The central charge in three dimensional anti-de Sitter space

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Abstract

This paper collects the various ways of computing the central charge $c = 3l/2G$ arising in 3d asymptotically anti-de Sitter spaces, in the Chern-Simons formulation. Their similarities and differences are displayed.

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The recently conjectured duality [1] between anti-de Sitter space and conformal field theory has opened the door to many surprising connections between those two theories. An adS/CFT correspondence was first obtained by Brown and Henneaux [2] who proved that the asymptotic symmetry group of anti-de Sitter space in three dimensions is the conformal group with a classical central charge
\[ c = \frac{3l}{2G}. \] (1)
This observation led Strominger [3] (see also [4]) to consider the degeneracy of states with \( L_0 \) and \( \bar{L}_0 \) fixed and he found the surprising result that its logarithm agrees exactly with the Bekenstein-Hawking entropy of a three dimensional asymptotically anti-de Sitter black hole [5]. The connection between CFT and black hole entropy can be seen by looking at the semiclassical form of the free energy \( f \) of a non-rotating three dimensional black hole with inverse temperature \( \beta \) (in units of \( l \)),
\[ -\beta f = -\beta e + \frac{\pi c}{6\beta} \] (2)
where \( e \) is proportional to the black hole mass and \( c \) is given by (1). This coincides with the free energy of a strip of width \( \beta \) (with periodic boundary conditions) in a 1+1 CFT [6] with central charge \( c \).

The Virasoro algebra discovered in Ref. [2] arose by studying the asymptotic Killing equations for anti-de Sitter spacetimes. It was then realized [7] that using the Chern-Simons formulation of 2+1 gravity and its connection to WZW theories, a simple realization of the conformal generators (by means of a twisted Sugawara construction) was possible. The central charge is associated in [7] to diffeomorphisms with a component normal to the boundary. The underlying boundary theory in this approach is a chiral WZW model having a Kac-Moody symmetry. The Virasoro algebra is then a sub-algebra of the Kac-Moody enveloping algebra. Since in this space there is a family of Virasoro operators with different classical central charges, in order to fix \( c \), or equivalently, to fix the relevant twisted Sugawara operator more information is needed. Fixing the Virasoro algebra to leave the asymptotic form of anti-de Sitter space invariant fixes \( c = 6k \) [8], where \( k \) is the level of the Chern-Simons theory. Since the equivalence between the gravitational and Chern-Simons actions fixes \( k = l/4G \), the central charge is then the same as that calculated in [2]. We shall come back to this point below.
A different Chern-Simons formulation for the results of [2] was proposed in [9] by making use of the known reduction from WZW_{SL(2,\mathbb{R})} theory (obtained at the boundary from the Chern-Simons theory) to Liouville theory [10]. Liouville theory has two commuting Virasoro operators with a classical central charge equal to \( c = 6k \), where \( k \) is again the level of the Chern-Simons theory, and the central charge is therefore again the same as that calculated in [2]. It is remarkable that the conditions that reduce WZW_{SL(2,\mathbb{R})} to Liouville theory are indeed satisfied by anti-de Sitter space asymptotically. In this approach the underlying Hilbert space then corresponds (at least ignoring zero modes) to that of a Liouville field and the central charge is uniquely determined to be \( c = 6k \). (This approach has recently been extended to supergravity in [13].)

The existence of a Kac-Moody algebra at the boundary was thought to be useful in giving a microscopical derivation of the entropy of three dimensional black holes [14] (see also [7, 15, 8]) independently of the connection with the recent work of Strominger, and provides a natural arena to explore 2+1 quantum gravity. It is surprising that Carlip's approach to computing the black hole's entropy does not depend so much on the value of \( c \) as on the Kac-Moody structure. Conversely, Strominger's idea [3] depends crucially on the value of \( c \), while the underlying Kac-Moody symmetry seems to be unimportant, and from this point of view can even be interpreted as predicting a state counting that disagrees with the Bekenstein-Hawking result [8, 16]. For this reason, we find it useful to have a clear relation between the boundary theories found in [7] and [9].

This letter is a collection of different known results including elements of [10, 17] (and the work that followed those papers) applied to 2+1 dimensional gravity and its relation to chiral WZW models. Our main goal is to compare and exhibit the similarities and differences between the approaches followed in [7] and [9] to describe the conformal generators and the central charge discovered in [2].

Three dimensional adS gravity can be recast as a \( SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \) Chern-Simons theory [11]. Consider first one \( SL(2,\mathbb{R}) \) [12] Chern-Simons theory formulated on a manifold with the topology of a solid cylinder. We use polar coordinates \( \{r, t, \varphi\} \) on the cylinder. The boundary is located at \( r \to \infty \) and has the topology of a cylinder. On the cylinder we use coordinates \( x^\pm = \pm t + \varphi \). The Chern-Simons action needs boundary conditions on the cylinder to be well defined. We take these boundary conditions from anti-de
Sitter space. Firstly we require that the radial component of the gauge field be given by

\[ A_r = \gamma H \]  

where \( H \) is the generator of the Cartan sub-algebra and \( \gamma \) is a real constant. This condition follows directly from the gauge field representing anti-de Sitter space. Note that this fixes \( (A_r)^2 = \gamma^2/2 \) (in the notation of [7, 8], this is \( \alpha^2 \)) and that \( \gamma \) parametrises a rescaling of the radial coordinate at infinity. Although we could fix \( \gamma \) by working with proper radial coordinates at infinity [8] (the relation between this radial coordinate and the one used in [9] is \( r = \log \tilde{r} \)), we keep an explicit dependence on \( \gamma \) to emphasise the fact that the final central charge is independent of it. Secondly, we impose the condition

\[ A^\alpha_\pm = 0 \]  

which also follows from classical \( \text{adS} \) space. (The corresponding boundary condition for the other \( SL(2, \mathbb{R}) \) copy is \( \tilde{A}_\pm = 0 \).)

It is well known that these boundary conditions lead to a Kac-Moody algebra for the currents \( A_\alpha(x^+) \). The group of gauge transformations leaving (3) and (4) invariant are characterised by parameters \( \lambda(x^+, r) = \exp(-rH)\lambda(x^+)\exp(rH) \) (they do not depend on \( x^- \) and the radial dependence can be gauged away), and are generated by Kac-Moody currents. In the following we shall denote the current \( A_\alpha \) simply by \( A \).

The point raised in [7] is that by considering the subset of gauge transformations with parameters

\[ \lambda^\alpha = -f' H^\alpha + f A^\alpha = -\frac{f'}{\gamma} A_r^\alpha + f A^\alpha \]  

with \( f \) an arbitrary function of \( x^+ \) (prime denotes derivative), the associated algebra is the Virasoro algebra with \( c = 6k \). (Note that the way this formula was presented in [7, 8] might have given the impression that the value of \( c \) depends on \( \gamma \) but this was because the formula given there assumed the use of a proper radial coordinate at infinity.) This can be seen simply by noticing that the associated generator is a twisted Sugawara operator [7]

\[ L = \text{Tr} \left( \frac{1}{2} A^2 + HA' \right) \]  

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with $A^a$ being the $SL(2, \mathbb{R})$ Kac-Moody currents. This is not a coincidence. The parameters of the form (5) can be interpreted as diffeomorphisms via the relation $\lambda^a = \xi^i A^a_i$. Comparing with (5) we find $\xi^i = \{-f'/\gamma, f\}$ which was shown in [8] to be exactly the form of the residual diffeomorphisms found in [2] (up to a radial reparameterisation). We thus arrive at the conclusion that the residual group of diffeomorphisms (asymptotic Killing vectors) found in [2] has a natural action in the connection representation through a twisted Sugawara construction.

However, as stressed above, the symmetry algebra leaving invariant the boundary conditions (3) and (4) is the whole Kac-Moody symmetry within which (6) only generates a sub-algebra of the enveloping algebra which explicitly leaves invariant the asymptotic form of the metric. There is in fact a whole family of Virasoro generators with different central charges which can be obtained by multiplying the first term in (5) by a constant. The particular form of (6) is selected because it generates diffeomorphisms that coincide with the asymptotic symmetries found in [2]. The value of the central charge, $c = 6k$, is thus associated with the invariance properties of the asymptotic metric. In this sense, this procedure can be viewed as re-deriving [2] from the connection point of view (see [8] for a detailed discussion on this point), rather than as using any intrinsic property relating the Kac-Moody and Virasoro algebras.

In [9] a different route was taken to relating the Kac-Moody and Virasoro algebras. First it was proved that the two $SL(2, \mathbb{R})$ chiral WZW models (arising in 2+1 dimensional gravity with negative cosmological constant) can be glued together into a single non-chiral one. It then follows that imposing the extra conditions [10]

$$A^- = 1, \quad \tilde{A}^+ = 1,$$  

plus $A^H = 0 = \tilde{A}^H$ leads to a Liouville theory (up to zero modes contributions that have not been calculated so far, and using the Gauss decomposition to parametrize the group manifold). It is remarkable that the conditions (7) are satisfied asymptotically by the anti-de Sitter gauge field [9]. Liouville theory is conformally invariant and it has two commuting Virasoro operators with a classical central charge $c = 6k$ [10]. This procedure thus fixes the value of $c$ with no ambiguity, although of course the use of (7) in the present context depends on its remarkable relation to the asymptotics of adS$_3$. 

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Another derivation of the two Virasoro operators with \(c = 6k\), without resorting to Liouville theory, is obtained by working directly with each chiral \(SL(2, \mathbb{R})\) WZW model. As shown in [17] the conditions

\[
A^- = 1, \quad A^H = 0, \quad (8)
\]

transmute an affine \(SL(2, \mathbb{R})_k\) algebra into a Virasoro algebra with \(c = 6k\). This can be seen as follows [17]. The parameters of the most general set of gauge transformations leaving (8) invariant have the form (in the basis \(\{J_\pm, H\}\))

\[
\lambda = \begin{pmatrix}
(1/2)f' & fA^+ - (1/2)f'' \\
-1/2 & f
\end{pmatrix}
\]

(9)

where \(f\) is an arbitrary function of \(x^+\). It is straightforward to check (using Dirac brackets or otherwise) that this subset of gauge transformations acting on the remaining component \(A^+\) yields a Virasoro algebra with \(c = 6k\) [17] (see [18] for a generalization). An advantage of this procedure is that no coordinates on the group manifold have been used. The Virasoro generator \(A^+\) is thus well adapted to deal with global issues, although no action has been written for it. Details of this procedure as applied to the supergravity/superCFT correspondence can be found in Ref. [13].

Our last step is to prove that once (8) is imposed, the twisted Virasoro operator (6) reduces to \(A^+\), and that the parameters (5) reduce to (9). In other words, on the constraint surface defined by (8), the quadratic Virasoro operator (6) is equivalent to \(A^+\) and, of course, they satisfy the same algebra. First we need to study the transformations \(\delta A^a = D\lambda^a\) with \(\lambda^a\) given by (5) and see whether they are consistent with the restrictions (8). If they are not consistent, then we need to modify the transformation (5). This is analogous to the reduction from Kac-Moody to Virasoro via (8). One cannot just plug (8) into the Kac-Moody algebra and expect to get the Virasoro algebra for the remaining component \(A^+\). The right thing to do is to construct the Dirac bracket whose dynamics is consistent with the constraints. Then, one finds the Virasoro algebra (in the Dirac bracket) with a non-zero central charge.

The variation of \(A^a\) with respect to the gauge transformation (5) in components is

\[
\delta A^+ = (fA^+)' + A^+ f'
\]

(10)
\[ \delta A^+ = f(A^-)' \]  \hspace{1cm} (11)
\[ \delta A^3 = (fA^3)' - f'' \]  \hspace{1cm} (12)

We see that the condition \( A^- = 1 \) causes no problems since it is preserved by (11) (this can also be seen from the fact that \( A^- \) commutes (weakly) with the modified Sugawara operator). On the other hand \( A^3 = 0 \) is not preserved by (12) due to the \( f'' \) term.

The analogue of constructing the Dirac bracket in this calculation is to modify the parameter (8). Consider gauge transformations with an improved parameter

\[ \lambda_i = -f'H + fA - \frac{1}{2}f''J_+ . \]  \hspace{1cm} (13)

This transformation is a combination of a diffeomorphism and a gauge transformation [8]. Studying the variation of the connection under these improved transformations, one finds

\[ \delta_i A^+ = (fA^+)' + A^+ f' - (f''/2)A^3 - (1/2)f'' \]  \hspace{1cm} (14)
\[ \delta_i A^- = f(A^-)' \]  \hspace{1cm} (15)
\[ \delta_i A^3 = (fA^3)' + f''(-1 + A^-) \]  \hspace{1cm} (16)

We see that the improved transformations are consistent with the constraints (8) and the residual function \( A^+ \) transforms as a Virasoro generator with a classical central term \( c = 6k \). Now, we can see that after imposing (8) the parameter (13) reduces exactly to the residual symmetries (9). Similarly, the Sugawara operator (6) reduces directly to \( A^+ \), as expected.

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References


[12] We use the $SL(2, \mathbb{R})$ basis $\{J_\pm, H\}$ satisfying $[J_+, J_-] = 2H$, $[H, J_\pm] = \pm J_\pm$ and $\text{Tr}H^2 = 1/2$. The $SL(2, \mathbb{R})$ gauge field $A$ is decomposed as $A = A^+ J_+ + A^- J_- + A^H H$.


