On the Phase-Space Volume of Primordial
Cosmological Perturbations

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Abstract

We show how to determine the typical phase space volume Γ for primordial gravitational waves produced during an inflationary stage, which is invariant under squeezing. An expression for Γ is found in the long wavelength regime. The quasi-classical entropy of a pure vacuum initial state defined as the logarithm of Γ modulo a constant remains zero in spite of the generation of fluctuations (creation of real gravitons).

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1 Introduction

As well known, the inflationary scenario offers an elegant solution to some of the outstanding problems of the big bang cosmology like the horizon and the flatness problems. In this scenario primordial scalar (adiabatic) perturbations [1] and primordial gravitational waves [2] are produced as a consequence of the anavoidable quantum fluctuations of some field(s). The scalar (adiabatic) fluctuations eventually lead to the formation of stars, galaxies and the large scale structure in the Universe, while gravitational waves (GW) forms a stochastic waves background that could be in principle presently observable. An intriguing property of the produced fluctuations is their quantum origin. It turns out that the inflationary scenario provides us, in addition to the above mentioned achievements, with a physical system, namely the primordial fluctuations, undergoing a quantum to classical transition solely as a result of its dynamics [4]. We will show that the effective volume in phase-space $\Gamma$ occupied by the primordial GW generated during inflation is constant, it is not the naive product of variances of the canonically conjugate variables $y$ and $p$. The latter definition yields a large phase-space volume for large squeezing and we show that it corresponds to a complete loss of the classical correlation between the quantities $y$ and $p$ for large squeezing.

The determination of the typical phase space volume of cosmological perturbations is closely connected with the question of their entropy. In particular, we use the semiclassical definition for the entropy $S = \ln \Gamma + \beta$, ($\beta = \text{constant}$). Note that generation of fluctuations in the inflationary scenario may be considered as a particular case of particle creation in a Friedmann-Robertson-Walker (FRW) isotropic cosmological model. Pioneering papers on the entropy of particles created in a FRW model [3] define the entropy through the average number of created particles neglecting phase correlation between them at late times (this just results in the transformation of a pure quantum state into a mixed one). After that, there were numerous attempts [5, 6] to apply this approach to cosmological perturbations and to find their entropy either by neglecting the phase correlations or using another ad hoc coarse graining procedure. Though coarse graining and neglection of phases is possible in the case of creation of massive particles which become thermalized at very early stage of the evolution of the Universe, it is not justified in the case of cosmological perturbations which are massless (as GW) or almost massless (as the inflaton field) and very weakly interacting. We have stressed already that the actual phase space volume of perturbations is much less than the one that would arise as a result of such a coarse graining [4] and we want now to consider this question in more detail.

2 Evolution of primordial quantum gravitational waves

We will consider for simplicity primordial GW produced during an inflationary stage [4]. They originate from vacuum fluctuations of the quantized tensorial metric perturba-
tions. Each polarization state $\lambda$, where $\lambda = 1, 2$, and the polarization tensor is normalized to $\epsilon_{ij} \epsilon^{ij} = 1$, has an amplitude $h_\lambda$ given by

$$ h_\lambda = \sqrt{32\pi G \phi_\lambda} \quad (1) $$

where $\phi_\lambda$ is a real massless scalar field. We will omit in the following the polarization subscript $\lambda$ and introduce the rescaled field $y \equiv a \phi$, $a$ being the scale factor. Let us consider the dynamics of a real massless scalar field on a flat FRW universe. Actually, the field can be massive and all the arguments presented here will go through. As a result of the coupling with the gravitational field, which plays here the role of an external classical field, the state $|0, \eta_0\rangle$ which is the vacuum state of the field at some given initial time $\eta_0$ will no longer be the vacuum at later time. Indeed field quanta are produced in pairs with opposite momenta, we get a two-mode squeezed state. Though the state at late time is no longer annihilated by the annihilation operators $a(k)$, it is still annihilated by the following time-dependent operator

$$ \left\{ y(k, \eta_0) + i \gamma_k^{-1}(\eta) p(k, \eta_0) \right\} |0, \eta\rangle_S = 0 \quad (2) $$

Therefore the state at late time is still Gaussian, its wave functional in the infinite volume case (continuous $k$’s) being

$$ \Psi[y(k, \eta_0), y(-k, \eta_0)] = \sqrt{N_k} \exp \left( -\frac{1}{2} \int d^3 k \ \gamma_k \ y(k, \eta_0) y(-k, \eta_0) \right) \quad (3) $$

where $N_k$ is a normalization coefficient. We note the persisting symmetry under reflections in $k$-space, it is present in the initial state and the evolution doesn’t spoil it, we get a two-mode squeezed state.

The dynamics of the system is particularly transparent in the Heisenberg representation. We have, using the field modes $f_k(\eta)$ with $\Re f_k \equiv f_{k1}$ and $\Im f_k \equiv f_{k2}$, $f_k(\eta_0) = 1/\sqrt{2k}$, (we adopt a similar notation for all quantities)

$$ y(k, \eta) \equiv f_k(\eta) \ a(k, \eta_0) + f_k^*(\eta) \ a^\dagger(-k, \eta_0) = f_{k1}(\eta) \ e_y(k) - f_{k2}(\eta) \ e_p(k) \quad (4) $$

and the momentum modes $g_k(\eta)$, $g_k(\eta_0) = \sqrt{\frac{k}{2}}$

$$ p(k, \eta) \equiv -i [g_k(\eta) \ a(k, \eta_0) - g_k^*(\eta) \ a^\dagger(-k, \eta_0)] = g_{k1}(\eta) \ e_p(k) + g_{k2}(\eta) \ e_y(k) \quad (5) $$

The time independent operators $e_p(k)$, resp.$e_y(k)$, satisfy

$$ \langle e_y(k) \ e_y^\dagger(k') \rangle = \langle e_p(k) \ e_p^\dagger(k') \rangle = \delta^{(3)}(k - k') \quad e_{y,p}(k) = e_{y,p}(-k) \quad (6) $$

They obey the commutation relations

$$ [e_i(k), \ e_j^\dagger(k')] = 2i \ \delta_{ij} \ \delta^{(3)}(k - k') \quad , \quad i, j = y, p \quad (7) $$

The operators $e_y$ and $e_p$ correspond to standing, not running quantum waves.
The canonical commutation relations (7) express the fact that the quantities $e_y, e_p$ are operators and that the system is a quantum mechanical one. Furthermore, 
\[
\gamma_k = \frac{1}{2|f_k|^2} (1 - i2F(k)) \quad \text{with} \quad F(k) = f_{k1} g_{k2} - f_{k2} g_{k1},
\]
hence at $\eta = \eta_0$, we start with a minimum uncertainty wave function. The field modes obey the Klein-Gordon equation
\[
f''_k - \frac{a''}{a} f_k + k^2 f_k = 0,
\]
and are constrained to satisfy the condition
\[
2(f_{k1} g_{k1} + f_{k2} g_{k2}) = 1,
\]
which can be viewed as either the Wronskian condition for (9) or the commutation relations imposed by canonical quantization. We note that special interest is attached to the quantity $B(k) \equiv f_k^* g_k$, with
\[
\Re B(k) = \frac{1}{2}, \quad \Im B(k) = F(k).
\]
The evolution of the system is conveniently parametrized by the squeezing parameter $r_k$, the squeezing angle $\varphi_k$ and the phase $\theta_k$ [7]. A crucial property is that for large squeezing, $|r_k| \gg 1$, the phase and squeezing angles do not evolve independently of each other, we can impose instead with a proper choice of the initial conditions [4]
\[
\theta_k + \varphi_k \simeq 0.
\]
Wavelengths much larger than the Hubble radius at the end of the inflationary stage are in a WKB quasi-classical state corresponding to $|F(k)| = \frac{1}{2} \sin 2\theta_k \sinh 2r_k \gg 1$, $|r_k| \gg 1$, $f_k$ almost real and $g_k$ almost imaginary, and use has been made of (13) for the last property. We then get for the primordial GW field modes
\[
\sqrt{2k} f_k \simeq \cos \varphi_k e^{r_k}, \quad \sqrt{2k} |f_k| \gg 1,
\]
\[
\sqrt{2k} g_k \simeq \sin \varphi_k e^{r_k} \approx 0.
\]
The primordial GW dynamics can then be described by two real functions, namely $r_k$ and $\varphi_k$ or equivalently $f_{k1}$ and $g_{k2}$, instead of three. In the limit $|r_k| \to \infty$, the dynamics of the system corresponds to a classical stochastic process: on one hand, $y(k, \eta) = f_k(\eta) e_y(k) \equiv y_c(k)$ where $f_k$, resp. $g_k$, is real resp. imaginary, while $e(k)$ is a time-independent Gaussian stochastic function of $k$ with zero average and unit dispersion; on the other hand, the Fourier transform $p(k, \eta)$ of the momenta take their classical values $p_c(k, \eta)$
\[
p_{c}(k, \eta) = -i \frac{g_k}{f_k} e_y(k) = -i \frac{g_k}{f_k} y(k, \eta)
\]
for each realization of the stochastic field \( y(k, \eta) \) and \( y_{cl}, \) resp. \( p_{cl}, \) is the classical amplitude, resp. momentum, for the corresponding initial conditions when the decaying mode is negligible. A generic outcome of the inflationary stage is the very high squeezing of the primordial GW on cosmological scales till after the last Hubble radius crossing.

3 The phase-space volume \( \Gamma \)

What is the typical volume \( \Gamma \) occupied by the GW in phase-space? This question is closely connected to the problem of the entropy as might be expected. It is instructive to look at it in the Heisenberg representation using eqs. (4,5). According to these equations, in the limit that our system is equivalent to a classical stochastic process, the volume occupied in phase-space should have measure zero! Indeed, while the amplitude is typically found inside a domain \( \delta y \) which is of order of \( \Delta y, \) the variance of \( y, \) for each value of \( y \) there corresponds only one momentum \( p_{cl}(y) \). This can be expressed mathematically in the following way. If we define the conditional probability density \( P(p, \eta \mid y_1, \eta_1) \) for a certain value \( p \) of the momentum \( p \) at time \( \eta \) provided the amplitude has a value \( y_1 \) at time \( \eta_1 \), we find for \( |r_k| \to \infty \)

\[
P_{cl}(p, \eta \mid y, \eta) = \delta(p - p_{cl}(y)),
\]

with \( p_{cl}(y) \equiv \frac{\sqrt{2}}{\hbar} y. \) This is graphically illustrated in Fig.1 where the thin solid circle represents the initial vacuum state, the strongly elongated ellipse is the squeezed vacuum state, and the straight line - the major axis of the ellipse - just gives the approximation of the quantum GW by a classical GW with a stochastic amplitude, a fixed phase and a probability distribution in phase-space given by (17). It is crucial to properly take into account the stochasticity of the operators \( y \) and \( p \) as expressed by the correlation (16) between these operators. Note that this analysis does not depend on a specific squeezed state, only the dynamics of the field modes, namely the vanishing of the decaying mode given by the second term in eq. (26) below, is taken into account. Therefore the conclusions reached apply to more general squeezed states than just the squeezed vacuum and it could apply as well to other squeezed systems possessing a growing and a decaying mode [8].

Equation (17) is in sharp contrast to the phase space volume (we consider here and below our system to be enclosed in a finite volume)

\[
\Delta^2 y \Delta^2 p = \langle y y^\dagger \rangle \langle p p^\dagger \rangle = F^2(k) + \frac{1}{4} = 4 \Delta y_1 \Delta p_1 \Delta y_2 \Delta p_2.
\]

The latter tends to \( |F(k)| \) for \( |F(k)| \gg 1, \) the fact that it becomes very large just reflects the semiclassical limit of the system. It implies in particular that in the Schrödinger representation the probability density \( \rho(y_0, \eta) = |\Psi|^2 \) is spreading (in amplitude space) and that it moves along classical trajectories. Taking the quantity \( \Delta^2 y \Delta^2 p \) for the phase-space volume occupied by the system corresponds to the case
where, for given $y$ inside $\Delta y$ we neglect information, which is available as a result of large squeezing, about the different momenta inside $\Delta p$. We will come back to this important point at the end of this article. In other words, with this definition one is not taking into account the true origin of the stochasticity of $y(k, \eta)$ and $p(k, \eta)$ expressed by (4,5). From the above discussion it turns out that, for $|r_k| \gg 1$, an ansatz of the kind

$$\Gamma \sim \Delta^2 y \Delta^2 P \quad \text{where} \quad P \equiv p - \frac{g_{k2}}{f_{k1}} y,$$  \hspace{1cm} (19)

would be more adequate as it possesses the correct behaviour in the limit $|r_k| \to \infty$. We shall now show that this choice comes naturally out when use is made of the Wigner function.

Indeed, in order to find $\Gamma$ one needs to consider a probability density in phase-space and not just in amplitude space. Of course, a quantum system does not (and cannot) possess a true probability density in phase-space, one can at best construct some candidates which have basic desired properties. The Wigner function [9] is such a well-known example. It is not positive definite in general but it has this property for any initial coherent state and in particular for an initial vacuum state as these are Gaussian states. Therefore, we will take the Wigner function as the probability density of our system in phase-space. The following Wigner function is obtained for any initial coherent state (the subscript 0 applies to quantities at time $\eta_0$ and we insert here $\hbar$ back)

$$W \propto N_k^2 e^{-\frac{|y_0 - \langle y \rangle|^2}{2|f_k|^2}} \left| \frac{F(k)}{|f_k|^2} \right|^2 \delta \left( p_0 - \frac{F(k)}{|f_k|^2} y_0 \right) \delta \left( p_{02} - \frac{F(k)}{|f_k|^2} y_{02} \right).$$  \hspace{1cm} (20)

Note that this Wigner function is calculated with the exact wavefunction [10]. For the initial vacuum state $\langle y \rangle = \langle p \rangle = 0$ at all times, the (product of) Gaussians in amplitude space is the probability density $\rho(y_{01}, y_{02})$ with variances

$$\sigma_{y_i}^2 = \frac{1}{2} |f_k|^2 = \Delta^2 y_i, \quad i = 1, 2,$$  \hspace{1cm} (22)

while the Gaussians in momentum space are centered around $\frac{F(k)}{|f_k|^2} y_i$ with variance $\sigma_{p_i}$ given by

$$\sigma_{p_i}^2 = \frac{1}{8 |f_k|^2}  \neq \Delta^2 p_i, \quad i = 1, 2.$$

We note that $\sigma_{p_i}^2$ is independent of $y_i$. From (21) it is natural to define $\Gamma$ as being of the order of the product of variances

$$\Gamma \sim \sigma_{y_1} \sigma_{p_1} \sigma_{y_2} \sigma_{p_2} = \frac{1}{16},$$  \hspace{1cm} (24)

with some (unimportant) proportionality constant. Hence the typical volume occupied by the system in phase-space is invariant under squeezing, it is conserved and of
the order of the minimal value of the quantity \( \Delta y_1 \Delta y_2 \Delta p_1 \Delta p_2 \) allowed by the uncertainty principle. It is certainly negligible compared to \( F^2(k) \gg 1 \). Note that this corresponds to \( \Delta^2 y_1 \Delta^2 p_1 = \Delta^2 y_2 \Delta^2 p_2 = \frac{1}{4} \) where \( y_{1,2}, p_{1,2} \equiv 2p_{1,2} \) are canonically conjugate. From the Wigner function given above we see that the Gaussian in \( p \) space multiplying \( |\Psi|^2 \) must be interpreted as the conditional probability \( P(p, \eta \mid y, \eta) \) introduced earlier. In the limit of infinite squeezing eq. (17) is indeed recovered as is seen from (21).

If we define now the entropy \( S \) as we would do in classical statistical mechanics, we get

\[
S = - \int \int d\Gamma W \ln(\alpha W) = \ln \Gamma + \beta, \tag{25}
\]

with \( d\Gamma \equiv 4 \, dy_{01} \, dy_{02} \, dp_{01} \, dp_{02} \). It is seen that the entropy is the familiar expression in terms of \( \Gamma \), it is constant under squeezing and can be put equal to zero by a proper choice of \( \alpha \) as is desirable for any pure state. The entropy must remain zero under squeezing which just describes the unitary time evolution of our pure state. Therefore the idea to connect the entropy with the volume in phase-space using a formula like (25) is fruitfull, however \( \Gamma \neq \Delta p_1 \Delta y_1 \Delta p_2 \Delta y_2 \).

For \( k \ll aH \), when the GW are in the quasi-static regime, the GW modes have the asymptotic expression

\[
\tilde{f}_k = a \tilde{\phi}_k = C_1 a + C_2 a \int_{\eta}^{\eta'} a^{-2} d\eta', \quad C_1 \Im C_2 = -\frac{1}{2} \tag{26}
\]

where \( C_1 \) can be made real and positive, both \( C_1 \) and \( C_2 \) are constants for a given wavelength. We have further

\[
P = \frac{1}{2f_{k_1}} e_p \tag{27}
\]

\[
|\tilde{r}_k| \to \infty \quad g_{k_1} e_p = -\frac{\Im C_2}{a} e_p. \tag{28}
\]

The Wigner function then becomes

\[
W \propto |\Psi(y_0)|^2 \exp \left(-\frac{|P_0|^2}{\Delta_p^2} \right). \tag{29}
\]

A crucial point is that while

\[
\Delta^2 y \Delta^2 p \simeq f_{k_1}^2 g_{k_2}^2 \langle e_y^2 \rangle \simeq 1, \quad k \ll aH, \tag{30}
\]

we have on the other hand

\[
\Delta^2 y \Delta^2 p \simeq f_{k_1}^2 g_{k_2}^2 \langle e_y^2 \rangle \langle e_p^2 \rangle \simeq \frac{1}{4}, \tag{31}
\]

\[
\simeq \Delta^2 \tilde{\phi} \Delta^2 \tilde{p}_{\tilde{\phi}} \tag{32}
\]

\[
\simeq C_1^2 |\Im C_2|^2. \tag{33}
\]

Note that \( p_{\tilde{\phi}} \) is solely associated with the decaying mode. Equation (30) depends only on the growing mode and shows explicitly that \( \Delta^2 y \Delta^2 p \) cannot be a true measure for
the volume occupied in phase-space and as said above the fact that this expression becomes very large has little to do with $\Gamma$. In contrast, equation (31) makes use of both growing and decaying modes and is therefore a relevant quantity in this respect. The presence of the decaying mode may induce a quantum “signature” in the power spectrum $|f_k|^2$, as it avoids the presence of zeroes in the spectrum [11]. However, this effect is very small since the spectrum is essentially determined by the growing mode. In contrast, the role played by the decaying mode here is crucial, once it is neglected the phase-space volume $\Gamma$ vanishes. If it is taken into account, $\Gamma$ is constant even though vanishingly small compared to $\Delta^2y \Delta^2p$.

4 Summary and discussion

We have shown that the typical phase-space volume $\Gamma$ of primordial GW generated during an inflationary stage remains finite under squeezing as seen in eq.(24) and we give an expression for it in the long wavelength regime, eqs (31-33). It is enlightening to note in this connection that the canonical commutation relations just express Liouville’s theorem on the conservation of volume in phase-space for the underlying classical system. Indeed classical GW will also satisfy

$$g_k f_k^* + g_k^* f_k = 1,$$

(34)
a fact which just expresses Liouville’s theorem on the conservation of volume in phase space for Hamiltonian dynamics. Were the primordial GW classical and stochastic ab initio, they would also occupy a constant typical volume in phase space. For the primordial GW of quantum origin generated by inflation, the fact that $\Delta^2 y \Delta^2 p$ remains finite and constant is just an expression of quantum coherence between the growing and decaying modes, so both of them are needed in order to determine the phase-space volume.

It was further shown that the quasi-classical entropy can be defined as the logarithm of $\Gamma$ modulo a constant, eq.(25) so that this entropy vanishes at all times for our pure initial vacuum state. Finally, we would like to stress the observational implications of the coarse graining procedure which is adopted for the calculation of the entropy. In a number of papers [5, 6] the entropy of cosmological perturbations was calculated assuming that as a result of the interaction of a quantum perturbation, which was initially in the pure vacuum state, with the environment (or as a result of coarse graining), non-diagonal elements of the density matrix of the perturbation calculated in some particular basis quickly become zero. Formal neglection of non-diagonal elements of the density matrix in both $N$-particles and coherent states basis leads to the same value of the entropy $S_k \simeq 2|r_k|$ for $|r_k| \gg 1$ for each mode $k$. However, as a result of the procedure adopted, the correlation (16) gets completely lost. In Fig.1, the neglection of non-diagonal elements corresponds to the replacement of the ellipse by a large circle with radius equal to the major semi-axis of the ellipse. When (16) holds, we have trajectories of measure zero in phase-space, so that the perturbations have a fixed temporal phase after the last Hubble radius crossing. The latter is a crucial property. In the case of adiabatic perturbations, as is well known, it
leads to the appearance of Sakharov oscillations in the power spectrum of matter and to multiple acoustic, or Doppler, peaks in the multipole spectrum $C_l$ of the microwave background temperature anisotropies at small angular scales. The latter are now the subject of intense investigation in connection with the future satellite missions, like COBRAS-SAMBA, which will be able to measure these peaks.

We believe that a more physical coarse graining should be used. It should be based on a realistic model of the interaction of the perturbations with the environment. A simple toy model is depicted in Fig.1 by the dashed circle which represents a stochastic GW background emitted by matter after the last Hubble radius crossing. This background does not experience squeezing. Therefore, it has both stochastic amplitudes and stochastic phases. The radius of the dashed circle is much bigger than the radius of the thin solid circle. As a result, for the sum of the two GW backgrounds - squeezed primordial and non-squeezed secondary - the noise in all directions in phase-space turns out to be much bigger than for the vacuum state. Thus, as we pointed out in [4], such a state is not “squeezed” anymore from the observational point of view. On the other hand, the radius of the dashed circle is much less than the major semi-axis of the ellipse, so all predictions about $rms$ values of perturbations and their essentially fixed temporal phase remain unaltered. When applied to adiabatic perturbations, such a model does not destroy the multiple acoustic peaks of the $C_l$ multipoles. The entropy defined as the logarithm of the effective phase-space volume $\Gamma$ for the sum of these two backgrounds, though non-zero anymore, is still much smaller than $2|r_k|$ as is clear from Fig.1. Work on this issue is currently under progress.

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