et al. (1998) and Subramanian and Barrow (1997) in the diffusion and free-streaming limits; however, we have bridged the two regimes by truncating the radiation field moment expansion above quadrupole moment. We also note that Thompson and Blaes (1998) have considered radiation damping for waves in the context of gamma ray bursts.

2. Equations

Our aim in this section is to derive a dispersion relation for MHD waves in the presence of a background radiation field. In a sense, this is not a fully self-consistent approach since the linearized equations are only appropriate when the turbulent velocities are small in the fluid frame, yet the dissipation of significant turbulent motions is the source of energy for the radiation. Nonetheless, we believe our approach should lead to a reasonable approximation to the truth. Simulations show that, when the only damping is numerical, the turbulence spectrum declines sharply toward shorter wavelengths. Thus, the short wavelength modes are legitimately in the linear regime, relative to the “equilibrium” background provided by larger amplitude, longer wavelength fluctuations, except that there exist non-linear couplings which cause the cascade of energy to smaller scales. A linear dispersion relation should at least provide a qualitative indication of the major effects.

2.1. Photon Damping

We first begin with a qualitative description of the different regimes of photon damping. When radiation pressure in a fluid is significant compared to gas pressure, momentum and energy can be transported by radiation in such a way as to damp out perturbations in the fluid. There are two relevant length scales: \( k^{-1}_T = 1/n_e \sigma_T \) (\( n_e \) is the electron number density and \( \sigma_T \) is the Thomson scattering cross-section), the photon mean free path, and \( k^{-1}_D \simeq k^{-1}_T c/c_s \), the diffusion length (\( c_s \) is the phase speed of long-wavelength acoustic perturbations). These two wavelengths define three characteristic regimes:

1) Optically thin regime: When the wavenumber \( k > 2\pi k_T \), photons can travel freely across a wavelength. The Doppler shift due to fluid motion creates a flux in the fluid rest frame that acts as a headwind for the electrons. As we will show later, this effect leads to a damping rate that is independent of \( k \).
2) Non-diffusive regime: This is the range of wavenumbers $k_D < k < 2\pi k_T$. In this regime, although a single wavelength is optically thick, photons can diffuse out of a fluctuation in a single wave period. This effect will prove especially important to compressive waves.

3) Optically thick diffusive regime: When $k < k_D$, photons are effectively dragged along with the fluid oscillations. Their diffusion can be described well by conventional transport coefficients (Weinberg 1972). If one thinks of the system as a single fluid, these correspond to shear viscosity and (a version) of heat conduction. Mihalas and Mihalas (1984), and references therein, have derived these coefficients in the diffusion approximation.

With these wavelength distinctions in mind, we now derive the exact dispersion relations for MHD modes damped by radiation transport.

### 2.2. Radiation transfer equation

Mihalas and Mihalas (1984) derived the equations of radiation viscosity in the limit of time-steady, diffusive behavior. They additionally assumed pure absorptive opacity and LTE, in contrast to our assumption of pure isotropic scattering; however, this does not affect radiation viscosity. Our case involves time-dependent behavior and gradients that may be so sharp as to completely invalidate the diffusion approximation. Consequently, we must rederive the equations of radiation viscosity in a way that is appropriate for our regimes of interest.

We write down the radiation transfer equation in a quasi-inertial “lab” frame which travels along with the local mean orbital velocity. We neglect rotation because we will be interested only in fluctuation wavelengths very short compared to a radius (in fact, for some purposes to make rotation negligible requires a stronger constraint to wavelengths very short compared to a disk thickness). We also neglect the thermal source function, absorption opacity, and stimulated scattering. The source function is then solely due to electron scattering. Evaluated in the lab frame and averaged over frequency, it is (Pomraning 1973, equation 6.1):

$$
S_T(n_f) = \frac{1}{\sigma_T} \int d\nu_f d\nu_i d\Omega_i \frac{\nu_f}{\nu_i} \frac{d\sigma_T}{d\Omega_i} (\nu_i \rightarrow \nu_f, n_i \rightarrow n_f) I(n_i, \nu_i)
$$

(2)

where $I(n, \nu)$ is the specific intensity in the direction $n$ at frequency $\nu$, $i$ and $f$ subscripts indicate the initial and final photon respectively, $\sigma_T$ is the Thomson cross section, and $n$ is the direction of photon motion in the lab frame. If the fluid moves with velocity $\beta$ (in units of $c$) relative to the lab frame, we have the following relations, correct to first order in $\beta$:

$$
\frac{\nu_f}{\nu_i} = 1 - \beta \cdot (n_i - n_f),
$$

(3)

$$
\frac{d\sigma_T}{d\Omega_i} (\nu_i \rightarrow \nu_f, n_i \rightarrow n_f) = [1 + \beta \cdot (n_i - n_f)] \delta [\nu_f (1 - \beta \cdot n_f) - \nu_i (1 - \beta \cdot n_i)] \frac{\sigma_T}{4\pi}.
$$

(4)
where the first relation is the familiar frequency shift due to Compton scattering, in which we have neglected terms of order $h\nu/m_e c^2$; the second is the transformation between frames for the Thomson scattering cross section, where we have made the approximations of isotropic scattering and negligible electron recoil.

The first four moments of the frequency integrated specific intensity are:

$$J = \frac{1}{4\pi} \int d\Omega I(n)$$

$$H = \frac{1}{4\pi} \int d\Omega n I(n)$$

$$K_{ij} = \frac{1}{4\pi} \int d\Omega n_i n_j I(n)$$

$$L_{ijk} = \frac{1}{4\pi} \int d\Omega n_i n_j n_k I(n),$$

where $d\Omega$ is $\sin \theta d\theta d\phi$, $n$ is the unit vector pointing in the $(\theta, \phi)$ direction, and $I(n)$ is the frequency integrated specific intensity.

Integrating the source function (2) over solid angle and frequency and keeping only terms of order $\beta$, we get:

$$S_T(n) = (1 + 3\beta \cdot n)J - 2\beta \cdot H.$$  

(9)

The full frequency-integrated radiation transfer equation in the lab frame, including only terms first order in $\beta$ and neglecting emissivity and absorption is then

$$\frac{1}{ckT} \frac{\partial I(n)}{\partial t} + \frac{n}{kT} \cdot \nabla I(n) = (1 + 3\beta \cdot n)J - 2\beta \cdot H - (1 - n \cdot \beta)I(n),$$  

(10)

The last term on the RHS of this equation is due to electron scattering opacity, boosted from the fluid frame to the lab frame. This equation agrees with Psaltis and Lamb (1997), except that we have dropped terms second order in $\beta$ and have ignored the temperature of the electrons. Taking the first moment of this equation $(1/4\pi \int d\Omega)$, we get:

$$\frac{1}{ckT} \frac{\partial J}{\partial t} + \frac{n}{kT} \cdot \nabla \cdot H = -\beta \cdot H.$$  

(11)

Next, taking the second moment $(1/4\pi \int d\Omega n)$ gives:

$$\frac{1}{ckT} \frac{\partial H_i}{\partial t} + \frac{n}{kT} \cdot \nabla_i K_{ji} = \beta_i J + \beta_j K_{ji} - H_i.$$  

(12)

Finally, taking the third moment of the transfer equation, we find:

$$\frac{1}{ckT} \frac{\partial K_{ij}}{\partial t} + \frac{n}{kT} \cdot \nabla_k L_{ijk} = \delta_{ij} \frac{J}{3} - \frac{2}{3} \delta_{ij} \beta \cdot H - K_{ij} + \beta_k L_{ijk}.$$  

(13)

Now, to close these equations, we must make some assumption about the form of the radiation field. The Eddington approximation is equivalent to setting the quadrupole and higher moments
to zero. However, we want to consider the effect of radiation viscosity, which is only present if there is shear, and this requires a term of quadrupole order or higher in the radiation field. We therefore set all higher moments to zero, but retain the monopole, dipole, and quadrupole moments:

\[ I(n) = I_1 + I_2 n \cdot n_D + \frac{I_4}{2} [3(n \cdot n_Q)^2 - 1], \]  

(14)

where \( n_D \) is the direction of the dipole moment and \( n_Q \) is the direction of the quadrupole moment, and \( I_{1,2,4} \) are independent of \( n \). Using this multipole expansion for the intensity, the fourth moment of the radiation field (equation 8) can be expressed in terms of the flux:

\[ L_{ijk} = \frac{1}{5} \left[ \delta_{ij} H_k + \delta_{ik} H_j + \delta_{jk} H_i \right], \]  

(15)

using the relation \( \int (d\Omega / 4\pi) n_i n_j n_k n_l = (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) / 15 \). This result allows us to express the third moment of the transfer equation (13) as:

\[ \frac{1}{c k T} \frac{\partial K_{ij}}{\partial t} + \frac{1}{5 k T} [\delta_{ij} \nabla \cdot H + \nabla_i H_j + \nabla_j H_i] = \frac{\delta_{ij}}{3} (J - 2 \beta \cdot H) - K_{ij} + \frac{1}{5} [\delta_{ij} \beta \cdot H + \beta_i H_j + \beta_j H_i], \]  

(16)

Finally, we need to calculate the effect of the photons on the electrons. The rate of momentum transfer from the photons to the electrons via Compton scattering is:

\[ C_i = \int d\nu d\Omega' d\Omega d^3 \beta' \Delta p e \frac{\sigma_T}{\hbar \nu} f(\beta')(1 - \beta' \cdot n') \frac{I_\nu(\theta', \phi')}{\hbar \nu}, \]  

(17)

where primes denote the particles before scattering in the lab frame, \( \beta' \) is the electron velocity, \( f(\beta) \) is the electron distribution function, and \( \Delta p \) is the momentum transferred during the scattering. We make the assumption that all electrons move with the fluid velocity \( \beta \), i.e. \( f(\beta') = \delta^3(\beta' - \beta) \). In the limit of non-relativistic electron speeds, the momentum equation would be unchanged if we had instead averaged over a finite-width velocity distribution. We again assume the scattering is isotropic and we ignore terms of order \( \hbar \nu / m_e c^2 \) and higher. Performing the integral in equation (17) and keeping only terms of order \( \beta \), we find

\[ C_i = -\frac{4 \pi k T}{c} [\beta_i J + \beta_j K_{ji} - H_i], \]  

(18)

which is proportional to the RHS of equation (12). We can then use this electron-photon momentum transfer rate in the fluid momentum equation. Ignoring all other forces, the fluid momentum equation is:

\[ \rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{C}, \]  

(19)

where \( \mathbf{v} = c \beta \). In most cases, \( \mathbf{C} \cdot \beta \) is negative, and thus there is generally a drag on the fluid due to collisions with photons.

We assume that in the equilibrium state the radiation is uniform, time independent, and isotropic so that \( J = I \). The higher order moments of the unperturbed radiation field are then
simply \( H_i = 0, K_{ij} = \delta_{ij} J / 3 \), and \( L_{ijk} = 0 \). We also assume that the unperturbed fluid is at rest in the lab frame. These assumptions greatly simplify the equations, and retain most of the physics of the waves. We assume that all perturbations vary with space-time dependence \( e^{i(k \cdot x - \omega t)} \), e.g. the perturbed mean intensity is \( J + \delta J e^{i(k \cdot x - \omega t)} \); for this reason we must also restrict attention to modes with \( kh \gg 1 \). The perturbed radiation transfer equations are:

\[
\delta J = \frac{c}{\omega} k \cdot \delta \mathbf{H} \tag{20}
\]

\[
- \frac{i \omega}{k T c} \delta H_i = - \frac{ik_j \delta K_{ij}}{k T} + \frac{4}{3} \delta \beta_i J - \delta H_i \tag{21}
\]

\[
- \frac{i \omega}{k T c} \delta K_{ij} = - \frac{i}{5k T} [\delta_{ij} k \cdot \delta \mathbf{H} + k_i \delta H_j + k_j \delta H_i] + \frac{\delta_{ij} J}{3} \delta J - \delta K_{ij}. \tag{22}
\]

Solving these equations for \( \delta J \) in terms of \( \delta \beta \) yields:

\[
\delta J = \frac{4}{3} J (1 - i \varpi) \left[ \varpi (1 - i \varpi)^2 + \frac{i}{3} k^2 \left( 1 - \frac{9}{5} i \varpi \right) \right]^{-1} \frac{k}{2} \delta \beta. \tag{23}
\]

where we have defined normalized variables \( \overline{k} \equiv k / k T \), and \( \varpi \equiv \omega / k T c \), so that the optically thin regime corresponds to \( |\overline{k}| > 2\pi \). The perturbed mean intensity disappears for modes with \( k \perp \delta \beta \) since there is no compression of the radiation field.

Next, the perturbed flux is:

\[
\delta \mathbf{H} = \frac{4}{3} J (1 - i \varpi) \left[ \varpi (1 - i \varpi)^2 + \frac{i}{5} k^2 \left( 1 - \frac{9}{5} i \varpi \right) \right]^{-1} \frac{k}{2} \delta \beta. \tag{24}
\]

The perturbed collision integral is:

\[
\delta C = - \frac{4\pi k T}{c} \left[ \frac{4}{3} \delta \beta J - \delta \mathbf{H} \right] \tag{25}
\]

For incompressive waves, in the limit that \( \varpi \ll k^2 \ll 1 \) (low frequency, optically thick limit), the momentum transfer rate is

\[
\delta C = - \frac{4}{5} k T P_{\text{rad}} k^2 \delta \beta. \tag{26}
\]

Thus, the photon-fluid friction is proportional to \( -k^2 \delta \beta \), which looks like a \( \nabla^2 v \) term, with a constant of proportionality \( 4P_{\text{rad}} k T / 5c \). This is the same as the radiation viscosity term derived by multiple authors, e.g. Mihalas and Mihalas (1984). Indeed, the photon viscosity computed by Loeb and Laor (1992) for a steady shear flow gives exactly this viscosity term, except with a factor of 10/9 from considering the exact differential Thomson cross section (rather than assuming isotropic scattering as we did above). Including polarization introduces another small correction factor (Hu and Sugiyama 1996). In the large \( k/k T \) (optically thin) limit, the friction approaches a constant, which is what one would expect for electrons in a uniform radiation field; since the wavelength is much shorter than a photon mean free path, each electron sees the averaged radiation from several wavelengths. The friction changes when \( \omega \neq 0 \) to take into account time-dependent diffusion of the radiation as the wave oscillates.
2.3. MHD equations

With the perturbed radiation quantities in hand, we now write down the perturbed MHD equations of motion. We define the $z$ axis as the direction of the magnetic field, and ignore both gravitational potential gradients and rotation, in keeping with our restriction to modes with $kh \gg 1$. Ignoring rotation is valid for $\omega \gg \Omega$, or $kh \gg c_s/v_A$. By omitting rotation effects, we restrict our attention to wavenumbers short enough that the Balbus-Hawley instability does not operate. The effects of vertical gravity on radiation waves in accretion disks has been considered by Gammie (1998), who found unstable “photon bubble” modes. Our equations ignore these modes; however, we do include radiation viscosity, which Gammie (1998) ignored. The MHD equations are

\[
\frac{\rho \partial \mathbf{v}}{\partial t} + \nabla \left( P_{\text{gas}} + \frac{B^2}{8\pi} \right) - \frac{1}{4\pi} (\mathbf{B} \cdot \nabla)\mathbf{B} - \mathbf{C} = 0
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0
\]  \hspace{1cm} (27)

where $P_{\text{gas}} = Nk_BT$ and $k_B$ is Boltzmann’s constant.

Assuming an equilibrium state with $\rho$ and $\mathbf{B}$ constant, and $\mathbf{v} = 0$, the perturbed equations are

\[-i\omega \rho \delta\mathbf{v} + ik \delta P_{\text{gas}} + \frac{i}{4\pi} \left| \mathbf{k}(\mathbf{B} \cdot \delta\mathbf{B}) - (\mathbf{B} \cdot \mathbf{k})\delta\mathbf{B} \right| - \delta \mathbf{C} = 0
\]

\[\frac{\delta \rho}{\rho} = \frac{\mathbf{k} \cdot \delta\mathbf{v}}{\omega}
\]

\[\omega \delta\mathbf{B} = \mathbf{B}(\mathbf{k} \cdot \delta\mathbf{v}) - (\mathbf{B} \cdot \mathbf{k})\delta\mathbf{v}
\]  \hspace{1cm} (28)

We will assume that $\delta P_{\text{gas}}/\delta \rho = c_g^2$ is constant in what follows. Combining these equations gives:

\[\varpi^2 \delta\mathbf{v} - \left( \frac{c_g}{c} \right)^2 (\mathbf{k} \cdot \delta\mathbf{v})\mathbf{k} - \left( \frac{v_A}{c} \right)^2 \left[ (\tilde{k} \cdot \delta\mathbf{v})\tilde{k} + \tilde{k}^2 \delta\mathbf{v} - \tilde{k} \cdot \delta\mathbf{v} \tilde{k} - \delta\mathbf{v} \tilde{k} \cdot \mathbf{k} \right] - \frac{i}{\varpi} \frac{v_A}{\rho k_T c} \delta \mathbf{C} = 0
\]  \hspace{1cm} (29)

where $v_A = \sqrt{B^2/4\pi \rho}$, the Alfvén speed.

Now, define $\mathbf{k} = k(\sin \theta \mathbf{\hat{x}} + \cos \theta \mathbf{\hat{z}})$. Setting the determinant of equations (29) to zero, and using equations (24) and (25), we find the following dispersion relation:

\[A_1 \left[ (A_1 + A_3 \mathbf{\hat{k}} \cos \theta)^2 + (A_1 A_2 - A_3^2) \mathbf{\hat{k}}^2 \right] = 0
\]  \hspace{1cm} (30)

where we have defined the auxiliary quantities:

\[A_1 = \left( \varpi^2 - \frac{v_A^2}{c^2} \mathbf{\hat{k}}^2 \cos^2 \theta + i \varpi \mathbf{\hat{k}} \right) D_1 - i \varpi \mathbf{\hat{k}} \left( 1 - i \varpi \right)
\]

\[A_2 = -\left( \frac{c_g^2}{c^2} + \frac{v_A^2}{c^2} \right) D_1 - \varpi \mathbf{\hat{k}} \left( 5 - 6i \varpi \right)(1 - i \varpi)/D_2
\]

\[A_3 = \frac{v_A^2}{c^2} \mathbf{\hat{k}} \cos \theta D_1
\]
\[ D_1 = (1 - i \omega)^2 + \frac{1}{5} \kappa^2 \]
\[ D_2 = 15 \omega (1 - i \omega)^2 + i \kappa^2 (5 - 9i \omega) \]
\[ \nu_A = v_A/c \]
\[ c_g = c_g/c \]
\[ \Gamma = \Gamma/kpc = 4P_{rad}/\rho c^2 = 3c_A^2/c^2 \]

(31)

In the limit \( P_{rad} = 0 \), this dispersion relation becomes the MHD dispersion relation for Alfvén modes and magnetosonic modes (e.g. Jackson 1975). For nonzero \( P_{rad} \), this dispersion relation admits a variety of modes: modified versions of Alfvén modes (incompressive); fast and slow magnetosonic modes (compressive); and radiative (electromagnetic) modes. On account of the complexity of the full dispersion relation, we will discuss some simplified limits. The dispersion relation for Alfvén-like modes factors out separately, \( A_1 = 0 \). We will discuss this branch in the next subsection. When \( \theta = 0 \) or \( \theta = \pi/2 \), the part of the dispersion relation in brackets simplifies considerably; we will look at the compressive modes for these propagation directions in §2.5.

2.4. Dispersion relation for incompressible waves

The case of \( A_1 = 0 \) yields the modified Alfvén modes, for which \( k \cdot \delta v = 0 \) and \( \delta v_z = 0 \). Since these modes are incompressible, \( \delta J = 0 \) and \( \delta H \) simplifies drastically. The equation \( A_1 = 0 \) becomes:

\[ \omega^2 - k^2 v_A^2 \cos^2 \theta + i\Delta_\omega = 0. \]

(32)

where

\[ \Delta_i \equiv \Gamma - \frac{1}{5} \kappa^2 - i \omega - k^2 \]

(33)

If \( v_A \ll c \), then \( |\omega| \ll \kappa \) (since \( |\omega| \approx kv_A \)) and we can expand:

\[ \Delta_i = \Gamma \left[ \frac{k^2}{5 + k^2} - 5i \omega \left( \frac{5 - k^2}{(5 + k^2)^2} \right) \right] + \mathcal{O} \left( \frac{\omega}{k} \right)^2. \]

(34)

Substituting this expression into equation (32), we can solve for \( \omega \):

\[ \omega = \pm \sqrt{k^2 v_A^2 \cos^2 \theta A_k - \Gamma^2 \left( \frac{k^2}{5 + k^2} \right)^2 \left( \frac{k^2}{5 + k^2} \right)^2 \left( \frac{\Gamma}{5 + k^2} \right) A_k - \frac{1}{2} \frac{\Gamma k^2 A_k}{5 + k^2}}. \]

(35)

where \( A_k = (1 + 5 \Gamma (5 - k^2)/(5 + k^2)^2)^{-1} \). This dispersion relation is valid for any \( \Gamma \) and for \( v_A \ll c \). For \( \Gamma \ll v_A \), it simply describes damped Alfvén waves, and it agrees with Jedamzik et al. (1998) in the diffusion, optically thin, and overdamped limits. Note that the damping rate we have computed bridges the optically thin and thick limits, which have been examined separately by other authors (Subramanian and Barrow 1997, Jedamzik et al. 1998).
We also find higher frequency modes with $|\omega| \gg \Gamma$, illustrated in figure 1. These modes are non-propagating in the optically-thick limit and damped on a timescale $\sim k T c^{-1}$, but travel at $c/\sqrt{5}$ when their wavelengths are shorter than a photon scattering length. Their dispersion relation is to first order the solution of $D_1 = 0$, which means that $\delta \beta \to 0$, so that these modes are simply electromagnetic. The speed of these modes is likely an artifact of the particular form taken for the moment hierarchy closure; their speed increases when we include moments higher than quadrupole.

### 2.5. Dispersion relation for compressible waves

Next, we consider the opposite limit of the dispersion relation: strongly compressible waves for which $k || \delta \beta$. This condition permits two sorts of waves: $\theta = 0 \ (k || B)$ which is simply a radiation damped sound wave; and $\theta = \pi/2 \ (k \perp B)$, the fast magnetosonic wave. The slow magnetosonic wave disappears when we impose $k || \delta \beta$ since it has a velocity component perpendicular to the wave vector. The damping rate of the slow magnetosonic wave is intermediate between the Alfvén wave and the fast magnetosonic wave, which is why we do not treat it here. The fast wave also has a small $k \perp \delta v$ component when it propagates at an angle $0 < \theta < \pi/2$; we ignore these waves here since they will have damping rates in between the $\theta = 0$ and $\theta = \pi/2$ cases. The dispersion relation for the $\theta = \pi/2$ case is:

$$\omega^2 - k^2 (c_g^2 + v_A^2) + i \Delta_\omega \omega = 0$$

where we have defined

$$\Delta_\omega = \Gamma \left[ 1 - \frac{1}{D_1} \left( 1 - \frac{i k^2}{D_2} (5 - 6 i \bar{\omega}) \right) \right]$$

In the diffusive limit, $k \ll k_D$, the fast magnetosonic dispersion relation becomes:

$$\omega^2 \left( 1 + 3 \frac{c_s^2}{c_f^2} \right) - k^2 (c_g^2 + v_A^2 + c_s^2) + i \Gamma \omega \left( - \frac{2}{5} k^2 + \frac{k^4}{9 \bar{\omega}^2} + \bar{\omega}^2 \right) + \mathcal{O} \left( \frac{1}{k T} \right)^3 = 0.$$  \hspace{1cm} (38)

where $c_s^2 = 4 P_{rad}/3 \rho$ is the sound speed due to radiation, and we have assumed that all velocities are smaller than $c$. Since $k \ll k_D$, $k$ is very much smaller than $k T$, so it makes sense to expand in terms of $k/k T$. In terms of this expansion, the zeroth order solution to equation (38) is

$$\omega_0 = \sqrt{\frac{c_g^2 + v_A^2 + c_s^2}{1 + \frac{3 c_s^2}{c_f^2}}} k.$$  \hspace{1cm} (39)

We now substitute this approximate solution into the next higher order term in equation (38) and obtain a quadratic equation for $\omega$ with the solution:

$$\omega = \pm \omega_0 - i \frac{c k^2}{6 k T (1 + R)^2 (1 + c_g^2/v_A^2 R)} \left[ R^2 + \frac{4}{5} (1 + R) + 6 \left( \frac{4}{5} - \frac{R}{5} \right) \frac{(c_g^2 + v_A^2)}{c^2} R + \frac{9(c_g^2 + v_A^2)^2 R^2}{c^4} \right],$$  \hspace{1cm} (40)
Fig. 1.— Dispersion relation for $k \perp \delta\theta$ waves with $v_A = c_r = 0.3c$ and $\cos \theta = 1$. The solid lines are Alfvén modes, while the dotted lines are the electromagnetic modes. The dashed line is the damping rate from equation (35).
where $R = 1 / \Gamma = \frac{\rho c^2}{4P_{\text{rad}}} = \frac{c^2}{6c^2}$, and we expect $R \gg 1$ in all applications of interest here. In the limit $c_g = v_A = 0$, this equation has the same form as equation (52) in Peebles and Yu (1970). In the limit that all velocities are smaller than $c$, this equation becomes:

$$\omega = \pm k \sqrt{c_g^2 + v_A^2 + c_r^2} - i \frac{ck^2c_r^2}{2kT(3c_r^2 + c_g^2 + v_A^2)}.$$  \hspace{1cm} (41)

Thus, in the diffusive limit, the damping rate has the usual $\propto k^2$ dependence.

In the optically thin limit ($k \gg k_T$), the dispersion relation becomes:

$$\omega = \pm k \sqrt{c_g^2 + v_A^2} - \frac{i\Gamma}{2},$$  \hspace{1cm} (42)

giving the same damping rate as for incompressible waves in the optically thin limit. Compressible waves are damped at the optically thin rate when the time for photons to diffuse across a wavelength is comparable to or shorter than the wave oscillation period. We do not have an analytic formula for the dispersion relation for $k_D \sim < k \sim < k_T$, so the full dispersion relation must be solved numerically, as shown in figure 2. A more accurate formula for $k_D$ can be found by equating the diffusive and non-diffusive damping rates:

$$k_D = \sqrt{3(3c_r^2 + c_g^2 + v_A^2)kT/c},$$  \hspace{1cm} (43)

which is where the compressive damping rate reaches its near-maximum.

When $\rho c^2 \gg P_{\text{rad}} \gg P_{\text{gas}}$ (i.e. $c_r \ll c$, $c_g \ll c_r$) and $v_A \ll c_r$ the damping rate for magnetosonic waves at small $k$ is $ck^2/6kT$. The damping rate for Alfvén modes, however, for small $k$ is $\sim \Gamma k^2/10 = 3(c_r/c)^2ck^2/(10kT)$; i.e. it is smaller by a factor $\sim (c_r/c)^2$. Thus, magnetosonic modes are damped much more strongly than Alfvén modes in the small $k$, small $\Gamma$ limit. The reason is that compressional waves continually compress the radiation field, which diffuses out of the wave, causing the wave to lose its pressure support, and thus damping it out. Since there is no compression in the Alfvén modes, photons only diffuse out of the wave perpendicular to $\delta v$, creating a quadrupole moment in the radiation field which leads to viscous damping, and that is much weaker than diffusive damping. A comparison of the dispersion relation for Alfvén modes and magnetosonic modes is shown in figure 2 with $v_A = c_r = 0.1c$, $c_g = 0$ ($\Gamma = 0.03$). In the optically thin limit, both waves are damped at a rate $\Gamma/2$, since the radiation is isotropic and uniform and thus the damping is just due to the dipole moment of Doppler shifted photons in the fluid frame. The phase speed of magnetosonic waves with these parameters is somewhat greater in the diffusion ($k < k_D$) limit than at larger $k_T$ because the radiation adds to the restoring force. As a corollary, the waves are mildly dispersive for $k \sim k_D$. Also shown are the analytic approximations to the damping rates, equations (35) and (40).

More complicated behavior can occur when $v_A \ll c_r$ because the removal of radiation pressure support when $k$ exceeds $k_D$ is a relatively more important effect. In figure 3, we show the compressive dispersion relation (equation 36) for $v_A = 0.001c$, $c_r = 0.01c$, and $c_g = 0$. There are three separate cases to consider for the fast magnetosonic modes:
Fig. 2.— Plot of the real and imaginary parts for Alfvén (solid lines) and magnetosonic (dotted lines) with $v_A = c_r = 0.1c$, $c_g = 0$. The dot-dash line in the top panel is equation (40); the dashed line (which overlaps the solid line) is equation (35).
1) In the diffusion limit, the modes are simply sound waves supported by radiation pressure since the diffusion timescale is much greater than the wave period.

2) In that portion of the non-diffusive regime in which $k_D < k < \Gamma/(2v_A)$, the drag due to radiation escaping from the waves is greater than the magnetic restoring force, so the wave becomes overdamped. This is the region with $Re(\omega) = 0$ in figure 3.

3) For $k > \Gamma/(2v_A)$, the radiation isotropizes on a timescale much shorter than the wave period. In this range of wavenumbers, fast magnetosonic waves propagate with phase speed $\sqrt{v_A^2 + c_s^2}$, but damp due to radiation drag.

There are additional overdamped modes which occur when the velocity perturbation is so small that the fluid reaches a terminal velocity and thus the mode damps out before it can oscillate (Subramanian and Barrow 1997). In the diffusive limit when $Re(\omega) = 0$, the radiation always has time to isotropize, so the photon damping rate is the same as in the optically thin case. The magnetic restoring term in equation (29) with $k_z = 0$ and $k||\delta \beta$ balances the optically thin radiation collision term when

$$\omega = -\frac{k^2 c}{3kT} \frac{v_A^2}{c^2 + v_A^2}.$$ (44)

This agrees exactly with the damping rate for the lower dotted curve in the upper panel of figure 3. For $k \sim 0.1k_T$, the upper dotted curve is the solution of $D_2 \sim 0$ (for very large $\overline{\kappa}$, $\omega \sim -i5k_Tc/9$), which means $\delta \beta \to 0$, so that this mode becomes electromagnetic (but non-propagating) in the optically thin limit.

We also find propagating electromagnetic modes (not plotted) whose velocities approach $c/\sqrt{5}$ and $c\sqrt{3/5}$ in the optically thin limit. These speeds are again artifacts of our closure relation.

2.6. The nature of the transport

Why is it that photon transport is so much more effective in damping compressive waves than incompressible ones? One way to understand the contrast is to take a closer look at equations (24) and (25). In the incompressible case ($\mathbf{k} \cdot \delta \beta = 0$), $\delta \mathbf{H} \simeq (4/3)J\delta \beta$ when $\overline{\sigma}$ and $\overline{\kappa}$ are both $\ll 1$. That is, the perturbed flux is simply $4/3$ times the mean intensity shifted by $\beta$. However, the force felt by the electrons is the difference between $\delta \mathbf{H}$ and $(4/3)J\delta \beta$, so the two very nearly cancel. The remainder after this near-cancellation is the retarding force due to the photon shear viscosity, and has magnitude $\sim (\overline{\kappa}^2 + \overline{\kappa^2})J$. However, the relative importance of the photon drag is characterized by the ratio $\overline{\Gamma}$, the ratio of the photon inertia to the fluid inertia, and this is $\ll 1$.

On the other hand, in the compressible case, there is an additional contribution to $\delta \mathbf{H}$ (and therefore $\delta \mathbf{C}$) that is, in the very low frequency limit, $\sim (4/3)J\delta \beta$, much greater than the contribution of shear viscosity when $\overline{\sigma}$ and $\overline{\kappa}$ are $\ll 1$. This new contribution is the result of photon diffusion. In a single-fluid picture this can be thought of as a sort of thermal conductivity
Fig. 3.— Plot of the real and imaginary parts for magnetosonic waves with $v_A = 0.001c$, $c_r = 0.01c$, $c_g = 0$. The dotted lines are overdamped waves, while the solid lines are magnetosonic waves.
(Weinberg 1972), but with respect to a peculiar equation of state (cf. equation 23). It damps the waves much faster than shear viscosity because the diffusing particles in this case are exactly those responsible for the wave’s restoring force.

In principle, photon-electron scattering could also lead to a magnetic diffusivity by creating a new source of electrical resistivity. In the optically thin limit, the resistivity is $\eta_r \simeq 1.4 \times 10^{-19} (c_r/c)^2$ s. In most parameter regimes this photon resistivity is small compared to the resistivity due to ordinary electron-ion Coulomb scattering, $\eta_p \simeq 1.4 \times 10^{-7} T^{-3/2}$ s; however, in the corona, where $T_c \sim 10^9 K$ and $c_r \sim 0.1 c$, photon resistivity may compete with Coulomb resistivity. Due to flux-freezing, the damping of the turbulent motions also damps the magnetic field fluctuations; however, due to the small resistivity, magnetic flux is still conserved.

3. Applications to conventional accretion disks

3.1. Context

In the preceding section we characterized the effects of photon diffusion and viscosity in terms of the rate at which they cause damping of linear MHD waves. In this section we will evaluate how effective these processes may be in dissipating fluctuations in accretion disks. To gain a sense of scale, we begin this section by estimating the corresponding damping rate for several other proposed dissipation mechanisms. Although the natural unit of time for the dispersion relation was the photon scattering time, in the context of disks the natural unit is (the inverse of) the orbital frequency $\Omega$, so we will quote all rates in that unit. As a further set of reference rates, in this subsection we will also establish the relevant standards of comparison for several different questions of interest.

Ordinary molecular viscosity (due to ion-ion collisions) creates a damping rate

$$\Gamma_{mol} = \frac{1}{3} \left( kh \right)^2 \frac{\sigma_T c_g}{\sigma_{coll} c_s} \Omega, \quad (45)$$

where $h$ is the (half) disk thickness and $\sigma_{coll}$ is the collision cross section. If $\sigma_{coll}$ is the Coulomb cross section, for example, $\sigma_T/\sigma_{coll} \sim (k_B T/m_e c^2)^2/\ln \Lambda$, where $\ln \Lambda$ is the usual Coulomb logarithm, $\simeq 30$. $\Gamma_{mol}/\Omega$ is generally a very small number. Ordinary viscosity is rendered even less effective because the magnetic field suppresses transport perpendicular to the field.

Transit-time damping (and the associated Landau damping) has been suggested by Quataert (1998) as the dissipational mechanism in accretion disks, particularly when the ion temperature is much greater than the electron temperature as in advection dominated accretion flows. As a fiducial point, we quote its rate (as calculated by Quataert 1998) for a single-temperature plasma:

$$\Gamma_{td} \simeq 0.2 \cos \theta \sin^{5/2} \theta \left( kh \right)^{5/2} c_A \left( \frac{c_p}{c_s} \right)^{5/2} \left( \frac{\Omega}{\Omega_i} \right)^{5/4} \Omega, \quad (46)$$
where $\Omega_i$ is the ion Larmor frequency. When $k$ is greater than an inverse ion Larmor radius, $\Gamma_{td} \propto (kh)^{1/2}$. Unless the magnetic field is exceedingly strong, $\Omega/\Omega_i \ll 1$, so that $\Gamma_{td} \ll \Omega$.

Depending on the question being asked, any candidate damping rate should be compared to one of three fiducial rates: the growth rate (absent dissipation) of the MHD waves (as, for example, due to magneto-rotational instability as in Balbus and Hawley 1991); the inverse time for waves to cross a disk scale-height; and the “nonlinear frequency” or inverse “eddy turnover time,” the rate at which energy moves between modes due to nonlinear coupling.

If the damping rate exceeds the non-dissipative growth rate, the fluctuations are unable to grow at all. This is a strong statement, for the growth rate of the magneto-rotational instability is generally $\sim \Omega$.

When the damping time is short compared to a disk scale-height crossing time, waves cannot carry significant energy from the midplane to the disk surface (see §4.2). This, too, may require very rapid damping, for the wave crossing time can be as short as $\sim \Omega^{-1}$ (for diffusive regime fast magnetosonic modes). The time for other modes to traverse a disk thickness is somewhat slower: $\sim (\Omega v_A/c_s)^{-1}$ for pure Alfvén modes, $\sim (\Omega \sqrt{v_A^2 + c_s^2/c_s})^{-1}$ for fast magnetosonic modes with $k > k_D$ and $\omega \neq 0$.

Thirdly, as emphasized by Gruzinov (1998), exceeding the nonlinear frequency at some wavenumber is the relevant criterion for deciding whether the damping can cut off the “inertial range” of turbulence at short wavelengths. The nonlinear frequency is defined by $\omega_{nl}(k) \equiv \epsilon/k E_k$ where $E_k$ is the energy density per unit wavenumber in the turbulent spectrum. The rate of energy dissipation per unit volume, $\epsilon$, is determined solely by the accretion rate (equation 1) and disk height, while the total energy in fluctuations may be related to the accretion rate if we know the ratio between the trace of the fluctuations’ stress tensor and its $r - \phi$ component (under the assumption that it is this last quantity which accounts for angular momentum transport in the disk). That is, the total energy density in fluctuations, volume averaged, is

$$\frac{1}{2} \langle Tr(T) \rangle = \int_{k_{\text{min}}}^{k_{\text{max}}} dk E_k = \frac{Tr(T) \dot{M} \Omega}{8\pi h}.$$  \hspace{1cm} (47)

where $T_{ij} = \langle \rho(\delta v_i \delta v_j - \delta v_{A,i} \delta v_{A,j}) \rangle$, and $\langle \rangle$ denotes volume averaging (Balbus et al. 1994).

Suppose the fluctuation spectrum is a power-law $E_k \propto k^{-n}$ from $k_{\text{min}} = \pi/h$ to $k_{\text{max}}$. Then,

$$\omega_{nl} = \frac{3}{(n-1)} \frac{Tr(T)}{Tr(T)} \left[ \left( \frac{kh}{\pi} \right)^{n-1} - \left( \frac{k}{k_{\text{max}}} \right)^{n-1} \right] \Omega.$$  \hspace{1cm} (48)

If $k_{\text{max}} \gg k_{\text{min}}$ and $n > 1$, then

$$\omega_{nl} \approx \frac{3}{(n-1)} \frac{Tr(T)}{Tr(T)} \left( \frac{kh}{\pi} \right)^{n-1} \Omega.$$  \hspace{1cm} (49)

In the simulations of Brandenburg et al. (1995) and Stone et al. (1996), $T_{r\phi}/Tr(T) \sim 0.1$, very roughly.
### 3.2. Radiation-pressure dominated disk

First consider radiation-pressure dominated disks. In this case, the Shakura-Sunyaev solution (in which $T_{r\phi}$ is set equal to $\alpha p$) yields two important results about the equilibrium. The disk aspect ratio is

$$\frac{h}{r} = \frac{3 \dot{m}}{2x},$$

(50)

where $\dot{m}$ is the accretion rate in Eddington units (for unit efficiency) and $x = rc^2/GM$ is the radius in gravitational units. We have ignored all relativistic factors. In addition, the (half) optical depth is

$$\tau = \frac{2c}{\alpha \Omega \dot{m} h} = \frac{4}{3} x^3 \alpha \dot{m},$$

(51)

where, in consonance with the result of simulations, we have ignored any contribution of magnetic pressure to disk vertical support.

We may now use these facts to evaluate the rate of photon damping. In this case,

$$\frac{c_r}{c} = \frac{3}{2} \dot{m} x^{-\frac{3}{2}}.$$  

(52)

Typically $c_r \ll c \left(\Gamma \ll 1\right)$ in thin accretion disks. For $k < k_D$ and radiation pressure larger than magnetic or gas pressures, compressive modes damp at a rate

$$\Gamma_{\text{comp}} = \frac{\alpha}{12} (kh)^2 \Omega.$$  

(53)

Incompressible modes damp more slowly at small $k$:

$$\frac{\Gamma_{\text{inc}}}{\Gamma_d} \simeq \frac{4 \dot{m}^2}{x^3}.$$  

(54)

So only for large $\dot{m}$ and very small radii will the incompressive damping rate equal or exceed the compressive. In the optically thin ($k > 2\pi k_T$) limit, both damping rates are constant and equal to $\Gamma_d^{\text{thin}} = 3\alpha^{-1} \Omega$.

Thus, we immediately see that compressible modes damp extremely rapidly. At the longest wavelengths the damping time is, not surprisingly, the same as the thermal time, $(\alpha \Omega)^{-1}$. They are, in fact, the same process—photon diffusion out of a region $\sim h$ in size. These modes damp so quickly because in this regime photons provide most of the pressure; consequently, it is their diffusion rate, not the ions’, which controls the damping rate.

To see just how rapid the photon damping is, we may compare it to, for example, the rate of transit-time damping. In this context of radiation-dominated accretion disks,

$$\Gamma_{\text{ttd}} \simeq 6.3 \times 10^{-13} \cos \theta \sin^{2/3} \theta (kh)^{5/3} (c_g/c_s)^{2/3} M_8^{-1/2} \Omega,$$

(55)

where $M_8$ is the mass of the central black hole in units of $10^8 M_\odot$. Because $c_g/c_s \ll 1$ when radiation pressure is dominant, $\Gamma_{\text{ttd}}$ is very slow indeed compared to even $\Gamma_d^{\text{inc}}$. 


So long as $\alpha < 1$, the damping rate for compressible modes does not exceed the Balbus-Hawley growth rate. However, the damping rate may well exceed the nonlinear frequency even for the longest wavelength modes, for

$$\frac{\Gamma_{d,\text{comp}}}{\omega_{nl}} = \frac{\pi^2(n-1)}{36} \frac{Tr(T)}{p} \left( \frac{kh}{\pi} \right)^{3-n}. \quad (56)$$

This expression follows from the fact that the Shakura-Sunyaev parameter $\alpha \equiv [Tr\phi/Tr(T)][Tr(T)/p]$, where $p$ is the total pressure. If the spectrum has the Kolmogorov slope ($n = 5/3$), the photon damping rate is greater than the nonlinear frequency for wavenumbers not much greater than $\pi/h$ unless $Tr(T)/p$ very much less than one (in the simulations of Stone et al. 1996 this quantity was $\sim 0.01 – 0.1$).

Even if photon damping does not overcome the fluctuations at longer wavelengths, it is still likely to end the inertial range of turbulence. The maximum photon damping rate (the optically thin limit) is achieved at $k_D h = 6/\alpha$, where the damping rate is $\Gamma_{d,\text{max}}^{\text{comp}} \approx 3\Omega/\alpha$. Comparing this rate to $\omega_{nl}$, we find

$$\frac{\Gamma_{d,\text{max}}^{\text{comp}}}{\omega_{nl}(k_D)} = (n-1) \left( \frac{\pi}{6} \right)^{n-1} \left[ \frac{Tr\phi}{Tr(T)} \right]^{n-3} \left[ \frac{Tr(T)}{p} \right]^{n-2}. \quad (57)$$

So long as $n < 2$, it is almost guaranteed that $\Gamma_{d,\text{comp}} > \omega_{nl}$ at some wavenumber. For instance, for $n = 5/3$, we find that

$$\frac{\Gamma_{d,\text{max}}^{\text{comp}}}{\omega_{nl}(k_D)} = 43 \left[ 10 \frac{Tr\phi}{Tr(T)} \right]^{-\frac{4}{3}} \left[ \frac{100Tr(T)}{p} \right]^{-\frac{1}{3}}, \quad (58)$$

where we have normalized to fiducial values in the ballpark of what is seen in simulations. We also emphasize that this equation is independent of $x$, $\dot{m}$, and mass of the black hole. Thus, radiation damping of compressive modes can be quite strong whenever radiation pressure dominates.

If there is any significant azimuthal field, i.e. $v_{A\phi} \approx c_s$, all long wavelength (i.e. $k$ not too much larger than $\Omega/v_A$) modes are at least partly compressible (Blaes & Balbus 1996). Simulations indicate that $v_A \ll c_s$, so the Balbus-Hawley unstable modes are very nearly incompressible. However, there can still be significant coupling between incompressible and compressible modes. In simulations of nonlinear magnetohydrodynamic turbulence by Stone et al. (1996), the density fluctuations $\langle \delta \rho^2 \rangle^{1/2}/\rho \approx 5 – 8\%$, while the velocity fluctuations $\langle \delta v^2 \rangle^{1/2}/c_s \approx 15\%$. Since $\delta \rho/\rho = k \cdot \delta v/\omega \approx k \cdot \delta v/c_s$, then $k \cdot \delta \hat{v} \approx 0.5$. Thus, the waves in the turbulence spectrum have a rather large compressive component. Also, in these simulations, pressure waves are seen which are not present in the non-turbulent state, indicating that the turbulence does create compressive waves (John Hawley, private communication). Another way to quantify the fraction of compressive turbulence is to take the power spectrum of the vortical and compressive components of the velocity, $\mathbf{v} = \mathbf{v}_{\text{vort}} + \mathbf{v}_{\text{comp}}$ such that $\nabla \cdot \mathbf{v}_{\text{vort}} = 0$ and $\nabla \times \mathbf{v}_{\text{comp}} = 0$. MHD shearing box simulations by Brandenburg et al (1995) show that the power spectrum amplitude of $\mathbf{v}_{\text{comp}}$ is about 10% of that of $\mathbf{v}_{\text{vort}}$ (Brandenburg 1998).
In addition to the canonical solution for radiation-dominated disks, there are also several extensions of this solution to which photon damping is relevant. At high accretion rates, $\dot{m} > 1$, the radiation pressure causes the disk to puff up, creating a “slim disk” (Abramowicz et al. 1988). Slim accretion disks have larger luminosities than thin accretion disks, making the effects of radiation damping much stronger. The standard thin disk equations cannot be applied to slim disks, since some of the radiation is carried radially inward through the disk rather than being radiated locally. However, the slim disk solutions look similar to the Shakura and Sunyaev (1973) solution in the limit of large $\dot{m}$ (Szuszkiewicz et al. 1996), so we expect our criterion for radiation dissipation to apply, under the assumption that slim disks can be approximately described by the thin disk equations.

Radiation pressure-dominated disks in which $T_{r\phi} = \alpha P_{tot}$ are viscously and thermally unstable (Shakura and Sunyaev 1976, Lightman and Eardley 1974). To cure this, some have suggested that the viscosity is proportional to the gas pressure rather than the total pressure, i.e. $T_{r\phi} = \alpha P_{gas}$. Indeed, as we will discuss in §5, photon diffusion may decouple the radiation pressure from the MHD fluctuations, leading to just this sort of result. If so, such disks effectively have a much smaller $\alpha$, and consequently radiation viscosity is much more efficient at damping perturbations since 1) the turbulent velocities are much smaller, reducing the nonlinear frequency and 2) the optical depth of the disk is much larger, so the radiation pressure at disk center is larger. Since the disk is still supported by radiation pressure, the height is the same as for $\alpha P_{tot}$ disks, and thus $\Gamma$ also remains the same. However, the non-linear frequency is reduced by a factor of $P_{g}/P_{tot}$, which is given by

$$\frac{P_{g}}{P_{tot}} = \frac{32 x^{3/2}}{27 \alpha \tau \dot{m}}. \quad (59)$$

In these disks, even the incompressible damping rate can beat the nonlinear frequency at $k \sim k_T$, the wavenumber at which $\Gamma_{d}^{inc}$ reaches its maximum value. For example, for $n = 5/3$ and $Tr(T)/P_{tot} = 10$,

$$\frac{\Gamma_{d}^{thin}}{\omega_{nl}(k_T)} = 9 \tau^{5/3} \dot{m}^2 x^{-3}. \quad (60)$$

The optical depth in these disks is given by

$$\tau = 5 \times 10^5 \alpha^{-\frac{4}{9}} \dot{m}^{\frac{5}{9}} M_5^{\frac{1}{3}} x^{-\frac{2}{3}}. \quad (61)$$

Using this expression for $\tau$, we see that the incompressible damping rate beats the nonlinear frequency out to a radius of

$$x < 178 \dot{m}^{1/3} M_5^{4/37} \alpha^{-4/7}. \quad (62)$$

Since the compressive damping rate is always greater than the incompressible, radiation damping will be important for a large range of radii if the viscous stress scales with gas pressure rather than radiation pressure.

Our discussion of accretion disks so far has neglected the vertical stratification of density and radiation pressure since we have been using values computed from a one-zone model. Because the
radiation damping rate is proportional to the radiation pressure, we would expect the damping to be relatively more important in the interior of the disk, so that more dissipation occurs deep inside the disk (provided the nonlinear frequency is independent of height).

3.3. Gas pressure-dominated disks

The damping criterion we have discussed only applies to the $P_{\text{rad}} \gg P_{\text{gas}}$ regions of the accretion disk. When $P_{\text{gas}}$ or $P_{\text{mag}} \gg P_{\text{rad}}$, the damping rate becomes $(ck^2/2kT)c_r^2/(3c_r^2 + c_g^2 + v_A^2)$ (cf equation 40) in the diffusive regime, so radiation damping will not compete with the nonlinear frequency. Thus, we expect that the radiation damping will only be important in the radiation-pressure dominated part of an accretion disk. The radius at which radiation pressure equals gas pressure is given by:

$$x_{\text{trans}} = 188(\alpha M_8)^{\frac{8}{21}} \dot{m}^{\frac{1}{16}} (1 - f)^{\frac{2}{7}}$$

where $M_8$ is the black hole mass in terms of $10^8 M_\odot$, $f$ is the fraction of energy lost to a corona, and we have assumed Thomson scattering opacity. Since this radius is very insensitive to the black hole mass, we expect radiation dissipation to be important in the range of radii in which most of the luminosity is created for black hole X-ray binaries, Seyfert galaxies, and quasars. There is a rather strong dependence on the luminosity relative to Eddington, so radiation dissipation won’t play a role for objects with small $\dot{m}$. We have computed a disk model which includes both radiation and gas pressure, and compared the nonlinear frequency with the numerical root of the dispersion relation for compressive waves. We find that the radius at which the damping rate exceeds the nonlinear frequency (for some $k$) is typically at $P_{\text{gas}} \simeq a \times P_{\text{rad}}$.

3.4. Growth of the radiation field

So far our discussion of accretion disks has assumed that they are already radiating. Since the radiation is derived from the dissipation of kinetic energy into electron thermal energy or photon energy density, we have only showed that radiation damping provides a dissipation mechanism which gives a self-consistent disk solution. Another stronger question to ask is: if a disk is in a state in which radiation pressure is small relative to gas pressure, for what parameters will the radiation dissipation cause growth of the radiation field, causing the disk to find a radiation pressure-dominated equilibrium?

The rate of change of the radiation field is given approximately by:

$$\frac{\partial P_{\text{rad}}}{\partial t} = \frac{Q}{2\hbar} \min \left[ 1, \left( \frac{\Gamma_d}{\omega_{\text{nl}}^2} \right)_{\text{max}} \right] - \frac{cP_{\text{rad}}}{\hbar(1 + \tau)}$$

where the first term on the right hand side is the rate of creation of radiation due to photon dissipation of turbulence ($Q$ is given by equation 1), and the second term is the rate of escape of
radiation from the disk. Now, $Q \simeq c P_{rad}/(1 + \tau)$ in equilibrium, so a steady-state radiation field can only be achieved for $(\Gamma_d/\omega_{nl}) \gtrsim 1$. In general, the maximum damping for compressive waves $\Gamma/2$ occurs for $k \sim k_D \sim k_{Tc}/c$. For a general disk, the maximum ratio of radiation damping to nonlinear frequency is given by:

$$\left( \frac{\Gamma_{comp}}{\omega_{nl}(k_D)} \right) \propto \tau^{2-n} \left( \frac{h}{r_g} \right)^{1-n} x^{\frac{3n-6}{2}} \dot{m} \frac{Tr(T)}{Tr\phi},$$

(65)

where $r_g = GM/c^2$, and the constant of proportionality is of order unity. For incompressive modes, the maximum damping rate occurs for $k \sim k_T$, where

$$\left( \frac{\Gamma_{inc}}{\omega_{nl}(k_T)} \right) \propto \dot{m} x^{-\frac{1}{2}} \tau^{2-n} \frac{Tr(T)}{Tr\phi}.$$

(66)

These expressions are valid for optically thick disks which may be radiation pressure or gas pressure supported, and have $Q$ given by equation (1). Whether either of these is greater than unity depends on what state the disk begins in. We consider one such starting state in the next section: an advection-dominated disk.

4. Unconventional accretion disks

4.1. Advection-dominated disks

In an advection-dominated disk, the equilibrium depends on the fact that the cooling timescale is much longer than the accretion timescale and thus the heat is advected inwards rather than being radiated locally. If radiation damping is strong enough to cause growth of the radiation field, then the radiation will damp out the turbulence and most of the heat will go into radiation rather than proton thermal energy which gets advected.

To estimate when radiation pressure is subject to growth, we assume a steady state disk with electrons of a constant temperature in which the viscous stress is generated by magnetic fields which create a turbulent cascade to smaller wavelengths. Using the criterion of Narayan and Yi (1995) for the existence of an advection-dominated solution ($\dot{m} < 0.5\alpha^2$), we find $\tau < (\alpha/0.1)x^{-1/2}$, so advection-dominated disks are usually in the optically thin regime. Since the disk is optically thin, the radiation damping is given by $\Gamma_{d,thin}^{comp}$ for either compressive or incompressive modes. Comparing the damping rate to the slowest nonlinear frequency (at $kh = \pi$), we find that the criterion for radiation growth using equation (64) is $x < 0.45[\alpha^2 Tr(T)/Tr\phi]^{2/3}$, which means that radiation viscosity will not cause optically thin advection dominated disks to cool and radiate. This also means that whatever radiation is produced in an advection-dominated accretion flow cannot be produced by the photon damping mechanism.
4.2. Corona-dominated accretion disks

An alternative disk equilibrium has been proposed by Svensson and Zdziarski (1994), in which
all of the angular momentum transport occurs within the accretion disk, while a fraction \( f \) of the
associated heat released occurs above the disk in a corona. Their equilibrium relies on the idea
that the energy can be efficiently transported from the disk to the corona somehow, presumably
through magnetic or acoustic waves. Since there is no outgoing radiation flux within the disk,
its equilibrium density and gas pressure are much greater than in the radiation-supported case.
However, unless \( f \) is very close to unity, there will still be a significant region of the disk in which
radiation pressure dominates (see figure 2 of Svensson and Zdziarski 1994). In this case equations
(53) and (54) still apply, and the radiation damping time for compressive waves is less than the
wave crossing time, \( 2\pi\Omega^{-1} \) for

\[
kh \gtrsim \sqrt{\frac{12}{\alpha(1 - f)}}
\]

(67)

For incompressible modes, the crossing time \( 2\pi c_s/(\Omega v_A) \) is greater than the damping time for

\[
kh \gtrsim \sqrt{\frac{3v_A^3}{m^2\alpha(1 - f)c_s}}
\]

(68)

Thus, only a limited range of wavelengths can successfully carry energy to the corona.

When gas pressure dominates, if \( v_A \approx c_g \), the crossing time for hydromagnetic waves is about
\( 2\pi/\Omega \). For incompressive waves with wavelengths less than the mean free path of a photon, or for
compressive waves with \( k > k_D \), the damping rate will be \( \Gamma_{\text{thin}}^d \). For \( f \approx 1 \) disks,

\[
k_D h = 2 \times 10^5 \dot{m}^{\frac{7}{8}} x^{-\frac{9}{8}} \alpha^{-\frac{1}{2}} (1 - f/2)^{-\frac{1}{2}} m_8^{\frac{1}{4}}
\]

(69)

while

\[
k_T h = \tau = 1.8 \times 10^8 \dot{m}^{3/4} x^{-3/4} \alpha^{-1} M_8^{1/4} (1 - f/2)^{-1/4} (\kappa/\kappa_T).
\]

(70)

Now for waves to be damped by photon viscosity before they can escape from the disk requires
\( \Gamma/2\Omega > 1 \). This ratio is:

\[
\frac{\Gamma}{2\Omega} = \frac{4aT^4\rho c_T h c}{2\rho c^2 \Omega} = 0.8(1 - f/2)\dot{m} \left( \frac{x}{10} \right)^{-\frac{3}{2}}
\]

(71)

neglecting relativistic factors, where \( T \) and \( \rho \) are the density and temperature inside the disk,
and we have used the equations from the appendix of Sincell and Krolik (1997) to evaluate the
disk parameters for \( f \approx 1 \). Thus, only extremely short wavelength waves may be damped rapidly
enough, and then only in rather extreme conditions (relatively large \( \dot{m} \) and small \( x \)). If \( v_A/c_g \ll 1 \),
the requirements for damping incompressible Alfven waves may be relaxed somewhat, but unless
this ratio is very small, the qualitative conclusion is unlikely to be altered.
5. Discussion

We have shown in the previous sections that the effectiveness of radiation in damping fluid motions depends strongly on the ratio of radiation to gas pressure. As noted in equation (63), radiation tends to be most important in the inner parts of accretion disks, which are, of course, the most important for energy release. At least some part of the disk is radiation-dominated when

\[ \dot{m} > 1.0 \times 10^{-3} x_{\text{min}}^{21/16} \alpha^{-1/8} M_8^{-1/8}, \]  

(72)

where \( x_{\text{min}} \) is the inner radius of the disk. If the central object is a black hole or a weakly-magnetized neutron star, we may expect \( x_{\text{min}} \) to be the radius of the marginally stable orbit, \( = 6 \) in the limit of a spinless black hole, and \( \to 1 \) as the spin of the black hole approaches its maximum possible value. However, if the disk does not extend in so far, whether because the central mass is a strongly-magnetized neutron star, or a larger object such as a white dwarf, the minimum accretion rate for which at least part of the disk is radiation-dominated rises, and may become impossibly high.

The remainder of this section, in which we outline the consequences of radiation damping in accretion disks, is divided according to consequences applicable to radiation-dominated disks and those applicable to the gas pressure-dominated case. Whether one set or the other is relevant to a given disk depends on how it fares according to the criterion of equation (72).

5.1. Radiation pressure-dominated disks

Two qualitative physical consequences follow from the strength of radiation damping in photon pressure-dominated disks. First, dissipative heating is delivered to the electrons and photons through radiation scattering, and not to the ions. Because it is the electrons that cool the gas through the creation and upscattering of photons, the only energy exchange process involving the ions is Coulomb scattering. This mechanism should keep the ion temperature very close to the electron temperature. If the average energy of photons is less than \( \beta^2 m_e c^2 / 3 + 4k_B T_e \), where \( T_e \) is the electron temperature, then the photons will receive most of the energy from scattering (Psaltis and Lamb 1997). The \( \beta^2 m_e c^2 \) term represents a modification of the Compton temperature due to bulk Comptonization.

Second, the process by which these disks shine may be thought of as a sort of “bootstrap”: if the disk were initially free of radiation, any initial photon creation by the electrons would lead to wave dissipation that heats the electrons, and therefore leads to more radiation. The question of what makes near-Eddington accretion disks shine has a tautological answer: bright accretion disks shine because they are so bright.

That MHD fluctuations should be present at all is likely due to the operation of the magneto-rotational instability identified by Balbus & Hawley (1991). This instability grows at a
rate $\sim kv_A$ for wavenumbers $k \leq \sqrt{3}\Omega/v_A$ when the magnetic field is weak (i.e. $v_A < c_s$). The compressibility of the growing modes is slight, so the corresponding radiation damping rate should be a fraction of the pure compressive rate, as given by equation (54). If most of the torque in the disk is due to magnetic fluctuations, the ratio between the magneto-rotational growth rate and the radiation damping rate is then at least $\sim 10(c_s/v_A)/(kh)$. We therefore expect the linear growth of MHD fluctuations to proceed unaffected by radiation damping.

However, shorter wavelength waves are not amplified by the magneto-rotational instability. Instead, they are pumped by nonlinear coupling with the longer wavelength, growing modes. Because the radiation damping rate is $\propto k^2$ in the diffusive regime, compressive modes excited by nonlinear coupling will be strongly damped. In other words, provided only that the nonlinear coupling between incompressible and compressible modes is reasonably strong, the “inner scale” of the MHD turbulence will be not much shorter than its “outer scale.” Any turbulent “inertial range” will be severely limited.

This fact leads to several other results. At a purely technical level, if short wavelengths are all severely damped, the life of the numerical simulator is made much easier, for there is no need to strive for very fine spatial resolution.

More physically, radiation damping may play an important role in regulating the value of the “viscosity” parameter $\alpha$. The magnetic part of the stress causing angular momentum transport may be written in the form

$$T_{r\phi} = -\frac{1}{4\pi} \int d^3k \delta B_r(\vec{k}) \delta B_\phi^* (\vec{k}).$$

(73)

If there is little power in the fluctuations at wavenumbers much more than $\sim 1/h$, the angular momentum transport is reduced below what it would otherwise be. Disks in this situation would then maintain rather larger surface densities. Increased optical depth also leads to greater radiation pressure for fixed emergent flux.

Another consequence for turbulence in radiative disks is that the ratio of the sound speed to the Alfvén speed changes with wavelength. In the diffusive regime, $c_s \sim c_r \gg v_A$, leading to a large plasma $\beta \equiv P_{tot}/P_{mag}$. When the plasma $\beta$ is large, MHD fluctuations are generally close to incompressible because pressure waves can travel rapidly enough to smooth out density disturbances. However, for short wavelengths, the radiation field decouples from the fluid, and $c_s \sim c_g \ll v_A$, which means the plasma $\beta$ becomes effectively quite small. For these short wavelengths, then, we can expect the turbulence to exhibit much greater compressibility. In the compressible regime, the speeds of the magnetosonic and Alfvén waves are comparable, so they may couple much more easily. A similar effect happens for Alfvén waves near recombination, as discussed by Subramanian and Barrow (1997).

The slope and inertial range of the turbulent spectrum will also be affected by the plasma $\beta$ parameter, which is usually held fixed in compressive MHD simulations (Matthaeus et al. 1996). Analytic theory and simulations show that for compressible MHD, $\delta\rho/\rho \sim (\delta v_A/c_s)^2$
(where $\delta v_A \equiv |\delta B|/\sqrt{4\pi \rho}$), so when $c_s$ drops dramatically in the non-diffusive regime, compressive damping will become very effective. Simulations of turbulent cascades with small $\beta$ but with incompressible stirring will show how much energy can be transferred to compressible modes. Current simulations of compressible turbulence in the ISM (Charles Gammie, private communication) show that shocks form when $v_A \gg c_g$, so that if the incompressible cascade does not transfer energy to compressible modes before reaching the non-diffusive scale, the energy may be dissipated in shocks at that scale. The dissipation in these shocks may be partly due to ordinary plasma processes, and partly due to radiation scattering. Thus, we expect that $k_{\text{max}}$ will never be much greater than $k_D$. As the radiation pressure varies with disk radius, $k_D$ changes and thus $k_{\text{max}}$ changes, so the value of $\alpha$ may become a function of radius.

Although certain consequences of radiation damping are relatively clear (at least qualitatively), consideration of this process also raises a number of questions:

1) What is the nature of the coupling between compressive and incompressive modes? Is it large enough to allow the radiation damping rate to compete with the nonlinear frequency? Are the analytic estimates we have made useful in the nonlinear regime?

2) In the simulations done to date, in which radiation pressure and transport are equally ignored, the magnetic energy density is an interesting fraction of the pressure and the associated fluctuations lead to a stress which is also proportional to the pressure. The question naturally arises whether, in radiation pressure-dominated disks, the $r - \phi$ stress and the energy in the magnetic field scale with the total pressure, or just with the gas pressure. The photon bubble instability (Arons 1992) will likely affect the disk structure and stress (Gammie 1998). With explicit consideration of the quality of dynamical coupling between radiation fluctuations and fluid fluctuations, as outlined here, simulations should now be able to answer these questions.

3) Can thermal or viscous instabilities be suppressed by radiation damping? Or does the dependence of dissipation on the radiation pressure exacerbate these instabilities? In both cases, the most important modes have radial wavenumbers $< h^{-1}$, so the calculation here does not directly bear on them. However, one might expect that some of the same effects will qualitatively carry over.

4) The relativistic portions of accretion disks may trap a number of long wavelength (i.e. $kh < 1$) normal modes (Nowak & Wagoner 1991, 1992). Some of these grow in amplitude due to viscous dissipation (Nowak & Wagoner 1992). Modulo the caveat of point 3), will radiation damping enhance (or destroy) these modes?

5) Many seek the origin of disk coronal heating in the dissipation of rising MHD waves (e.g. Rosner, Tucker, & Vaiana 1978; Heyvaerts & Priest 1989; Tout & Pringle 1996). If radiation damping quenches short wavelength fluctuations, will this affect the rate at which magnetic flux rises to the disk surface?
5.2. Gas-pressure dominated disks

When gas pressure dominates over radiation pressure, radiation damping does not compete with the nonlinear frequency. The question of what causes the heating of the disk therefore remains open. This conclusion is equally true of conventional gas pressure-dominated disks and unconventional ones like ADAFs.

Finally, the contrast between the radiation pressure-dominated and gas pressure-dominated regimes may mean that interesting observable effects occur in disks whose accretion rate fluctuates around the critical value of equation (72). If the value of $\alpha$ and the radiative efficiency depend on whether radiation damping plays a role, there could be significant modulations in the luminosity and spectrum on a viscous timescale.

We would like to thank Omer Blaes, Steve Balbus, Axel Brandenburg, Charles Gammie, and John Hawley for useful discussions.

This work was partially supported by NASA Grant NAG 5-3929 and NSF Grant AST-9616922. Eric Agol thanks the Isaac Newton Institute for Mathematical Sciences where part of this work was completed.
REFERENCES

Jackson, J. D. 1975, Classical Electrodynamics (New York: John Wiley & Sons)


Weinberg, S. 1972, Gravitation and Cosmology (New York: John Wiley & Sons)