Quantum kinetics and thermalization in an exactly solvable model

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Abstract

We study the dynamics of relaxation and thermalization in an exactly solvable model with the goal of understanding the effects of off-shell processes. The focus is to compare the exact evolution of the distribution function with different approximations to the relaxational dynamics: Boltzmann, non-Markovian and Markovian quantum kinetics.

The time evolution of the occupation number or distribution function is evaluated exactly using two methods: time evolution of an initially prepared density matrix and by solving the Heisenberg equations of motion. The former allows to establish a connection with the stochastic nature of thermalization and the fluctuation dissipation theorem, whereas the latter leads to the interpretation of an interpolating number operator to ‘count’ quasiparticles. There are two different cases that are studied in detail: i) no stable particle states below threshold of the bath and a quasiparticle resonance above it and ii) a stable discrete exact ‘particle’ state below threshold.

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The exact solution for the evolution allows us to investigate the concept of the formation time of a quasiparticle and to study the difference between the relaxation of the distribution of particles and quasiparticles. In particular we compare the quasiparticle distribution for asymptotic times with the equilibrium canonical distribution. For the case of quasiparticles in the continuum (resonances) the exact quasiparticle distribution asymptotically tends to a statistical equilibrium distribution that differs from a simple Bose-Einstein form as a result of off-shell processes such as the strength of the quasiparticle poles, the width of the unstable particle and proximity to thresholds. In the case ii), the distribution of particles does not thermalize with the bath.

We study the kinetics of thermalization and relaxation by deriving a non-Markovian quantum kinetic equation which resums the perturbative series and includes off-shell effects. A Markovian approximation that includes off-shell contributions and the usual Boltzmann equation are obtained from the quantum kinetic equation in the limit of wide separation of time scales upon different coarse-graining assumptions. The relaxational dynamics predicted by the non-Markovian, Markovian and Boltzmann approximations are compared to the exact result of the model. The Boltzmann approach is seen to fail in the case of wide resonances and when threshold and renormalization effects are important. Implications for thermalization in field theory models are discussed.
1 Introduction and Motivation

The new generation of ultrarelativistic heavy ion colliders, RHIC at Brookhaven and LHC at CERN will hopefully probe the quark-gluon plasma (QGP) and chiral phase transitions within the next few years. These colliders will provide the first opportunity to study phase transitions predicted by QCD in controlled experiments. The main goal of the theoretical program associated with ultrarelativistic heavy ion collisions is to provide a thorough understanding of the experimental signatures associated with the formation and evolution of the QGP as well as the chiral phase transition[1]. A very important part of this program is to study the dynamical evolution from the initial state corresponding to the highly contracted nuclei through the thermalization stage and the onset of the QGP in local thermodynamic equilibrium. The initial state after the collision is strongly out of equilibrium and there are very few quantitative models to study its subsequent evolution. Current theoretical predictions for the dynamics of the initial stages of formation of a QGP are based on the hard and semihard aspects which are studied via perturbative parton cascade models. These assume that at large energy densities the nuclei can be resolved into their partonic constituents and the dynamical evolution can therefore be tracked by following the parton distribution functions through a relativistic transport equation which includes scattering through the perturbative parton-parton interactions which are dressed by medium effects[2]-[8].

The important ingredient in this program is a relativistic kinetic or Boltzmann equation for the phase space distribution function of partons (quarks and gluons) whose collision terms incorporate the perturbative parton cross sections including screening effects that cut-off the infrared behavior[6, 7]. Although relativistic kinetic transport equations provide a natural framework to study non-equilibrium phenomena and are based on successes in many different fields (mostly at the non-relativistic level), their validity in the extreme situations envisaged to arise during the early stages of heavy ion collisions at RHIC and LHC is questionable at best. Detailed microscopic derivations of relativistic transport equations[3, 5, 9, 10] reveal that several coarse-graining assumptions must be invoked to arrive at the kinetic equations beginning from the microscopic Schwinger-Dyson equations. The typical assumptions involve smooth and slow variations over microscopic time scales, which results in a condition that the time scales for relaxation be much longer than the typical microscopic scales. These assumptions had been discussed by Geiger[5] within the context of parton cascade models of the evolution and thermalization and more recently by Bass et. al.[10] within the framework of the relativistic quantum molecular dynamics approach. With time scales for thermalization predicted to be of order of \( \approx 1\text{fm}/c \) (chemical equilibrium is typically achieved on longer time scales) the regime of validity of coarse-grained relativistic Boltzmann equations may be restricted to hard partons with typical momenta \( \geq \text{few Gev} \). However for soft quarks or gluons or collective modes with typical scales determined by the Debye
(or magnetic) screening lengths (at $T \approx 300 - 500$MeV) $(m_p)^{-1} \approx 1/gT \approx 1$fm or wide resonances, such as vector mesons, whether this separation of scales is enough to justify the validity of the usual kinetic approach is questionable.

There is a rekindled interest on a deeper understanding of quantum kinetics from microscopic quantum field theory, rather than from a semi-phenomenological approach and critical analysis of transport and kinetic approaches are beginning to emerge\cite{11}\textsuperscript{-}\textsuperscript{14}. In particular recently the initial stages of pre-equilibration during which quasiparticle correlations begin to build had been investigated in a many body system\cite{14}. The pre-equilibrium stage cannot be studied within a Boltzmann approach because the early time dynamics depends on the initial preparation of the state and is determined by off-shell (non-energy conserving) processes\cite{14}.

Unlike non-relativistic cases in which the validity of kinetic equations can be tested either numerically or experimentally as in condensed matter physics, the reliability of the approximations involved in obtaining relativistic kinetic equations in the high energy and density regimes to be probed at RHIC-LHC can only be speculated at present. Therefore, it is important to obtain insight into the dynamics of thermalization and relaxation and the validity of a kinetic description in simple model theories so as to provide a yardstick to use as a guiding principle.

It is the goal of this article to study the description of thermalization and relaxational dynamics in a simple and exactly solvable many body theory to obtain a deeper understanding of the off-shell processes involved in thermalization and to provide a yardstick to test different approximations.

The aspects that we seek to study in this article are the following:

- How do off-shell effects modify the dynamics of thermalization and relaxation? By off-shell we here refer to processes that do not conserve energy on short time scales and threshold effects that are not incorporated in the usual Boltzmann equation. These are responsible for quasiparticle properties such as widths and residues in the quasiparticle propagators.

- A detailed understanding of the relaxation of quasiparticles vs. that of bare and dressed particles and to explore the definition of a quasiparticle distribution function that is valid beyond the narrow width approximation.

- A comparison of the validity of Markovian (coarse grained) approximations including the Boltzmann equation, to the most general non-Markovian description of relaxation.

Although we anticipate that the answer to these questions will in general depend on the
details of the microscopic model, we propose to study a model which bears many properties of realistic quantum field theories and allows us to provide the answers to these (and other) questions to build intuition into more complex situations.

In section 2 we introduce the model and analyze the dynamics from the point of view of the time evolution of an initially prepared density matrix and the solution of the Heisenberg equations of motion. This analysis allows us to establish contact with the fluctuation dissipation relation. In section 3 we study in detail the time evolution of the distribution function and distinguish between bare particle and dressed particle and quasi-particle distributions. In section 4 we discuss the exact solution in terms of normal modes and analyze the definition of the interpolating number operator that describes the relaxational dynamics. In section 5 we analyze the approximate relaxational dynamics in terms of i) the Boltzmann equation, ii) the non-Markovian quantum kinetic equation and iii) a Markovian approximation to the quantum kinetic equation. In section 6 we provide a numerical comparison of the exact and approximate kinetics. Our conclusions and extrapolations to realistic quantum field theories are summarized in section 7.

2 The Model

As stated in the introduction, we seek to study aspects of quantum kinetics in a model that allows to compare approximate treatments of the relaxational dynamics to exact solutions. Obviously even a simple interacting quantum field theory would not allow an exact treatment of the dynamics and the validity of the approximations leading to the kinetic description could not be tested. Instead we sacrifice generality and study a model that has already been used in the literature to study aspects of decoherence and dissipation phenomena, and which shares many of the important features of a quantum field theory, such as renormalization and off-shell phenomena.

The physical situation that we have in mind is that of an initial distribution of bare particles that become in contact with a medium of high energy density which we will take to be in equilibrium at a given temperature $T$ and will be denoted by the ‘bath’. Before interacting with the medium the particle(s) has a physical mass and travels freely, which in field theory would correspond to a description in terms of physical in fields of the physical mass. These bare particles will mix with the particles in the medium and form dressed states by shifting the mass and residues of the propagators. We will describe this scenario by a sudden coupling of the bare particle(s) to the bath at an initial time which is taken to be $t = 0$. The initial density matrix is then factorized into a tensor product of a density matrix for the bare particles and another for the bath.
The model that we choose to describe this situation is that of an oscillator of bare frequency $\omega_0$ (representing the physical mass of the in particle states before the interaction) coupled linearly with a bath with an infinite number of degrees of freedom given by harmonic oscillators with frequencies $\omega_k$. Although this is a drastic simplification of the theory that we are ultimately interested in describing, this model has already been used in studies of dissipation[17]-[23] decoherence[24], and also as models for entropy production in heavy ion collisions[25] and more recently within the context of baryogenesis[26].

The Lagrangian is given by

$$L[q, Q_k] = \frac{1}{2} \left( \dot{q}^2 - \omega_0^2 q^2 \right) + \frac{1}{2} \sum_k \left( \dot{Q}_k^2 - \omega_k^2 Q_k^2 \right) - q \sum_k C_k Q_k,$$

where the different coefficients of $\dot{q}^2$, $\dot{Q}_k^2$ (oscillator masses) had been absorbed by a canonical transformation into a redefinition of the couplings $C_k$. We now refer to the oscillator $q$ as the ‘system’ i.e. the degree of freedom whose dynamics we are interested in studying, and the oscillators $Q_k$ as the ‘bath’, these will be integrated out in the non-equilibrium effective action. The degree of freedom $q$ could be associated with a particle of a particular wave number, for example in the linear relaxation approximation to kinetics in which only one particular mode is displaced from equilibrium, whereas all of the other modes in the field theory remain in equilibrium. In this case the mode in question could be identified with the variable $q$ and the other modes in equilibrium with the bath.

We will eventually take the limit in which the bath oscillators are distributed continuously by introducing the bath spectral density $J(\omega)$ and where appropriate replacing the discrete distribution with a continuum one in the following manner:

$$J(\omega) = \frac{\pi}{2} \sum_k \frac{C_k^2 \omega}{\omega_k} \delta(\omega - \omega_k)$$

in such a way that

$$\sum_k C_k^2 f(\omega_k) \to \frac{2}{\pi} \int d\omega \omega J(\omega) f(\omega). \tag{1}$$

Our main goal is to study the evolution of the number of excitations or ‘particle distribution’ associated with the quanta of the system. Anticipating self-energy renormalization effects by the medium, we define a reference frequency $\Omega$ and introduce the operator that counts the number of quanta of the system’s degrees of freedom associated with this frequency.
\[
\dot{n}(t) = \frac{1}{2\Omega} \left[ p^2(t) + \Omega^2 q^2(t) \right] - \frac{1}{2},
\] (2)

where \(p(t)\) is the momentum of the particle. The reference frequency could either be taken to be the bare frequency \(\omega_0\), or the ‘in-medium’ frequency renormalized by the interaction with the bath and we will leave this choice unspecified for the moment.

Since the theory is quadratic we can resort to a number of different ways to study the dynamical evolution:

1. Given an initial density matrix we can evolve it in time exactly and obtain all of the non-equilibrium correlation functions.

2. The Heisenberg equations of motion for the operators can be solved exactly and again we can obtain any correlation function.

3. The normal modes can be found exactly, from which we can find the exact ground state and also obtain the operators that create the particle or quasiparticle states to study the asymptotic evolution of non-equilibrium states.

4. We compute exactly the expectation value of the proposed number operator in the canonical ensemble of the system plus bath and compare the result to the asymptotic form of the non-equilibrium distribution function. This allows an unequivocal description of thermalization in terms of the density matrix.

We will pursue all of the above different approaches, since each particular method provides different insights and the main goal is to understand this simpler model in detail to provide intuition into the more complicated case of field theories.

### 2.1 Time evolution of an initial Density Matrix:

The first method is to calculate the time evolution of the reduced density matrix, \(\rho_s(t)\), of the particle that has been prepared at some initial time \(t_i\).

This can be achieved by treating the infinite set of harmonic oscillators, \(Q_k\), as a ‘bath’ and obtaining an influence functional[20]-[22, 23] by tracing out the bath degrees of freedom. We assume that the total density matrix for the particle-bath system decouples at the initial time \(t_i\), i.e.

\[\rho(t_i) = \rho_s(t_i) \otimes \rho_R(t_i),\]
where \( \rho_B(t_i) \) is the density matrix of the bath which describes infinite set of harmonic oscillators in thermal equilibrium at a temperature \( T \) and \( \rho_s(t_i) \) is the density matrix of the particle which is taken to be that of a harmonic oscillator in thermal equilibrium at temperature \( T_0 \). More complicated initial density matrices, including correlations between system and bath degrees of freedom can be studied by following the methods found in[19].

The complete information of non-equilibrium expectation values and correlation functions is completely contained in the time dependent density matrix

\[
\rho(t) = U(t, t_i) \rho(t_i) U^{-1}(t, t_i)
\]

with \( U(t, t_i) \) the time evolution operator. Real time non-equilibrium expectation values and correlation functions can be obtained via functional derivatives with respect to sources of the generating functional[27]

\[
Z[j^+, j^-] = \text{Tr} \left[ U(\infty, t_i; j^+) \rho(t_i) U^{-1}(\infty, t_i; j^-) \right] /\text{Tr} \rho(t_i),
\]

where \( j^\pm \) are sources coupled to the particle coordinate. This generating functional is readily obtained using the Schwinger-Keldysh method which involves a path integral in a complex contour in time[27]

In the present situation, the non-equilibrium generating functional is given by

\[
Z[j^+, j^-] = \frac{1}{\text{Tr} \rho(t_i)} \int dq dq_1 dq_2 \langle q_1 | \rho_s(t_i) | q_2 \rangle \int dQ dQ_1 dQ_2 \langle Q_1 | \rho_B(t_i) | Q_2 \rangle \times \int Dq^+ Dq^- DQ^+ DQ^- \exp \left\{ i \int dt \left[ L[q^+ , Q_k^+] - L[q^- , Q_k^-] - q^+ j^+ + q^- j^- \right] \right\}
\]

with

\[
dQ = \prod_k dQ_k \quad \text{and} \quad DQ = \prod_k DQ_k
\]

and with the following boundary conditions on the fields: \( Q_k^+(t_i) = Q_{ik} \); \( Q_k^+(\infty) = Q_k^- (\infty) = Q_k; \) \( Q_k^-(t_i) = Q_{ik} \); \( q^+(t_i) = q_i \); \( q^-(\infty) = q^- (\infty) = q; \) \( q^-(t_i) = q_2 \). The signs \pm in the above expressions correspond to the fields and sources on the forward (+) and backward (-) branches. The contribution from the branch along the imaginary time is canceled by the normalization factor. Real time, non-equilibrium Green’s functions are now obtained as functional derivatives with respect to the sources. There are four types of free propagators[27]

\[
\begin{align}
\langle Q_k^+(t) Q_k^-(t') \rangle &= -i G_k^-(t, t') = -i [G_k^-(t, t') \theta(t - t') + G_k^+(t, t') \theta(t' - t)] \\
\langle Q_k^-(t) Q_k^+(t') \rangle &= -i G_k^+(t, t') = -i [G_k^+(t, t') \theta(t' - t) + G_k^-(t, t') \theta(t - t')] \\
\langle Q_k^-(t) Q_k^-(t') \rangle &= G_k^-(t, t') = G_k^-(t, t') \\
\langle Q_k^+(t) Q_k^+(t') \rangle &= G_k^+(t, t') = G_k^+(t, t')
\end{align}
\]  

(3)
where
\[ G_k^<(t, t') = \frac{i}{2\omega_k} \left[ (1 + N_k) \exp\{-i\omega_k(t - t')\} + N_k \exp\{i\omega_k(t - t')\} \right] \]
\[ G_k^>(t, t') = \frac{i}{2\omega_k} \left[ (1 + N_k) \exp\{i\omega_k(t - t')\} + N_k \exp\{-i\omega_k(t - t')\} \right] \]
\[ N_k = \frac{1}{\exp\{\beta\omega_k\} - 1}. \] (4)

2.2 The reduced density matrix

The reduced density matrix, \( \rho_r(t) \), is defined as
\[ \rho_r(t) = \frac{Tr_R \rho(t)}{Tr_R \rho(t_i)}, \]
where the subscript \( R \) in \( Tr_R \) refers to tracing over the bath degrees of freedom. Taking the trace over \( Q_k \), one obtains the reduced density matrix in terms of the influence functional, \( \mathcal{F}[q^+, q^-] \)
\[ \rho_r(q, q'; t) = \int dq_1 dq_2 \rho_0[q_1, q_2] \int Dq^+ Dq^- \exp \left\{ i \int dt (L_0[q^+] - L_0[q^-]) \right\} \times \mathcal{F}[q^+, q^-], \]
where \( \rho_0[q_1, q_2] \) is the initial density matrix of the particle and
\[ \mathcal{F}[q^+, q^-] = \exp \left\{ i \sum_k C_k^2 \int dt \int dt' \sum_{a,b} q^a(t) G_k^{ab}(t, t') q^b(t') \right\}; \quad a, b = +, - \]
\[ L_0[q^+] = \frac{1}{2} \left[ (\dot{q}^+)^2 - \omega_0^2 (q^+)^2 \right]. \]

We will choose the initial density matrix of the particle to be that of an harmonic oscillator of reference frequency \( \Omega \) in thermal equilibrium at temperature \( T_0 \) given by
\[ \rho_0[q_1, q_2] = \sqrt{\frac{1}{2\pi\sigma}} \exp \left\{ ip_i(q_1 - q_2) \right\} \times \exp \left\{ -\frac{\Omega}{2\sinh \left[ \beta_0 \Omega \right]} \left[ ((q_1 - q_i)^2 + (q_2 - q_i)^2) \cosh \left[ \beta_0 \Omega \right] - 2(q_1 - q_i)(q_2 - q_i) \right] \right\}, \]
where \( q_i \) and \( p_i \) are respectively the average position and momentum of the particle, \( \beta_0 = 1/T_0 \) and
\[ \sigma = \frac{1}{2\Omega} \coth \left[ \frac{\beta_0 \Omega}{2} \right] = \frac{1 + 2n(0)}{2\Omega} \quad \text{and} \quad n(0) = \frac{1}{e^{\beta_0 \Omega} - 1}. \]
The reference frequency $\Omega$ will allow to understand the different features of the dynamics of the dressed particle in the medium, rather than the bare particle with frequency $\omega_0$. We will specify this reference frequency below when we study the dynamics in detail.

At this stage it is convenient to introduce the center of mass and relative coordinates, $x$ and $r$ respectively, which are defined as

$$x(t') = \frac{1}{2} [q^+(t') + q^-(t')], \quad r(t') = q^+(t') - q^-(t').$$

These are recognized as the Wigner coordinates. In terms of these coordinates the reduced density matrix becomes

$$\rho_{x_f, r_f; t} = \int dx_i dr_i \rho_0[x_i, r_i] e^{i \hat{x}_i x_f - i \hat{x}_i x_i} \int Dx Dr e^{i S_{eff}},$$

where

$$x_i = \frac{1}{2} (q_1 + q_2) ; \quad r_i = q_1 - q_2 ; \quad x_f = \frac{1}{2} (q + q') ; \quad r_f = q - q'$$

$$\rho_0[x_i, r_i] = \sqrt{\frac{1}{2\pi\sigma}} \exp \left\{ -\frac{1}{2\sigma} (x_i^2 - 2q_i x_i + q_i^2) - \frac{\Omega^2 \sigma}{2} r_i^2 + ip_i r_i \right\}$$

$$S_{eff} = \int_0^t dt' \left\{ - [\ddot{x}(t') + \omega_0^2 x(t')] r(t') + \int_0^t dt'' \left( r(t') \Sigma(t' - t'') x(t'') + \frac{i}{2} r(t') K(t' - t'') r(t'') \right) \right\}$$

with

$$\Sigma(t' - t'') = \Theta(t' - t'') \sum_k \frac{C_k^2}{\omega_k} \sin \left[ \omega_k (t' - t'') \right]$$

$$K(t' - t'') = \sum_k \frac{C_k^2}{2\omega_k} \coth \left[ \frac{\beta \omega_k}{2} \right] \cos \left[ \omega_k (t' - t'') \right] .$$

The kernel $\Sigma(t' - t'')$ is the retarded self energy of the system degree of freedom $q$.

Since $S_{eff}$ is gaussian, the path integral can be evaluated exactly using the saddle point method where the coordinates $x(t')$ and $r(t')$ are expressed as variations around the extremum configurations $\vec{x}(t')$ and $\vec{r}(t')$ respectively. These satisfy the following equations of motion[19]
\[
\ddot{x}(t') + \omega_0^2 \dot{x}(t') - \sum_k \frac{C_k^2}{\omega_k} \int_0^{t'} dt'' \sin[\omega_k(t' - t'')] \ddot{x}(t'') = 0
\]  
(9)

\[
\ddot{r}(t') + \omega_0^2 \dot{r}(t') + \sum_k \frac{C_k^2}{\omega_k} \int_0^{t'} dt'' \sin[\omega_k(t' - t'')] \ddot{r}(t'') = 0
\]  
(10)

with

\[
x(0) = x_i \quad ; \quad \dot{x}(t) = x_f \quad ; \quad \ddot{r}(0) = r_i \quad ; \quad \ddot{r}(t) = r_f.
\]

There is in principle an imaginary inhomogeneity in the equation of motion for the center of mass variable (9), but this inhomogeneity is shown to be a surface term and combines with the end point contributions. Therefore only the real part of the classical trajectory is relevant. (For details see ref. [19] page 145.) The eqs. (9) and (10) can be solved by Laplace transform, and we obtain

\[
\ddot{x}(t') = x_i \dot{g}(t') + \dot{x}_i g(t')
\]

\[
\ddot{r}(t') = r_i \dot{g}(t - t') + r_f \frac{\dot{g}(t - t')}{g(t)} + r_f \frac{\dot{g}(t - t')}{g(t)}
\]

\[
g^{-}(t') = \frac{\dot{g}(t)}{g(t)} \frac{g(t) - g(t)}{g(t) - \dot{g}(t)}
\]

where the function \( g(t) \) and its first derivative \( \dot{g}(t) \) are solutions of (9) with initial conditions \( g(0) = \dot{g}(0) = 0 \) and \( \dot{g}(0) = 1 \) and \( \dot{x}_i \) is obtained from \( x_i, x_f \) and \( g(t) \). Such solution is given by the inverse Laplace transform of \( \tilde{g}(s) \) which is given by

\[
\tilde{g}(s) = \frac{1}{s^2 + \omega_0^2 + \tilde{\Sigma}(s)}
\]  
(11)

with the Laplace transform of the retarded self energy given by

\[
\tilde{\Sigma}(s) = -\sum_k \frac{C_k^2}{\omega_k^2} \frac{\omega_k}{s^2 + \omega_k^2} \rightarrow -\frac{2}{\pi} \int d\omega J(\omega) \frac{\omega}{s^2 + \omega^2}.
\]  
(12)

Here we have taken the limit of a continuum distribution of bath oscillators as given by eq. (1).

We will postpone the computation of the function \( g(t) \) to the next section and we will provide a particular spectral density for the bath in a later section wherein we will compare exact results to different approximations.
Finally, substituting $\tilde{x}(t')$ and $\tilde{r}(t')$ and evaluating the remaining gaussian integrals in eq. (5) we are led to the reduced density matrix

$$
\rho_{x_f, r_f; t} = \frac{1}{2 \sqrt{\pi} A(t)} \exp \left\{ -\frac{1}{2} \left[ \frac{\sigma}{(\varrho(t))^2} + R_{-+}(t) - \frac{B(t)}{2 A(t)} \right] r_f^2 - \frac{1}{4 A(t)} x_f^2 \right\}
+ i \left[ \frac{\dot{\varrho}(t)}{\varrho(t)} - \frac{B(t)}{2 A(t)} \right] x_f r_f + i \left[ \frac{B(t)}{2 A(t)} (p_g(t) + q_i \dot{\varrho}(t)) - \frac{q_i}{\varrho(t)} \right] r_f
+ \frac{1}{2 A(t)} (p_g(t) + q_i \dot{\varrho}(t)) x_f - \frac{1}{4 A(t)} (p_g(t) + q_i \dot{\varrho}(t))^2 \right\},
$$

where

$$
A(t) \equiv \frac{\Omega^2 \sigma}{2} - g^2(t) + \frac{1}{2} R_{-+}(t) + \frac{\sigma}{2} g^2(t)
$$

$$
B(t) \equiv \frac{\sigma}{g(t)} \dot{\varrho}(t) - R_{-+}(t)
$$

$$
R_{-+}(t) \equiv \int_0^t dt' \int_0^t dt'' g(t - t') K(t' - t'') g(t - t'')
$$

$$
R_{--}(t) \equiv \int_0^t dt' \int_0^t dt'' \frac{g(t')}{g(t)} K(t' - t'') \frac{g(t'')}{g(t)}
$$

$$
R_{+-}(t) \equiv \int_0^t dt' \int_0^t dt'' g(t - t') K(t' - t'') \frac{g(t'')}{g(t)}
$$

Having obtained the reduced density matrix, we can now obtain the expectation values of $\varrho^2(t)$ and $p^2(t)$ and to compute the expectation value of the number operator (2), which after some straightforward algebra is shown to be given by

$$
\langle n(t) \rangle = -\frac{1}{2} + \mathcal{R}(t) + \frac{\Omega}{2} \left( \frac{p^4 + \Omega^2 \sigma}{2} g^2(t) + \frac{q^2 + \sigma}{2} \varrho^2(t) \right)
+ \left( \frac{p^2 + \Omega^2 (q^2 + 2 \sigma)}{2 \Omega} \right) \varrho^2(t)
+ \frac{p q_i}{\Omega} \left[ \dot{\varrho}(t) + \Omega^2 g(t) \right] \dot{\varrho}(t),
$$

where we have introduced the shorthand notation

$$
\mathcal{R}(t) \equiv \frac{1}{2 \Omega} \left\{ R_{--}(t) + \frac{2}{\varrho(t)} \frac{\dot{\varrho}(t)}{g(t)} R_{+-}(t) + \frac{\dot{\varrho}(t)^2}{g^2(t)} \right\}.
$$

The expression for $\mathcal{R}(t)$ can be simplified by introducing the functions

$$
h(\omega, t) \equiv \int_0^t d\tau e^{-i \omega \tau} g(\tau)
$$

$$
k(\omega, t) \equiv \int_0^t d\tau e^{-i \omega \tau} \dot{g}(\tau)
$$

$$
= \dot{\omega} h(\omega, t) + e^{-i \omega t} g(t).
$$
In terms of these functions, \( \mathcal{R}(t) \) can be written as
\[
\mathcal{R}(t) = \frac{1}{4\Omega} \sum_k \frac{C_k}{\omega_k} \left[ 1 + 2N(\omega_k) \right] \left( |h(\omega_k, t)|^2 + \Omega^2 |h(\omega_k, t)|^2 \right)
\]
(15)
and in the limit of a continuum spectrum of bath oscillators
\[
\mathcal{R}(t) = \frac{1}{2\pi\Omega} \int d\omega J(\omega) \left[ 1 + 2N(\omega) \right] \left( |h(\omega, t)|^2 + \Omega^2 |h(\omega, t)|^2 \right).
\]

The expectation value of the number operator (13) in the non-equilibrium density matrix has two contributions: one that is completely determined by the initial state of the system (proportional to \( p_i ; q_i ; \sigma \)) and the other, determined by the bath and given by \( \mathcal{R}(t) \).

### 2.3 Fluctuation-Dissipation

The main advantage of studying the time evolution of the reduced density matrix is that it allows to establish a direct relationship between the relaxation of the occupation number of the system and the fluctuation dissipation theorem. The connection between the fluctuation-dissipation and the Boltzmann equation has been investigated recently in the semi-classical regime[12].

This relationship is established by re-writing the term quadratic in the relative variable \( \mathbf{r}(t) \) in \( S_{\text{eff}} \) given by the last term in eq.(6) in the following form
\[
\exp \left\{ -\frac{1}{2} \int dt \int dt' \mathbf{r}(t) \mathbf{K}(t-t') \mathbf{r}(t') \right\} = \int \mathcal{D}\xi \exp \left\{ -\frac{1}{2} \int dt \int dt' \xi(t) \mathbf{K}^{-1}(t-t') \xi(t') \right. \\
+ \left. i \int dt \xi(t) \mathbf{r}(t) \right\}.
\]

The path integrals in eq.(5) can now be written in the following form[21]
\[
\int \mathcal{D}x \mathcal{D}r \, e^{iS_{\text{eff}}(x,r)} = \int \mathcal{D}x \mathcal{D}t \mathcal{D}\xi \, P[\xi] \, e^{i\tilde{S}_{\text{eff}}(x,r,\xi)}
\]
\[
\tilde{S}_{\text{eff}}(x, r, \xi) = \int_{t_0}^{t} dt' \mathbf{r}(t') \left\{ -\ddot{x}(t') - \omega_0^2 x(t') + \int dt'' \Sigma(t' - t'') x(t'') + \xi(t') \right\},
\]
\[
P[\xi] = \exp \left\{ -\frac{1}{2} \int dt \int dt' \xi(t) \mathbf{K}^{-1}(t-t') \xi(t') \right\}.
\]

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The path integral over the relative coordinate leads to a non-Markovian Langevin equation for \( x(t) \) in the presence of a stochastic Gaussian (but colored) noise term \( \xi \). The noise correlation function is determined by \( K(t-t') \) given by eq. (8).

The fluctuation-dissipation relation is established in the following manner. In the limit of a continuum distribution of the bath oscillators we find the time Fourier transform of the retarded self-energy \( \Sigma(t) \) (7) to be given by the analytic continuation of the Laplace transform \( s \to \omega - i\epsilon \), i.e.

\[
\tilde{\Sigma}(\omega - i\epsilon) = -\frac{2}{\pi} \int d\omega' \frac{\omega' J(\omega')}{[\omega'^2 - (\omega - i\epsilon)^2]}.
\]

Then we find (for \( \omega > 0 \))

\[
\text{Im} \left[ \tilde{\Sigma}(\omega) \right] = J(\omega)
\]

and the Fourier transform in time of the kernel \( K(t) \) (8) is given by

\[
\tilde{K}(\omega) = \frac{1}{2\pi} \text{Im} \left[ \tilde{\Sigma}(\omega) \right] \coth \left[ \frac{\beta \omega}{2} \right].
\]

This is the usual fluctuation-dissipation relation[22]. Finally we obtain the bath contribution to the non-equilibrium occupation number, which is determined by \( R(t) \) in a form that displays clearly its relationship to the fluctuation dissipation relation

\[
R(t) = \frac{1}{\Omega} \int d\omega \tilde{K}(\omega) \left( |k(\omega, t)|^2 + \Omega^2 |h(\omega, t)|^2 \right),
\]

where \( \tilde{K}(\omega) \) is the power spectrum of the bath. This expression makes explicit the stochastic nature of thermalization and establishes a direct relationship with the fluctuation dissipation theorem.

Detailed understanding of the particle number relaxation requires the knowledge of the dynamical function \( g(t) \) which will be studied in a later section for a particular choice of the spectral density of the bath.

### 2.4 The Heisenberg Operators

The above results can be understood in an alternative manner by obtaining the real time evolution of the Heisenberg picture operators, from which the expectation value of the number operator can be obtained by providing an initial density matrix.
The equations of motion of the Heisenberg operators \( q(t) \), \( p(t) \) and \( Q_k(t) \) are given by

\[
\ddot{q}(t) + \omega_0^2 q(t) = - \sum_k C_k Q_k(t) \tag{16}
\]

\[
\ddot{Q}_k(t) + \omega_k^2 Q_k(t) = - q(t) C_k. \tag{17}
\]

Solving eq.(17) using the retarded Greens function of the operator \( d^2/dt^2 + \omega^2 \) given by

\[
G(t, t') = \frac{1}{\omega} \sin[\omega(t - t')] \theta(t - t'),
\]

we find

\[
Q_k(t) = Q_k^{(0)}(t) - \frac{C_k}{\omega_k} \int_0^t dt' \sin[\omega_k(t - t')] q(t'),
\]

where \( Q_k^{(0)}(t) \) is the homogeneous solution of eq.(17). Substituting \( Q_k(t) \) back in eq.(16), we obtain the equation of motion of the operator \( q(t) \)

\[
\ddot{q}(t) + \omega_0^2 q(t) - \sum_k \frac{C_k^2}{\omega_k} \int_0^t dt' \sin[\omega_k(t - t')] Q_k^{(0)}(t') = - \sum_k C_k Q_k^{(0)}(t).
\]

Taking the expectation value of the above equation in the initial density matrix we recognize the equation of motion for the center of mass Wigner variable, eq.(9) obtained in the evolution of the density matrix.

The above equation is solved using Laplace transform, and the operator solution with the initial condition \( q(t = 0) = q(0); \dot{q}(t = 0) = p(0) \) is found to be given by

\[
q(t) = p(0) g(t) + q(0) \dot{g}(t) - \sum_k C_k \int_0^t d\tau Q_k^{(0)}(t - \tau) g(\tau), \tag{18}
\]

where \( g(t) \) is the same function which was defined in the previous section and whose Laplace transform is given by eq.(11).

It is convenient to write

\[
Q_k^{(0)}(t) = \frac{1}{\sqrt{2\omega_k}} \left[ a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t} \right],
\]

so that the integral in the last term in eq.(18) becomes

\[
\int_0^t d\tau Q_k^{(0)}(t - \tau) g(\tau) = \frac{1}{\sqrt{2\omega_k}} \left[ a_k^\dagger e^{i\omega_k t} h(\omega_k, t) + h.c. \right],
\]

where

\[
h(\omega_k, t) = \sin[\omega_k(t - t')] \theta(t - t'),
\]

is the retardation function.
where \( h(\omega, t) \) is defined in eq.(14).

Since the initial density matrix describes a thermal distribution for the quanta of a harmonic oscillator of reference frequency \( \Omega \), it is convenient to write the initial position and momentum operators in terms of the creation and annihilation operators of a quanta of frequency \( \Omega \) as

\[
q(0) = \frac{1}{\sqrt{2\Omega}} [b + b^\dagger] ; \quad p(0) = -i \sqrt{\frac{\Omega}{2}} [b - b^\dagger].
\]

Gathering all terms, \( q(t) \) and \( p(t) \) become

\[
q(t) = \frac{1}{\sqrt{2\Omega}} [b (\dot{q}(t) - i \Omega \dot{q}(t)) + b^\dagger (\dot{b}(t) + i \Omega \dot{b}(t))] - \sum_k \frac{C_k}{\sqrt{\omega_k}} [a_k^\dagger e^{i\omega_k t} h(\omega_k, t) + h.c.],
\]

\[
p(t) = \frac{1}{\sqrt{2\Omega}} [b (\dot{q}(t) - i \Omega \dot{q}(t)) + b^\dagger (\dot{b}(t) + i \Omega \dot{b}(t))] - \sum_k \frac{C_k}{\sqrt{\omega_k}} [a_k^\dagger e^{i\omega_k t} k(\omega_k, t) + h.c.],
\]

where \( h(\omega_k, t) ; k(\omega_k, t) \) are defined in eq.(14). The expectation value of the occupation number operator \( \hat{n}(t) \) in eq.(2) can be evaluated using an initial density matrix which is diagonal in the basis of the bare number operators for system and bath. Assuming a continuum spectrum of the bath oscillators, using eq.(1) and considering for simplicity the case of vanishing expectation values of \( q(0) ; p(0) \) in the initial density matrix, we find

\[
\langle \hat{n}(t) \rangle = -\frac{1}{2} + \frac{1 + 2n(0)}{4\Omega^2} [\ddot{g}^2(t) + 2\Omega^2 \dot{g}^2(t) + \Omega^4 g^2(t)]
\]

\[
+ \frac{1}{2\pi\Omega} \int d\omega J(\omega) \left[ 1 + 2N(\omega) \right] (|k(\omega, t)|^2 + \Omega^2 |h(\omega, t)|^2). \quad (19)
\]

Setting \( q_i = 0 \); \( p_i = 0 \) in the result (13) we find that eq.(19) reduces to the expression obtained by the time evolution of the density matrix (13) and the last term is identified with \( R(t) \).

The operator method allows to compute any correlation function of operators in the initial density matrix at arbitrary times, whereas the time evolution of the density matrix would require the introduction of external sources and taking functional derivatives with respect to those to obtain unequal time correlation functions.

3 Real Time Evolution: \( g(t) \) and \( n(t) \)

Before specifying a choice of the spectral density of the bath \( J(\omega) \) we can obtain more insight by analyzing the real time behavior of \( g(t) \) and consequently of \( \langle n(t) \rangle \) in general. Having
determined the general features of the evolution, we will then specify a particular choice of \( J(\omega) \) and provide a detailed numerical study comparing with different approximations in a later section. In general the spectral density fulfills
\[
J(\omega) = \begin{cases} 
\neq 0 & \text{for } \omega_{th} < |\omega| < \omega_c \\
0 & \text{otherwise}
\end{cases}
\]
where \( \omega_{th} ; \omega_c \) are threshold and cutoff frequencies respectively.

The real time evolution of \( g(t) \) is given by the inverse Laplace transform
\[
g(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{st} \tilde{g}(s) \, ds ,
\]
where \( \tilde{g}(s) \) is given by eq.(11) and \( \Gamma \) refers to the Bromwich contour running along the imaginary axis to the right of all the singularities of \( \tilde{g}(s) \) in the complex \( s \) plane. Therefore we need to understand the analytic structure of \( \tilde{g}(s) \) to obtain the real time dynamics of the particle occupation number.

From the expression (12) for the Laplace transform of the retarded self-energy, we find that \( \tilde{\Sigma}(s) \) has cuts along the imaginary \( s \)-axis for \( s = i\omega ; \omega_{th} < |\omega| < \omega_c \) as can be seen from
\[
\tilde{\Sigma}(s = i\omega \pm 0^+) = \Sigma_R(\omega) \pm i \Sigma_I(\omega)
\]
with
\[
\Sigma_R(\omega) = \frac{2}{\pi} \mathcal{P} \int d\omega' \frac{\omega' J(\omega')}{\omega^2 - \omega'^2} \\
\Sigma_I(\omega) = 2 \text{sign}(\omega) \int d\omega' J(\omega') \omega' \delta(\omega'^2 - \omega^2) \\
= \text{sign}(\omega) J(|\omega|) .
\]

As in field theory, it is convenient to introduce a renormalized frequency by performing a subtraction of the self-energy. Clearly the subtraction point is arbitrary, and we choose to subtract at \( s = 0 \). We thus introduce the renormalized frequency as
\[
\omega_R^2 = \omega_0^2 + \tilde{\Sigma}(s = 0) = \omega_0^2 - \frac{2}{\pi} \int d\omega \frac{J(\omega)}{\omega} ,
\]
and the once subtracted self energy is given by
\[
\tilde{\Sigma}_s(s) = \Sigma(s) - \Sigma(s = 0) = \frac{2}{\pi} \int_0^\infty d\omega \frac{J(\omega)}{\omega} \frac{s^2}{s^2 + \omega^2} .
\]
Isolated poles of \( \tilde{g}(s) \) are at the values \( s_p \) which satisfy
\[
\frac{s_p^2}{2} + \omega_R^2 + \sum s(s_p) = 0.
\]
These are purely imaginary when they are below the threshold of the bath. These correspond to exact stable states of the particle-bath interacting system.

If the imaginary part of the pole (in the \( s \)-variable) \( \omega_p \) is above threshold \( (\omega_p > \omega_{th}) \), then the pole is in the second (unphysical) Riemann sheet and for weak couplings the spectral density \( S(\omega) \) will feature a Breit-Wigner resonance shape where the width of the resonance is related to the imaginary part of the kernel \( \tilde{\Sigma}_S \) and the peak of the resonance is at \( \omega_p \).

The position of these complex poles can be parametrized in terms of real and imaginary parts as
\[
s_p = i\omega_p - \Gamma.
\]

These correspond to unstable states and are not eigenstates of the interacting Hamiltonian. If the width \( \Gamma \ll \omega_p \), these are long-lived resonances and are almost energy eigenstates. These states will be identified with quasiparticles in the next section.

Depending on the strength of the coupling with the environment, \( J(\omega) \), and the value of \( \omega_R \), the imaginary part of the pole, \( \omega_p \), can be above or below the threshold, \( \omega_{th} \).

1) Consider first the case in which the pole is above threshold, i.e. \( \omega_p > \omega_{th} \). Since there are no isolated singularities below threshold, only the cut will contribute to the integral (20). The Bromwich contour \( \Gamma \) in the complex \( s \)-plane is chosen as the one shown in fig. 1 where all the singularities of \( \tilde{g}(s) \) are to the left of the contour. Evaluating the integral along the contour, we obtain
\[
\tilde{g}(t) = \frac{2}{\pi} \int_{\omega_{th}}^{\omega_c} d\omega \ S(\omega) \ \sin(\omega t),
\]
where the spectral density \( S(\omega) \) is given by
\[
S(\omega) = \Sigma_I(\omega) \ |\tilde{g}(s = i\omega + \epsilon)|^2 = \frac{\Sigma_I(\omega)}{[\omega^2 - \omega_R^2 - \Sigma_R(\omega)]^2 + [\Sigma_I(\omega)]^2}. \quad (24)
\]
From the initial condition \( \tilde{g}(0) = 1 \) we find the sum rule
\[
\frac{2}{\pi} \int_{\omega_{th}}^{\omega_c} d\omega \ S(\omega) = 1. \quad (25)
\]
For weak coupling, the spectral density can be approximated by a Breit-Wigner resonance
and asymptotically $g(t)$ is approximately given by\cite{28}
\[ \dot{g}(t) \sim Z \cos(\omega_p t + \alpha) e^{-\Gamma t}; \quad \Gamma \sim \frac{Z \Sigma_{R}(\omega_p)}{2\omega_p}; \quad Z = \left[ 1 - \frac{\partial \Sigma_{R}(\omega)}{\partial \omega^2} \right]^{-1}_{\omega = \omega_p} \quad (26) \]

with $\alpha$ a constant phase-shift\cite{28}. We identify this behavior with a typical quasiparticle which acquires a width through medium effects and whose residue at the quasiparticle pole is smaller than one as a consequence of the overlap between the initial bare particle state and the continuum of the bath. This interpretation will be further clarified when we study the exact normal modes in the next section.

**II** Consider next the case in which there is only a single isolated pole below the cut. In this case, there are two contributions to the integral (20); the pole contribution and the cut contribution. In this case we find
\[ \dot{g}(t) = Z \cos(\omega_p t) + \frac{2}{\pi} \int^{\omega_c}_{\omega_{th}} d\omega \ \omega \ S(\omega) \ \cos(\omega t) , \quad (27) \]
where we define the wave function renormalization $Z$ as in (26) above,
\[ Z = \left[ 1 - \frac{\partial \Sigma_{R}(\omega)}{\partial \omega^2} \right]^{-1}_{\omega = \omega_p} . \quad (28) \]

Asymptotically at long time, the cut contribution vanishes with a power law determined by the behavior of $S(\omega)$ near threshold\cite{28}, and $g(t)$ oscillates with the pole frequency $\omega_p$. Just as in the previous case, the bare particle has been dressed by the medium effects, and to distinguish from the bare or quasiparticle we call this state the dressed particle. The position of the dressed particle pole has been shifted and its residue is smaller than one as a result of the overlap with the continuum of states of the bath.

From the initial condition $\dot{g}(0) = 1$, we derive the important sum rule
\[ Z + \frac{2}{\pi} \int^{\omega_c}_{\omega_{th}} d\omega \ S(\omega) = 1. \quad (29) \]

Both cases of the sum rule (25) and (29) are a consequence of the canonical commutation relations. Since the spectral density $S(\omega)$ is positive semi-definite, the above sum rule determines that $Z \leq 1$.

The expression (27) allows us to explore the concept of the dressing time of the particle. At long times the contribution to $g(t)$ from the continuum vanishes typically as a power law determined by the behavior of the spectral density near threshold\cite{28} and the contribution
from the pole dominates the dynamics. This contribution results in an asymptotic oscillatory behavior of \( \dot{g}(t) \) with an amplitude determined by the residue \( Z \) at the particle pole. The formation time can be defined to be the time it takes for the amplitude of \( \dot{g}(t) \) to reach its asymptotic value \( Z \) (initially \( \dot{g}(0) = 1 \)). In the case in which the pole is embedded in the continuum (unphysical Riemann sheet) and we deal with quasiparticles, a similar concept can be introduced, now being the formation time of the quasiparticle. There are now two competing time scales: the formation time scale during which the quasiparticle pole dominates the dynamics and the contribution of the continuum becomes subleading, and the relaxation time scale which is determined by the imaginary part of the self energy at the quasiparticle pole, i.e. the width of the resonance. In this case, the two different time scales can only be resolved if they are widely separated which requires that the resonance be very narrow and the exponential relaxation associated with the decay of the quasiparticle allows many oscillations to occur. This condition can be quantified as \( \Gamma / \omega_p << 1 \) which requires a weak coupling to the bath. We will explore these situations numerically in a later section where a particular density of states of the bath will be proposed.

\[ \text{3.1 Asymptotic behavior of } n(t) \]

The asymptotic behavior of \( \langle n(t) \rangle \) is completely determined by the long time dynamics of \( g(t) \). We have shown that \( g(t) \) vanishes asymptotically for poles in the continuum while the contribution from the isolated pole dominates for the case in which the pole is below threshold. We will consider each individual case in detail.

1) \( \omega_p > \omega_{th} \) : in this case the function \( g(t) \) vanishes exponentially at asymptotically long times (26) and the asymptotic behavior of the particle occupation number is given by

\[ \langle n(\infty) \rangle = -\frac{1}{2} + \mathcal{R}(\infty) \]

with

\[ \mathcal{R}(\infty) = \frac{1}{2\pi i \Omega} \int_{\omega_{th}}^{\infty} d\omega \left[ 1 + 2N(\omega) \right] S(\omega) \left( \Omega^2 + \omega^2 \right) , \]

where we used eqs.(22) and (24), recognized the Laplace transform of \( g(t) \) in the long time limit for eq.(15) (using the vanishing of \( g(t) \) at long times)

\[ |h(\omega, \infty)|^2 = |\tilde{g}(s = i\omega + \epsilon)|^2 \]

\[ |k(\omega, \infty)|^2 = \omega^2 |h(\omega, \infty)|^2 . \]
It is clear that the asymptotic value of $\langle n(\infty) \rangle$ is different from the equilibrium occupation number of the bath $N(\omega_p)$.

Suppose that the spectral density $S(\omega)$ can be approximated by a narrow Breit-Wigner resonance with

$$S(\omega) = \frac{Z}{2\omega_p} \frac{\Gamma}{(\omega - \omega_p)^2 + \Gamma^2} \rightarrow \frac{\pi Z}{2\omega_p} \delta(\omega - \omega_p),$$

where

$$\Gamma = \frac{Z \Sigma f(\omega_p)}{2\omega_p}$$

as would be the case for weak coupling. Then the asymptotic occupation number becomes

$$\langle n(\infty) \rangle = Z \left( \frac{\Omega^2 + \omega_p^2}{2\Omega\omega_p} \right) \left[ N(\omega_p) + \frac{1}{2} \right] - \frac{1}{2},$$

which is different from the equilibrium value of the bath. We now see that choosing the reference frequency $\Omega = \omega_p$, the particle thermalizes completely with the bath, i.e. asymptotically the occupation number becomes the one predicted by the canonical ensemble as we will see below in eq.(45). This expression, thus reveals the importance of counting the quasiparticles instead of the bare particles. Even in the weak coupling limit the distribution of bare particles is not thermal whereas the true quasiparticles are described by a thermal distribution (that differs from that of Bose-Einstein by perturbatively small terms).

The asymptotic value of the distribution is approached exponentially. The thermalization time scale is given by $\tau_{th} = 1/2\Gamma$ since it is determined by $g^2(t)$ which is the dependence of the occupation number on the function that determines the real time evolution either of the density matrix or of the Heisenberg operators.

Even when the occupation number is defined in terms of the true ‘in medium’ pole, there will be departures from the Bose-Einstein distribution for non-negligible width $\Gamma$ and when the strength of the pole $Z$ is substantially smaller than one. These corrections will arise in the case of broad resonances and may lead to large departures from the Bose-Einstein distribution. This situation will be explored numerically later.

In the case of a wide resonance, the product $N(\omega) S(\omega)$ is sensitive to the width of the resonance. For bath temperature $T \ll \omega_{th}$ the Bose-Einstein distribution will only probe the tail of the broad spectral density closer to threshold and the product is only sensitive to the threshold behavior of $S(\omega)$[26].
In particular if near threshold \( S(\omega) \approx (\omega - \omega_{th})^\alpha \) then for temperatures \( T < \omega_{th} \) the temperature dependence of the equilibrium abundance of unstable particles in the bath is approximately given by

\[
n(T; t = \infty) - n(0; t = \infty) \approx e^{-\frac{\omega_{th}}{T}} T^\alpha + 1
\]

which reveals threshold corrections to the Boltzmann exponential suppression. This result has been anticipated in[26] within a different context.

In the opposite limit when \( T >> \omega_p \) the product is sensitive to the width of the resonance and the details of the spectral density. Thus in the case of a broad resonance the departures from a Bose-Einstein distribution function for the quasiparticles will be important. Clearly this is the regime in which a Boltzmann approximation could be unreliable.

These corrections originate in off-shell effects that will depend on the particular spectral density of the bath and the coupling between the particle of the bath. We will quantify the corrections for a particular choice of \( J(\omega) \) in a following section.

Moreover, the asymptotic value of the particle number does not depend on the initial condition of the particle; e.g. initial expectation values of position and momentum, temperature or occupation number.

**II) \( \omega_p < \omega_{th} \) :** In this case the asymptotic time dependence of the function \( g(t) \) is completely determined by the isolated pole below the continuum and the function rings with this frequency with asymptotic amplitude determined by the wave function renormalization \( Z \) given by (28). The asymptotic behavior of the particle occupation number defined at a reference frequency \( \Omega \) is now given by

\[
\langle n(\infty) \rangle = -\frac{1}{2} + R(\infty) + \frac{Z^2 \sin^2(\omega_pt)}{2\Omega^2} \left[ p^2 + (\Omega^4 + \omega_p^4)\sigma + \omega_p^4 \right] + \frac{pq_i Z^2 \Omega}{2} \left( \frac{\omega}{\omega_p} - \frac{\omega_p}{\Omega} \right) \sin(2\omega_p t) + \frac{Z^2 \cos^2(\omega_p t)}{2} \left( \frac{p^2}{\Omega} + 2\Omega\sigma + \Omega q_i^2 \right),
\]

where \( R(\infty) \) is the limit value of \( R(t) \). For \( \Omega = \omega_p \), i.e. the position of the dressed particle pole, the asymptotic value of the occupation number obtains the simple form

\[
\langle n(\infty) \rangle = -\frac{1}{2} + R(\infty) + Z^2 \left[ n(0) + \frac{1}{2} \right] + \frac{Z^2}{2\Omega} \left[ p^2 + \Omega q_i^2 \right].
\]

The last term can be identified as the contribution from the expectation values of \( p(0); q(0) \) in the initial density matrix.
Unlike the case in which the pole is in the continuum, the asymptotic value of the particle occupation does depend on how the particle was prepared initially since expression (36) depends on \( p_i, q_i \) and \( n(0) \).

In this case, \( \mathcal{R}(\infty) \) has contributions from both the continuum cut and the isolated pole below the continuum.

In order to compare the results to those obtained from an approximate quantum kinetic equation obtained via a perturbative expansion in the next section, it is useful to obtain an expression for \( \mathcal{R}(0) \) up to first order in \( J(\omega) \). The expression for \( \mathcal{R}(\infty) \) (31) is proportional to the spectral density \( S(\omega) \) given by (24). When the pole is below the continuum, the contribution from the cut is proportional to \( J(\omega) \) and perturbatively small when \( J(\omega) \) is small. Furthermore the continuum contribution dephases rapidly at long times, and asymptotically the relevant contribution to \( g(t) \) arises from the isolated pole. After some straightforward algebra we find for \( \Omega = \omega_p \) that at long times

\[
|k(\omega, t)|^2 + \Omega |h(\omega, t)|^2 = Z^2 \left\{ \frac{1 - \cos(\omega + t)}{\omega^2} + \frac{1 - \cos(\omega - t)}{\omega^2} \right\}
\]

with

\[
\omega_{\pm} \equiv \Omega \pm \omega
\]

and to lowest order in \( J(\omega) \), the asymptotic contribution \( \mathcal{R}(\infty) \) is given by

\[
\mathcal{R}(\infty) = \frac{Z^2}{2\pi \Omega} \int d\omega \: J(\omega) \left[ 1 + 2N(\omega) \right] \left( \frac{1}{\omega^2_{+}} + \frac{1}{\omega^2_{-}} \right) + O(J^2),
\]

which for easier comparison with the results from kinetics, can be written in the following form

\[
\mathcal{R}(\infty) \approx \frac{1}{2} (1 - Z^2) + \frac{Z^2}{\pi \Omega} \int d\omega \: J(\omega) \left\{ \frac{1 + N(\omega)}{\omega^2_{+}} + \frac{N(\omega)}{\omega^2_{-}} \right\} + O(J^2),
\]

where the term \( (1 - Z^2) \approx 2(1 - Z) \) and we have used the sum rule (29) to lowest order.

Setting \( p_i = q_i = 0 \) in eq.(36), the asymptotic occupation number becomes

\[
\langle n(\infty) \rangle = Z^2 \left[ n(0) + \frac{1}{\pi \Omega} \int d\omega J(\omega) \left\{ \frac{1 + N(\omega)}{\omega^2_{+}} + \frac{N(\omega)}{\omega^2_{-}} \right\} \right] + O(J^2).
\]

We have purposely kept \( Z \) in the above expression to compare it to the results from the quantum kinetics approximation to be obtained later.

Clearly this result depends on the initial distribution of the particle and the details of the spectral density of the bath, leading to the conclusion that in the case in which the particle pole is real (below threshold), the particle does not thermalize with the bath.
4 Collective Normal Modes and Quasiparticles

In a many body problem, the poles of the exact two particle Green’s functions are identified with the collective modes. In general the poles are complex resulting in the damping of the collective excitations. We can make contact with this many body concept by studying the normal modes of the total Hamiltonian for the particle-bath system under consideration.

Since the Hamiltonian is quadratic, it can be diagonalized by a canonical transformation in terms of the normal modes. In order to establish a correspondence with the continuum distribution of bath oscillators it is convenient to write the Hamiltonian in the continuum form

\[
H = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \frac{1}{2} \int_{\omega_{th}}^{\omega} d\omega \left[ P^2(\omega) + \omega^2 Q^2(\omega) \right] + q \int_{\omega_{th}}^{\omega} d\omega \; C(\omega) \; Q(\omega)
\]

\[
J(\omega) = \pi \frac{C^2(\omega)}{2\omega} .
\]

The Hamiltonian of this rather simple model can be diagonalized by finding the normal modes. Let us write the linear change coordinates and momenta (canonical transformation) to the normal modes as [17, 26]

\[
q = S_\lambda \alpha(\lambda) Q(\lambda) \quad ; \quad p = S_\lambda \alpha(\lambda) P(\lambda)
\]

\[
Q(\omega) = S_\lambda \beta(\omega, \lambda) Q(\lambda) \quad ; \quad P(\omega) = S_\lambda \beta(\omega, \lambda) P(\lambda),
\]

where the symbol \( S_\lambda \) stands for the sum over the discrete and integral over the continuum normal mode eigenvalues \( \lambda \) that render the Hamiltonian in diagonal form

\[
H = \frac{1}{2} S_\lambda \left[ P^2(\lambda) + \lambda^2 Q^2(\lambda) \right] .
\]

The vectors \( V(\lambda) = (\alpha(\lambda), \beta(\omega, \lambda)) \) obey the normal mode eigenvalue equation which in components reads

\[
\omega_0^2 \alpha(\lambda) + \int_{\omega_{th}}^{\omega} d\omega \; C(\omega) \; \beta(\omega, \lambda) = \lambda^2 \alpha(\lambda)
\]

\[
C(\omega) \alpha(\lambda) + \omega^2 \beta(\omega, \lambda) = \lambda^2 \beta(\omega, \lambda)
\]

and the \( \lambda \)'s are the exact eigenenergies of the Hamiltonian.
Solving for $\beta(\omega, \lambda)$ in terms of $\alpha(\lambda)$ in eq. (42) and inserting the solution back into (41) we find the solution for the coefficients and the secular equation for the eigenvalues to be given by

$$\beta(\omega, \lambda) = \frac{C(\omega) \alpha(\lambda)}{(\lambda - i\epsilon)^2 - \omega^2} + B \delta(\lambda - \omega)$$

$$\left[ \chi^2 - \omega_0^2 - \frac{2}{\pi} \int_{\omega_{th}}^{\omega_c} d\omega \frac{\omega J(\omega)}{(\lambda - i\epsilon)^2 - \omega^2} \right] \alpha(\lambda) = B \ C(\lambda) \ ,$$

where we used ‘retarded’ boundary conditions (with the $i\epsilon$ prescription) to establish contact with the previous results, and $B$ is determined by normalizing the eigenstates.

There are two distinct possibilities: I) an isolated pole below the continuum threshold of the bath corresponding to a dressed stable particle, II) a continuum of states and a quasiparticle pole in the unphysical Riemann sheet (resonance).

I) Isolated poles: The condition for isolated poles below the bath continuum requires setting $B = 0$ since the spectrum of the bath has no support below threshold. The position of the pole is found from the secular equation

$$\omega_p^2 - \omega_0^2 - \frac{2}{\pi} \int_{\omega_{th}}^{\omega_c} d\omega \frac{\omega J(\omega)}{\omega_p^2 - \omega^2} = 0 \ .$$

This expression is identified as the condition for isolated poles in the Laplace transform $\tilde{g}(\omega)$ (see eq. (11)) for $s = i\omega_p$. The value of $\alpha(\omega_p)$ is determined from normalization and we find

$$\alpha(\omega_p) = \sqrt{Z}$$

with $Z$ the wave function renormalization given by eqs. (21) and (28). Normalization of the vectors is equivalent to the sum rule (29).

II) Continuum states:

For the continuum states we take $B = 1$ so that $Q(\lambda) \to Q(\omega)$ when $C(\omega) \to 0$ and we find the coefficients

$$\alpha(\lambda) = \frac{C(\lambda)}{(\lambda - i\epsilon)^2 - \omega_0^2} - \frac{2}{\pi} \int_{\omega_{th}}^{\omega_c} d\omega \frac{\omega J(\omega)}{(\lambda - i\epsilon)^2 - \omega^2}$$

$$\beta(\omega, \lambda) = \delta(\lambda - \omega) + \frac{C(\omega) \alpha(\lambda)}{(\lambda - i\epsilon)^2 - \omega^2} \ .$$

In this case the normalization results in the sum rule given by eq. (25).
Because of our choice of boundary conditions, the coefficients are complex and the resulting new coordinates are not Hermitian, this can be remedied by absorbing the phases by a trivial canonical transformation and defining the coefficients in terms of their absolute values. This phase carries the information of the boundary conditions (the \( ie \) prescription) and since it is removed by a canonical transformation the results are independent of these.

Let us consider the case of an isolated pole below the threshold of the bath continuum at \( \lambda = \omega_p \). This state is the one that evolves from the bare particle degree of freedom upon adiabatically switching-on the system-bath couplings \( C_k \) and is identified with the position of the isolated pole in the Laplace transform of the function \( g(t) \) given by (11).

Separating the contribution from the isolated pole we write

\[
q(t) = \sqrt{Z} Q_0(t) + Q_{\text{cont}}(t) \\
p(t) = \sqrt{Z} P_0(t) + P_{\text{cont}}(t),
\]

where the operators \( Q_{\text{cont}} \) : \( P_{\text{cont}} \) create excitations in the continuum of the bath out of the exact ground state. Writing \( Q_0 \) : \( P_0 \) in terms of creation and annihilation operators of the exact eigenstates, we see that asymptotically long times the operators \( q(t) \) : \( p(t) \) create an exact one dressed particle state out of the exact vacuum. In the limit of asymptotically long times and invoking the Riemann-Lebesgue lemma

\[
q(t)|0> \rightarrow \frac{\sqrt{Z}}{\sqrt{2\omega_p}} e^{\omega_p t}|1_p>,
\]

where \( |0> \) is the exact ground state and the contribution from the continuum states averages to zero at long times by the dephasing between modes.

The operator

\[
A_q^+(t) = \frac{1}{\sqrt{2\omega_p Z}} \left[ \omega_p q(t) + ip(t) \right]
\]

asymptotically at long times creates a dressed particle state with unit residue out of the exact vacuum. At any finite time the state created by this operator is not an eigenstate of the full Hamiltonian but has overlap with states in the continuum. We associate the operator (43) with dressed particles in the case of isolated poles or quasiparticles for resonances, in contrast to the normal (collective) modes of the system that are exact eigenstates.

Although a priori one would be tempted to define the dressed particle as the normal mode of frequency \( \omega_p \) associated with the creation and annihilation operators obtained from
the normal mode described by $Q_0$; $P_0$, these are of little use: these operators represent linear combinations of the particle and the degrees of freedom of the bath. Obviously the number operator associated with this normal mode is constant in time. Furthermore, in the case of a resonance, there is no operator that creates or destroys a quasiparticle and that can be understood as the limit of a bare particle operator upon adiabatic switching on of the interaction. Thus the interpolating operator (43) is the natural candidate for counting.

In an experimental situation such as for example a lepton in a plasma, or an electron in a metal, one would like to write down an evolution equation for the distribution function that describes the particle dressed by the medium effects. The interpretation of the quasiparticle creation operator is consistent with this physical situation since the added particle will move in the bath being dressed by the interaction with the medium, the resulting quasiparticle will have a new dispersion relation (given here by $\omega_p$) and in general a width, and the probability associated with this quasiparticle pole will be reduced by the overlap with the states of the bath. Whereas the collective modes in the bath are stationary states, this quasiparticle is not because it overlaps with the collective modes and its time evolution involves dephasing.

In the case in which the pole at $\omega_p$ has a value larger than the threshold for the bath oscillators, it has moved into the second (unphysical) Riemann sheet upon adiabatically switching-on the interaction and is no longer part of the eigenspectrum of the Hamiltonian. In this case it has become an unstable state and $\omega_p$ will have a small negative imaginary part given by $\Gamma$ [see eq.(26)]. In this case the overlap with the continuum results in an almost exponential decay of the quasiparticle distribution after the short formation time of the quasiparticle.

We then see that the interpolating number operator

$$\hat{n}_{\text{quasi}}(t) = \frac{1}{2\omega_p Z} \left[ p^2(t) + \omega_p^2 q^2(t) \right] - \frac{1}{2Z}$$

(44)

can be interpreted as either the dressed particle distribution function in the case of an isolated pole below the continuum of the bath or the quasiparticle distribution function in the case of a resonance. Besides setting the reference frequency $\lambda \equiv \omega_p$ in eq.(2) the wavefunction renormalization factor $Z$ accounts for the strength of the particle or quasiparticle pole.

**Interpretation of results:**

This analysis in terms of normal modes reveals several features of the exact solutions obtained in the previous sections.

- **Thermalization of resonances:** in the case in which the quasiparticle pole is above threshold, the asymptotic value of the quasiparticle distribution given by eqs.(30)-(31)
is a consequence of thermalization. Indeed by using the expansion of $q : p$ in terms of the normal mode coordinates and momenta given by eqs. (39)-(40) it is straightforward to prove that

$$< n_{\text{quasi}}(\infty) > = Tr \left[ \hat{n}_{\text{quasi}}(0) e^{-\beta H} \right]$$

with $\hat{n}_{\text{quasi}}(0)$ the quasiparticle number operator (44) at the initial time $t = 0$. This is a remarkable result: the density matrix, which initially was of a factorized form for particle and bath at different temperatures has evolved in time to the equilibrium density matrix for the total system at the temperature of the bath. However the distribution of quasiparticles is not given by the Bose-Einstein form. Furthermore, the contribution to $\hat{n}_{\text{quasi}}(\infty)$ that does not vanish as $T \to 0$ can be interpreted as a zero point contribution from the resonance. In the case in which the quasiparticle becomes a narrow resonance we see from eqs. (34) and (44) that

$$< n_{\text{quasi}}(\infty) > = N(\omega_p) + \frac{1}{2}(1 - \frac{1}{Z})$$

and the number of quasiparticles departs from a Bose-Einstein distribution at the temperature of the bath with the departure determined by the off-shell effects that result in $Z \neq 1$ through the sum rules. The last term, identified above with the zero point contribution is interpreted as the normalization borrowed from the continuum by the quasiparticle. Although in this simple case $Z$ does not depend on temperature and the last term in (46) can be subtracted out as a redefinition of the quasiparticle vacuum, in a general field theory, the wave function renormalization will be medium dependent and such subtraction would be unjustified.

- **Non-Thermalization of stable particles:** In the case in which the particle pole is below threshold the asymptotic oscillations in the expression (35) for $\Omega \neq \omega_p$ are a consequence of the interference between the state of arbitrary frequency $\Omega$ and the normal mode with frequency $\omega_p$. These oscillations disappear when the reference frequency ($\Omega$) is chosen to be the normal mode pole frequency ($\omega_p$) which is also the particle frequency, this fact has already been noticed within a different context[16]. The factor $Z^2$ in eq.(36) has the following origin: asymptotically at long times $q(t) \to \sqrt{Z}Q_0$; $p(t) \to \sqrt{Z}\mathcal{P}_0$ in the sense of matrix elements. But the $Q_0$; $\mathcal{P}_0$ create particle states out of bare states with amplitude $\sqrt{Z}$, therefore one of the factors $Z$ in eq.(36) arises from the asymptotic (weak) limit on the operators, and another factor $Z$ arises because the calculation of eq.(36) was performed in terms of the bare states overlap with the particle states given by the wave function renormalization. Using the expansion in terms of normal modes we find that $< \hat{n}_{\text{quasi}}(\infty) >$ given by eq.(44) does not coincide with

$$Tr \left[ \hat{n}_{\text{quasi}}(0) e^{-\beta H} \right]$$

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5 Kinetics

Having provided an analysis of the exact evolution of the distribution function and distinguished between that of particles and quasiparticles, we now proceed to obtain kinetic equations in several stages of approximation to compare with the exact results. Kinetic equations are obtained by truncating the hierarchy of Schwinger-Dyson equations under certain assumptions. The typical assumptions are those of slow relaxation as compared to the microscopic time and length scales and rely on a separation of scales, or a multi-time scale problem. To warrant this separation between scales clearly a perturbative parameter must be invoked and the resulting kinetic equations provide a resummation of the perturbative expansion. Different type of approximations result in different resummation schemes.

5.1 Boltzmann Equation

The simplest kinetic equation to describe the approach to equilibrium is the Boltzmann equation, which is obtained by writing a gain minus loss balance equation in which energy is conserved and using Fermi’s Golden Rule. Writing $q(t)$ and $Q_k(t)$ in terms of creation and annihilation operators, we identify the only terms that can contribute by energy conservation: the ‘gain’ term in which one quanta of the system’s oscillator is created and one quanta of the oscillator with label $k$ is annihilated, minus the ‘loss’ term in which a quanta of the system’s oscillator is annihilated and a quanta of the oscillator of label $k$ in the bath is created. The first term has probability given by

$$\text{gain} = \frac{C_k^2}{4\omega_0 \omega_k} (1 + n) N_k.$$  

The second term has a probability

$$\text{loss} = \frac{C_k^2}{4\omega_0 \omega_k} (1 + N_k) n.$$  

Thus using Fermi’s Golden Rule we find the usual Boltzmann equation

$$\langle \dot{n}_B(t) \rangle = (2\pi) \sum_k \frac{C_k^2}{4\omega_0 \omega_k} [(1 + n(t)) N_k - (1 + N_k) n(t)] \delta(\omega_k - \omega_0) \rightarrow$$

$$\int d\omega \frac{J(\omega)}{\omega_0} [(1 + n(t)) N(\omega) - (1 + N(\omega)) n(t)] \delta(\omega - \omega_0),$$

(47)
where we have assumed that the bath remains in thermal equilibrium with constant distribution functions. The solution is clearly

\[
\langle n_B(t) \rangle = N(\omega_0) + [n(0) - N(\omega_0)] e^{-\Gamma_B t} \quad \Gamma_B = \frac{J(\omega_0)}{2\omega_0} = \frac{\Sigma_I(\omega_0)}{2\omega_0},
\]

where we recognize the lowest order (Born approximation) to the decay rate which is given by eqs.(22) and (26). Obviously the Boltzmann equation predicts no relaxation in the case in which the pole is below the continuum threshold since in this case \( J(\omega_0) = 0 \).

Even when the bare frequency is in the continuum of the bath, the Boltzmann approximation predicts no relaxation if \( n(0) = N(\omega_0) \) as the gain and loss processes exactly balance.

As we will see explicitly numerically below the exact solution shows a non-trivial time dependence in this case because the bare particle is dressed by the medium and the asymptotic equilibrium distribution function reveals off-shell effects as discussed in the previous section.

### 5.2 Quantum Kinetic Equation:

The quantum kinetic equation is obtained by taking the expectation value of the number operator using the Heisenberg equations of motion and truncating the exact equations of motion within a particular approximation.

Since we want to obtain the kinetic equation for the relaxation of the distribution function of particles with frequency \( \Omega \) (for quasiparticles this is the pole frequency of the propagator, for bare particles it is simply \( \omega_0 \)) it is convenient to write the total Hamiltonian in terms of this frequency adding a counterterm of the form

\[
H_{ct} = \frac{\delta \omega^2}{2} q^2(t) \quad ; \quad \delta \omega^2 = \omega_0^2 - \Omega^2.
\]

As usual the counterterm is chosen appropriately in perturbation theory to cancel the contributions recognized as those arising from a shift in the frequency.

Taking the derivative of eq.(2) and using the equations of motion we obtain

\[
\dot{n}(t) = -\frac{1}{\Omega} \left\{ \sum_k C_k \mathcal{Q}_k(t) \dot{q}(t) + \frac{\delta \omega^2}{2} [\dot{q}(t) \dot{q}(t) + \dot{q}(t) q(t)] \right\}.
\]

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The expectation value of the time derivative of the occupation number is calculated by multiplying eq.(49) by \( \rho(0) \) and taking the trace

\[
\langle \dot{n}(t) \rangle = -\frac{1}{\Omega} \frac{d}{dt^\prime} \left\{ \sum_k C_k \langle q^+(t^\prime) Q_k^- (t) \rangle + \frac{\delta \omega^2}{2} \langle q(t)q(t^\prime) + q(t^\prime)q(t) \rangle \right\}_t^0, \tag{50}
\]

where

\[
\langle q^+(t^\prime) Q_k^- (t) \rangle = \text{Tr} \left[ q(t^\prime) \rho(0) Q_k(t) \right].
\]

We need to evaluate the non-equilibrium matrix element \( \langle q^+(t^\prime) Q_k^- (t) \rangle \). This can be achieved by treating the interaction term in perturbation theory. The zeroth order term in the perturbative series does not contribute because the initial density matrix commutes with the number operator at the initial time.

A simple diagrammatic analysis of the perturbative series reveals that the kinetic equation can be written exactly as

\[
\langle \dot{n}(t) \rangle = -\frac{1}{\Omega} \sum_k C_k \frac{d}{dt^\prime} \left\{ \int_0^t dt^\prime \left( \Sigma_k^- (t, t^\prime) \mathcal{G}^>(t^\prime, t) - \Sigma_k^+ (t, t^\prime) \mathcal{G}^<(t^\prime, t) \right) \right\}_t^0 + \frac{\delta \omega^2}{2} \left( \mathcal{G}^>(t, t^\prime) + \mathcal{G}^<(t, t^\prime) \right)_t^0,
\]

where \( \mathcal{G}^{<,>}_k \) are the exact Green’s functions for the system, defined similarly to those of the bath eq.(3) and \( \Sigma_{k}^{<,>} \) are the irreducible self-energy components, again defined similarly to eq.(3).

To first order in the interaction we use the free-field propagators and the lowest order contribution to the self-energy. It is straightforward to show that the counterterm contribution vanishes to this order and eq.(50) becomes

\[
\langle \dot{n}(t) \rangle = \frac{i}{\Omega} \sum_k C_k^2 \frac{d}{dt^\prime} \left[ \int_0^\infty dt^\prime \left( \langle q^+(t^\prime) q^+(t^\prime) \rangle \langle Q_k^- (t) Q_k^- (t^\prime) \rangle - \langle q^+(t^\prime) q^- (t^\prime) \rangle \langle Q_k^+ (t) Q_k^- (t^\prime) \rangle \right) \right]_t^0. \tag{51}
\]

Substituting the non-equilibrium Green’s functions from eq.(3) in the right hand side of eq.(51), taking the derivative with respect to \( t^\prime \) and arranging terms, we obtain

\[
\langle \dot{n}(t) \rangle = \frac{1}{\pi \Omega} \int_0^t dt^\prime \int d\omega J(\omega) \left\{ [1 + n(0) + N(\omega)] \cos [(\Omega + \omega)(t - t^\prime)] + [N(\omega) - n(0)] \cos [(\Omega - \omega)(t - t^\prime)] \right\} , \tag{52}
\]

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where \( n(0) \) is the distribution of quanta for the particle at the initial time and \( N(\omega) \) are the Bose-Einstein distributions of the bath which will be taken to be constant and given by eq.(4).

We now propose a scheme that provides a resummation of the perturbative series by replacing the initial distribution \( n(0) \) by self-consistently updating the distribution inside the integral in eq.(52) by replacing \( n(0) \rightarrow n(t') \). It will be shown explicitly below that this prescription leads to a Dyson summation of particular Feynman diagrams and the case where \( n \) is constant is understood as the lowest order term in this expansion. Within non-relativistic many-body quantum kinetics, this approximation is known as the generalized Kadanoff-Baym ansatz[29, 12]. The validity of this approximation in the weak coupling limit is confirmed by comparing the resulting evolution of the distribution function to the exact result obtained in the previous sections.

This resummation scheme has been recently invoked to study relaxation in the case of gauge theories in the hard-thermal loop limit[16] and to provide a resummation scheme that incorporates non-perturbatively the effects of instabilities in the photon production mechanism during the chiral phase transition [30]. The quantum kinetic equation is then given by

\[
\left\langle \dot{n}_{qk}(t) \right\rangle = \frac{1}{\pi \Omega} \int_0^t dt' \int d\omega \ J(\omega) \left\{ [1 + n(t')] + N(\omega) \right\} \cos \left[ (\Omega + \omega)(t - t') \right] \\
+ [N(\omega) - n(t')] \cos \left[ (\Omega - \omega)(t - t') \right].
\]

(53)

The resulting linear kinetic equation, eq.(53), can now be solved via Laplace transforms. The Laplace transform of \( \langle n_{qk}(t) \rangle \) is given by

\[
\tilde{n}_{qk}(s) = \frac{n(0) + \frac{1}{\pi \Omega} \int_{\omega_{th}}^{\omega_{ch}} d\omega J(\omega) \left\{ \frac{1 + N(\omega)}{s} \right\}}{s - \frac{1}{\pi \Omega} \int_{\omega_{th}}^{\omega_{ch}} d\omega J(\omega) \left\{ \frac{\omega_+}{s^2 + \omega_+^2} - \frac{\omega_-}{s^2 + \omega_-^2} \right\}},
\]

(54)

where \( n(0) \) is the initial occupation number of the particle and \( \omega_\pm \) are given by eq.(37). The dynamics of the occupation number of the particle is obtained by taking the inverse Laplace transform along the Bromwich contour. The analytic structure of \( \tilde{n}_{qk}(s) \) consists of cuts along the imaginary axis in the \( s \)-plane and poles. For the pole contributions, we distinguish two cases:

Case I: Poles in the continuum. In this case there are two poles: 1) a pole where the denominator of eq.(54) vanishes, i.e.

\[
s_p - \frac{1}{\pi \Omega} \int d\omega J(\omega) \left\{ \frac{s_p}{s_p^2 + \omega_+^2} - \frac{s_p}{s_p^2 + \omega_-^2} \right\} = 0.
\]

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For weak coupling, one can solve for the pole, $s_p$, in perturbation theory and one can show that the pole is given by (up to first order in $J(\omega)$)

$$s_p = -\frac{J(\Omega)}{\Omega} = -2\Gamma,$$

where we used the fact

$$\lim_{s \to 0} \frac{s}{s^2 + \omega_\pm^2} = \pi\delta(\omega_\pm),$$

and $\Gamma$ is given by eq.(33). The contribution from this pole vanishes exponentially for long times. 2) There is a second pole at $s = 0$ and the residue of this pole, using eq. (55), is $N(\Omega)$. The average occupation number is then given by the contribution of the two poles and the cut

$$\langle n(t) \rangle = N(\Omega) + \text{(residue at } s_p) e^{-2\Gamma t} + \text{ contribution from the cut}. $$

Asymptotically, the contribution from the last two terms vanish and the particle occupation number goes to the equilibrium occupation of the bath with frequency $\Omega$. Comparing the above result for $\Omega = \omega_p$ with the one obtained exactly in the small coupling regime, eq. (34), we see that they differ by a factor of order $J(\omega)$ which can be compensated for by considering higher orders in deriving the kinetic equation. Thus we see that for weak coupling ($\omega_p \approx \omega_0$), the solution of the quantum kinetic equation approaches that of the Boltzmann approximation given by eq.(48).

**Case II :** Poles below the continuum. Since $J(\omega_p)$ vanishes for poles below the continuum, there is only one pole at $s = 0$. The average occupation number is given by the sum of the residue of the pole and the cut contribution. At long times the cut contribution vanishes at least as a power law[28] and the asymptotic average occupation number is given by

$$\langle n_{\phi}(\infty) \rangle = \frac{n(0) + \frac{1}{\pi\Omega} \int_{\omega_{th}}^{\infty} d\omega \ J(\omega) \left\{ \frac{1 + N(\omega)}{\omega_+^2} + \frac{N(\omega)}{\omega_-^2} \right\}}{1 - \frac{1}{\pi\Omega} \int_{\omega_{th}}^{\infty} d\omega \ J(\omega) \left\{ \frac{1}{\omega_+^2} - \frac{1}{\omega_-^2} \right\}}.$$

The denominator of the above equation can be simplified considerably becoming simply $Z^{-2}$. The above equation is now written as

$$\langle n_{\phi}(\infty) \rangle = Z^2 \left[ n(0) + \frac{1}{\pi\Omega} \int_{\omega_{th}}^{\infty} d\omega \ J(\omega) \left\{ \frac{1 + N(\omega)}{\omega_+^2} + \frac{N(\omega)}{\omega_-^2} \right\} \right].$$

Comparing the above result with the one obtained exactly in the small coupling regime, eq.(38), we see that the two results coincide.
Obviously this quantum kinetic equation includes contributions from intermediate states that do not conserve energy and therefore provide off-shell corrections. Although for the case in which the quasiparticle pole is in the continuum, we see that asymptotically at long times the distribution becomes similar to that obtained in the Boltzmann approximation with the same relaxation rate, but at early times the solution of the quantum kinetic equation differs appreciably from the Boltzmann solution in that the relaxation rate vanishes at the initial time, whereas it is a constant for Boltzmann. The vanishing of the relaxation rate at the initial time is a consequence of the fact that the initial density matrix is diagonal in the number representation, thus whereas the quantum kinetic equation describes correctly the initial evolution, the Boltzmann equation has coarse grained over these time scales and misses the early time behavior.

5.3 Markovian approximation:

If the particle occupation number varies on time scales larger than the memory of the kernel in the kinetic equation, a Markovian approximation may be reasonable. In the Markovian approximation, the particle occupation number \( n(t') \) in eq.(53) is replaced by \( n(t) \) and taken outside the integral. This approximation would be justified in a weak coupling limit, in this case when the spectral density of the bath \( J(\omega) \) includes a small coupling (as it will be specified in the next section) \( \eta \). The rational behind this approximation is the realization of multi-time scales: a microscopic or short time scale given by \( t \approx 1/\omega_p, \approx 1/\omega_h \) and another relaxation or long time scale \( t_1 \approx \eta t \).

Thus in the Markovian approximation, eq.(53) becomes

\[
\left\langle n(t) \right\rangle = \frac{1}{\pi \Omega} \int d\omega J(\omega) \left\{ \frac{(1 + N(\omega)) \sin(\omega t_+)}{\omega_+} + \frac{N(\omega) \sin(\omega t_-)}{\omega_-} \right\} \\
+ \frac{n(t)}{\pi \Omega} \int d\omega J(\omega) \left\{ \frac{\sin(\omega t_+)}{\omega_+} - \frac{\sin(\omega t_-)}{\omega_-} \right\}.
\]

(56)

A computational advantage of this equation is that it provides a local update procedure. A connection with the Boltzmann approximation is made with a second stage of approximation, known in the Boltzmann literature as the ‘completed collision approximation’ and consists in taking the limit \( t \to \infty \) in the arguments of the sine functions in eq.(56). Using the limiting distribution

\[
\lim_{t \to \infty} \frac{\sin[\omega t_\pm]}{\omega_\pm} = \pi \delta(\omega_\pm)
\]

which is used in the derivation of Fermi’s Golden Rule, and noticing that only \( \omega_- \) could
vanish leads to the Boltzmann expression

\[
\langle n(t) \rangle = \frac{J(\Omega)}{\Omega} [N(\Omega) - n(t)]
\]

which coincides with the Boltzmann equation obtained above [eq.(47)] within the limit of validity of the perturbative expansion since perturbatively \( \Omega \approx \omega_0 \) if \( \Omega \) is taken as the position of the quasiparticle pole.

6 Numerical Analysis

In order to compare the particle number relaxation \( n(t) \) between the exact results eq.(13) and the various approximations to the kinetic description eq.(53), Boltzmann and Markovian, we have solved numerically for a particular choice of the spectral density of the bath.

We will choose the following model for \( J(\omega) \)

\[
J(\omega) = \eta (\omega - \omega_{th}) \theta(\omega - \omega_{th}) \theta(\omega_c - \omega).
\]  

This is a generalization of the Ohmic bath in which \( J(\omega) \) vanishes for frequencies below a threshold frequency \( \omega_{th} \) and above a cutoff frequency \( \omega_c \), and \( \eta \) is a coupling parameter. This is the simplest spectral density of the bath that allows us to model important features of a field theory and illuminates the main aspects of relaxational dynamics.

This form of the spectral density for the bath has been motivated by previous studies of decoherence and dissipation in similar model theories[17]-[23, 24, 26], including model descriptions of entropy production and decoherence in heavy ion collisions[25]. It is the simplest realization that allows us to vary parameters and investigate the different regimes for the phenomena discussed in the previous section. By varying the coupling \( \eta \) and the value of the bare (or renormalized) frequency we can test the different scenarios.

For the case of the quasiparticle pole embedded in the continuum the dimensionless parameter that determines the separation of time scales is given for the spectral density (57) by

\[
\frac{\Gamma}{\omega_p} \approx \frac{\eta}{2\omega_p^2}(\omega_p - \omega_{th})
\]

When this ratio is \(< 1\) the resonance is rather narrow and there are many oscillations before the decay, the time scales are widely separated. In the other limit when this ratio \(\approx 1\) the particle is strongly coupled to the bath, resulting in a wide resonance and a potential for
large off-shell effects including effects related to the proximity of the peak of the resonance to the threshold.

For $J(\omega)$ given by eq.(57), the real and imaginary parts of $\Sigma(\omega) = i\omega + \epsilon^+$ are given by

$$\Sigma_R(\omega) = \eta \frac{\pi}{\pi} \left\{ (\omega_{\text{th}} + \omega) \log \left[ \frac{\omega_{\text{th}}(\omega + \omega_c)}{\omega_c(\omega_{\text{th}} + \omega)} \right] + (\omega_{\text{th}} - \omega) \log \left| \frac{\omega_{\text{th}}(\omega_c - \omega)}{\omega_c(\omega_{\text{th}} - \omega)} \right| \right\}$$

$$\Sigma_I(\omega) = \eta (\omega - \omega_{\text{th}}) \theta(\omega - \omega_{\text{th}}) \theta(\omega_c + \omega_{\text{th}}).$$

The dynamical function $g(t)$ satisfies the equation of motion of the particle, eq.9. In terms of the renormalized frequency $\omega_R$ given by eq.23, $g(t)$ can be shown to satisfy the following equation

$$\ddot{g}(t) + \omega_R^2 g(t) + \frac{2}{\pi} \int_0^t dt' \int d\omega \frac{J(\omega)}{\omega} \cos \omega(t - t') \dot{g}(t') = 0; \ g(0) = 0; \ \dot{g}(0) = 1.$$

We scale our results to an arbitrary unit of frequency and refer all dimensionful quantities to this unit since the important physical quantities are dimensionless ratios (such as $\omega/T$ etc).

Now we study different scenarios in detail.

Figure 2 shows the case for which the dressed particle pole is below threshold. In this case the Boltzmann equation predicts that no relaxation occurs because the imaginary part of the self-energy evaluated on shell (damping rate) vanishes. The exact solution, and the quantum kinetic approximation along with the Markovian limit all predict non-trivial relaxation, and for this weak coupling case all agree to within few percent. Obviously in this case the relaxation is solely due to off-shell effects since the dissipative effects associated with processes that conserve energy (on-shell) vanish. The inset of the figures shows the dynamics of dressing of the particle and the time scales predicted by the exact result are well reproduced by both the quantum kinetic equation and its Markovian approximation.

In contrast, fig.3 shows the case for which the quasiparticle pole is in the continuum but with a narrow width $\Gamma/\omega_p \approx 0.02$. The bath temperature is fixed $T = 100$ and the initial temperature of the particle $(T_0)$, is varied in a wide range. We notice that in the case in which the temperature of the bath and that of the bare particle are the same, the Boltzmann equation predicts no relaxation because the gain and loss processes balance exactly, this is the straight line in the graph for bare particle temperature $T_0 = 100$. The exact solution as well as the kinetic and Markovian approximation predict relaxation, the kinetic and the Markovian approximations are very close to the exact expression. Analytically we know that the exact, Markovian and kinetic will asymptotically approach the Boltzmann result.
(with very small corrections) in this very narrow width case. Obviously the time scales for relaxation and the early time dynamics are features not reproduced by the Boltzmann equation and clearly a result of off shell effects, since all of the energy conserving detailed balance processes are contemplated by the Boltzmann equation.

Figure 4 compares two situations: the left figures correspond to the case of a pole just slightly below (dressed particle) and the right figures just slightly above threshold (quasiparticle). This case provides for strong renormalization effects because the wave function renormalization departs significantly from one. The left figure for \( \hat{g}(t) \) depicts clearly the dressing time of the particle, with \( \hat{g}(0) = 1 \) we see that after a short time the asymptotic value \( \hat{g}(t) \approx Z \cos(\omega_p t) \) is achieved. This figure thus reveals two time scales, one associated with the oscillation scale of the dressed particle \( 1/\omega_p \) and the other associated with the decay to the asymptotic form, this time scale determines the dressing time of the particle and for the case under consideration corresponds to just a few oscillations. This dressing time scale clearly depends on the details of the spectral density since it determines the early time dynamics after the preparation of the initial state. The right figure for \( \hat{g}(t) \) presents three different time scales: initially there is the time scale of formation of the quasiparticle, very similar to the left figure, the time scale associated with the quasiparticle pole \( \approx 1/\omega_p \) and finally the time scale associated with the exponential decay. The formation time scale and that of exponential decay can only be resolved in the narrow width approximation, in this particular example \( \Gamma/\omega_p \approx 0.005 \) and the time scales associated with the quasiparticle formation from the initial state and exponential relaxation can be resolved. These are clearly displayed in fig.5 where the logarithm of the maxima of \( \hat{g}(t) \) is plotted versus time. In fig.6 we show the expectation value of the number operator eq.(2) for \( \Omega = \omega_p \) for the same values of the parameters as in fig.4 (left figure corresponds to pole below threshold and right figure to the pole above threshold) and equal particle and bath temperature \( T_0 = T = 10 \). Whereas the Boltzmann equation predicts again no relaxation, in the left figure because the damping rate vanishes and in the right figure because the on-shell gain and loss processes balance each other, the exact and quantum kinetics description of relaxation both predict non trivial evolution of the dressed particle and quasiparticle distribution functions respectively. The left figure shows that whereas the quantum kinetic and Markovian evolution are not too different from the exact, asymptotically all of them depart significantly from Boltzmann. The early time dynamics predicted by the Markovian and quantum kinetics are very close to the exact expression. In the right figure, corresponding to a narrow resonance we see that asymptotically the quantum kinetic and Markovian evolution asymptotically approach the Boltzmann result but obviously the early and intermediate time dynamics is remarkably different. Furthermore the exact result reaches an asymptotic value that is very different from Boltzmann, as a result of the strong quasiparticle renormalization effects, with \( Z \) departing significantly from one [see fig. 4]. Despite the fact that the resonance is rather narrow, its proximity to threshold results in strong off-shell effects.
Figure 7 is perhaps one of the most illuminating. The parameters are the same as for the right part of fig.4, i.e. with the quasiparticle pole in the continuum and close to threshold, the bath temperature is $T = 10$ and the particle is initially at zero temperature. In the left figure we plot the Boltzmann, exact, quantum kinetic and Markovian evolutions respectively for the expectation value of the number operator (2) for $\Omega = \omega_p$, whereas the right figure corresponds to dividing by $Z$ the results of the exact, quantum kinetics and Markovian evolutions to make contact with the quasiparticle number operator (44). This figure clearly shows that the Boltzmann approximation coarse grains over the early time behavior and completely misses the formation time scales and the early details of relaxation.

Finally, fig.8 presents the evolution of the quasiparticle distribution for a case of a strong coupling regime resulting in a wide resonance: $\eta = 5; \omega_{th} = 5.0; \omega_c = 40; \omega_p = 9.58; Z = 0.982$ for a bath temperature $T = 200$ and zero initial particle temperature. The ratio $\Gamma/\omega_p \approx 0.1$ is comparable to that of a realistic vector meson, such as for example the neutral $\rho$ meson. The inset in the figure displays the early time behavior. We see in this figure that whereas the early time behavior is similar for the exact and approximate evolutions, which is a consequence of zero initial temperature for the particle, at times of the order of the relaxation time there is a dramatic departure. Furthermore the Boltzmann approximation predicts a very different early time evolution because it coarse grains over the formation time of the quasiparticle.

Whereas the quantum kinetic evolution and its Markovian approximation track very closely the Boltzmann, the exact evolution is approximately 15% smaller resulting in a smaller population of resonances asymptotically. The departures in the exact result are a consequence of off-shell effects associated with a large width of the resonance, since in the narrow resonance approximation and for $Z$ so close to one, the asymptotic limit of the exact solution coincides with that of the Boltzmann equation.

7 Conclusions and implications

The goal of this article is to study the dynamics of thermalization including off-shell effects that are not incorporated in a Boltzmann description of kinetics. In particular the focus is to assess the validity of the Boltzmann approximation as well as non-Markovian and Markovian quantum kinetic descriptions of relaxation and thermalization in a model that allows an exact treatment.

Although the model treated in this article allows an exact solution and therefore provides an arena to test the regime of validity of several approximate descriptions of kinetics and compare to an exact result, it is obviously not a full quantum field theory. Specific field
theory models used to obtain a microscopic description of thermalization and relaxation will certainly contain details that are not captured by the model investigated here. However, from the exhaustive analysis in this article we believe that some of the results obtained here are fairly robust and transcend any particular model. These are the following:

- **Boltzmann vs. quantum kinetics:** A necessary criterion for the validity of a Boltzmann approach is that there is a clear and wide separation of time scales between the microscopic time scales and the time scales of relaxation. This is typically the situation in which quasiparticles correspond to very narrow resonances in the spectral functions and their lifetime is much longer than the typical microscopic scales. If perturbation theory is applicable and the quasiparticle resonance is narrow and its position is far away from thresholds, then a Boltzmann description is likely to be reliable for time scales longer than the formation time of the quasiparticle. When there is competition of time scales or the early stages are physically relevant a full quantum kinetic equation must be obtained.

- **Microscopic time scales:** In order to determine the microscopic time scales the first step is to determine the position of the resonances or quasiparticles, i.e. the quasiparticle pole including the medium effects. The bare particle poles do not determine the microscopic time scales. Obviously for weakly interacting theories the position of the bare and quasiparticle poles will be very close and the microscopic time scales are similar.

- **Relaxational time scales:** An estimate of the relaxational time scale is determined by the width of the resonance, $\Gamma$, a wide separation of time scales that would provide a necessary condition for the validity of a Boltzmann approximation would require that $\Gamma/\omega_p << 1$.

- **Thresholds:** Although a wide separation of time scales is a necessary condition for the validity of a Boltzmann approach, it is not sufficient. In particular when the position of the resonance is too close to threshold, there will be important corrections to the long and short time dynamics arising from the behavior of the spectral density at threshold. Threshold effects can lead to strong renormalization of the amplitude of the quasiparticle pole (wave function renormalization) that results in sizable distortions of the equilibrium distributions as compared to the free particle ones. In particular we have seen how thermalization is achieved but with large corrections in the quasiparticle distribution functions from the usual Bose-Einstein form.

- **Formation times, Markovian vs. non-Markovian kinetics:** The model that we have studied allowed us to explore the concept of the formation time of a quasiparticle. This concept is simply unavailable within a Boltzmann approach, since the
Boltzmann equation coarse grains over the formation time scales. This is clearly revealed in figures 5 and 7. Both the non-Markovian quantum kinetics and its Markovian approximation include off-shell effects and capture the early time dynamics associated with the formation of the quasiparticle. The formation time of a quasiparticle becomes relevant if the initial state is very far from equilibrium, since in a non-linear evolution, large initial departures can result in large corrections in the asymptotic region. An important message learned in this work is that even in strongly coupled cases a non-Markovian quantum kinetic description provides a very good approximation to the correct dynamics for most of the relevant time scale. A Markovian approximation that is obtained by extracting the distribution functions from inside the non-local time integrals, but without taking the interval of time to infinity offers a viable description, which is close to the exact evolution and that of the non-Markovian quantum kinetics at weak and intermediate couplings. Its main advantage is computational because this approximation provides a local update equation.

• **Implications for Field Theory:** The lessons learned in this article allow us to provide some sound implications for realistic field theoretical models. In particular for example in the dynamics of thermalization of the quark-gluon plasma, we are led to the conclusion that a Boltzmann description of thermalization could be reliable for hard partons for which i) the perturbative (medium corrected) scattering cross sections and ii) the initial parton distribution seem to lead to a separation between the (short) microscopic scale (determined by the inverse of the hard momentum of the parton) and the relaxation time scale for partons with typical energies larger than say about 1 GeV\(^5\).

For soft degrees of freedom we envisage strong departures from a Boltzmann description and the necessity of a quantum kinetic description. This view is supported by recent studies of anomalous relaxation of quasiparticles in ultrarelativistic plasmas[15, 16, 30] in which infrared and threshold effects lead to very different relaxational dynamics as compared to the predictions based on quasiparticle resonances[15]. In this case, as advocated in ref.[16] a quantum kinetic description begins by resumming the perturbative expansion for the time evolution of the distribution function for the quasiparticles and keeping memory effects in a full non-Markovian description. A Markovian approximation that leads to local update equations can be obtained in weak coupling but without invoking energy conservation in intermediate states. This Markovian approximation has been shown in this article to be very close to the non-Markovian quantum kinetics. Unlike a fully coarse grained Boltzmann description, this intermediate Markovian equation describes rather well the initial stages of quasiparticle formation and offers a local update alternative to non-local kinetics which includes off-shell effects.

We expect large corrections to Boltzmann relaxation in the case of wide hadronic
resonances such as for example vector mesons near the chiral phase transition. For typical vector mesons both effects that lead to strong corrections are present: the ratio of the width of the resonance to the position of the resonance (quasiparticle pole) is not much smaller than one, for example in the case of the $\rho$ vector meson is about 0.2 in vacuum, furthermore in medium, the position of the resonance is not too far from thresholds. In these cases we conclude that a non-Markovian quantum kinetic description will be needed to understand reliably the abundance of resonances in the medium and their thermalization time scales. Departures from Boltzman abundances in medium could lead to experimentally observable effects in dilepton and photon spectra from hadronic resonances for example.

As presented in this article and advanced in previous work[16, 30], a non-Markovian, quantum kinetic description can be provided from a microscopic field theory model by beginning with perturbation theory, and resumming the perturbative expansion including off shell effects. The details of this program in a full quantum field theory applied to the case of resonance relaxation will be provided in future work.

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Figure 1: The complex contour used to evaluate $g(t)$ for the cases in which a) the pole is below the threshold and b) the pole is above threshold.
Figure 2: The expectation value of the particle occupation number given by eq.(2) for $\Omega = \omega_p$ for the case in which the pole is below threshold for different particle temperatures $T_0 = 100, 200$ and 10 and bath temperature $T = 100$. The pole is at $\omega_p = 1.95719$ and $Z = 0.95621$. The numerical parameters are $\eta = 0.85$, $\omega_c = 45$, $\omega_{th} = 5$ and $\omega_R = 2$. 
Figure 3: The expectation value of the particle occupation number given by eq.(2) for $\Omega = \omega_p$ for the case in which the pole is above threshold for different particle temperatures $T_0 = 100, 200$ and 10 and bath temperature $T = 100$. The pole is at $\omega_p = 9.83397$ and $Z = 0.99631$. The numerical parameters are $\eta = 0.85$, $\omega_c = 45$, $\omega_{th} = 5$ and $\omega_R = 10$, resulting in $\Gamma/\omega_p \approx 0.02$. 
Figure 4: The functions $S(\omega)$ and $\dot{g}(t)$ for the cases in which the pole is just below (left column) and just above (right column) the threshold frequency ($\omega_{th} = 5$). In the left column, $\omega_R = 5.52$, $\omega_p = 4.98083$ and $Z = 0.63845$ while in the right column $\omega_R = 5.65$, $\omega_p = 5.07373$ and $Z = 0.69959$. The numerical parameters are $\eta = 3.0$ and $\omega_c = 55$, with $\Gamma/\omega_p \approx 0.005$. 

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Figure 5: The logarithm of the maxima of $\dot{g}(t)$ vs $t$ for $\eta = 3.0$ and $\omega_c = 55$, $\omega_{th} = 5$, $\omega_R = 5.52$, $\omega_p = 4.98083$ and $Z = 0.63845$, corresponding to the right column of fig.4.
Figure 6: The expectation value of the number operator eq.(2) for the same values of the parameters as in figure(4)(left figure corresponds to pole below threshold and right figure to the pole above threshold) for the case of equal particle and bath temperature $T_0 = T = 10$. The insert shows the early time behavior.

Figure 7: The expectation value of the number operator eq.(2) for the same values of the parameters as for the right column in figure(4)(quasiparticle pole above threshold), $\eta = 3.0$ and $\omega_c = 55$, $\omega_{th} = 5$, $\omega_R = 5.65$, $\omega_p = 5.07373$ and $Z = 0.69959$. The temperature of the bath is $T = 10$ and zero initial particle temperature ($T_0 = 0$), the exact, Markovian and kinetic curves have been divided by the wave function renormalization $Z$ in the rightmost figure. The insert in the left figure shows the early time behavior.
Figure 8: The expectation value of the number operator eq. (2) for \( \eta = 5.0 \) and \( \omega_c = 40 \), \( \omega_{ph} = 5 \), \( \omega_p = 9.58 \), \( Z = 0.982 \) and bath temperature \( T = 200 \), corresponding to \( \Gamma / \omega_p \approx 0.1 \).