Quantum Fields in Curved Spacetime:
Quantum-Gravitational Nonlocality and Conservation of Particle Numbers

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Abstract

We argue that the conventional quantum field theory in curved spacetime has a grave drawback: The canonical commutation relations for quantum fields and conjugate momenta do not hold. Thus the conventional theory should be denounced and the related results revised. A Hamiltonian version of the canonical formalism for a free scalar quantum field is advanced, and the fundamentals of an appropriate theory are constructed. The principal characteristic feature of the theory is quantum-gravitational nonlocality: The Schrödinger field operator at time $t$ depends on the metric at $t$ in the whole 3-space. It is easily comprehended that the canonical commutation relations may be fulfilled only if that nonlocality takes place. Applications to cosmology and black holes are given, the results being in complete agreement with those of general relativity for particles in curved spacetime. A model of the universe is advanced, which is an extension of the Friedmann universe; it lifts the problem of missing dark matter. A fundamental and shocking result is the following: There is no particle creation in the case of a free quantum field in curved spacetime; in particular, neither the expanding universe nor black holes create particles.

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Introduction

The conventional quantum field theory in curved spacetime \([1,2]\) is based on the following representation of a scalar quantum field: \(\phi(p) = \sum_j \{f_j(p)a_j + f_j^*(p)a_j^\dagger\}\). Here \(p\) is a point of spacetime manifold; \(\{f_j\}\) and \(\{f_j^*\}\) are complete sets of positive and negative norm solutions to the Klein-Gordon, or generalized wave equation \((\Box + m^2)\chi = 0\); \(a_j\) and \(a_j^\dagger\) are annihilation and creation operators, \([a_j, a_j^\dagger] = 0\), \([a_j, a_j^\dagger] = \delta_{jj'}\). In the comoving reference frame, where \(p = (t, s)\), the conjugate momentum is \(\pi(p) = \frac{\partial}{\partial t} \phi(p) = \sum_j \{f_j(p)a_j + f_j^*(p)a_j^\dagger\}\).

The canonical commutation relations are: \(\{\phi(s, t), \phi(s', t')\} = 0, \{\pi(s, t), \pi(s', t')\} = 0, \{\phi(s, t), \pi(s', t')\} = i\delta(s, s')\). We obtain for the commutators: \(\{\phi(s, t), \phi(s', t)\} = \sum_j \{f_j(s, t)f_j^*(s', t) - f_j^*(s', t)f_j(s, t)\}, \{\pi(s, t), \pi(s', t)\} = \sum_j \{f_j(s, t)f_j^*(s', t) - f_j^*(s', t)f_j(s, t)\}\). In the generic case of a time-dependent metric, the canonical commutation relations do not hold. The reason is that the wave equation \((\Box + m^2)\phi = 0\) is local with respect to the metric: For a given operator \(\phi(s, t)\), it is possible to obtain an arbitrary operator \(\phi(s', t)\) by choosing an appropriate metric, which results in the violation of the relation \(\{\phi(s, t), \phi(s', t)\} = 0\). The violation leads to disastrous effects: It becomes possible to introduce an absolute notion of simultaneity.

Thus the conventional theory should be denounced and the related results revised.

In this paper, a consistent theory for a free scalar quantum field in curved spacetime is advanced. The basic outline of the theory is as follows.

Spacetime manifold is \(M = T \times S\) where \(T\) stands for time and \(S\) for 3-space. In the comoving reference frame, metric \(g\) is of the form \(g(t, s) = dt \otimes dt - h_{ij}, t \in T, s \in S, h_{ij} = h(t, s)\).

First and foremost, the canonical commutation relations must be fulfilled, so that we put in any picture \(\phi(t, s) = \frac{1}{\sqrt{2}} \sum_j \frac{1}{\sqrt{\omega_j(t)}} \{u_j(t, s)a_j + u^*_j(t, s)a_j^\dagger\}, \pi(t, s) = \frac{1}{\sqrt{2}} \sum_j \sqrt{\omega_j(t)}\) \times \{-u_j(t, s)a_j + u^*_j(t, s)a_j^\dagger\}\) where \(\{u_j\}\) is a complete set of functions on \(S\), such that \(u_j, u_{j'} = (u_j, u_{j'}) \equiv \int_S ds \sqrt{(h_t)} u_j^* u_{j'} = \delta_{jj'}, (h) = \det(h_{ik}), u_j^* = u_{p(j)}, p\) is a permutation, and \(\omega_{p(j)} = \omega_j\). Now the canonical commutation relations do hold.

We choose the functions \(u_j\) to be solutions to the equation \(\Delta u_j = -k^2_j u_j, \Delta \chi = \Delta_t(s)\chi = \frac{1}{\sqrt{(h)}} \partial_t [\sqrt{(h)} h^{ik}\partial_k \chi]\), and put \(\omega_j(t) = (m^2 + k^2_j)^{1/2}\). Then the Hamiltonian is \(H_t = \sum_j \omega_j(t)a_j^\dagger a_j\).

The principal characteristic feature of the theory is quantum-gravitational nonlocality: The Schrödinger field operator \(\phi_S(t, s)\) at time \(t\) depends on metric \(h_t(s)\) in the whole 3-space \(S\). In view of the failure of the conventional theory considered above, it is easily comprehended that the canonical commutation relations may be fulfilled only if the nonlocality is involved.

Applications to cosmology and black holes are given. Bearing the relation \(\omega_j = (m^2 + k^2_j)^{1/2}\) in mind, it is apparent that the results are in complete agreement with those of general relativity for particles in curved spacetime. A model of the universe is advanced, which is an extension of the Friedmann universe; the model lifts the problem of missing dark matter.

The expression \(H_t = \sum_j \omega_j(t)N_j, N_j = a_j^\dagger a_j\) for the Hamiltonian in the comoving reference frame implies the fundamental and shocking result: There is no particle creation in the case of a free quantum field in curved spacetime; in particular, neither the expanding universe nor black holes create particles.

2
1 Preliminaries

1.1 Spacetime

The employment of the comoving reference frame implies that spacetime manifold $M$ is a trivial bundle [3], so that we assume from the outset that

$$M = T \times S, \quad M \ni p = (t,s), \quad t \in T, \quad s \in S,$$

(1.1.1)

holds, where $T$ is time and $S$ is 3-space.

Metric $g$ in the comoving reference frame is of the form

$$g = g(t,s) = dt \otimes dt - h_{ik}(t,x)dx^i dx^k.$$

(1.1.2)

1.2 Classical field dynamics

The Lagrangian density for a real free scalar field $\varphi$ is

$$\mathcal{L} = \frac{1}{2} \left\{ \partial^\mu \varphi \partial_\mu \varphi - m^2 \varphi^2 \right\} = \frac{1}{2} \left\{ \partial_t \varphi \partial_t \varphi - h^{ik}\partial_i \varphi \partial_k \varphi - m^2 \varphi^2 \right\}.$$

(1.2.1)

The related dynamical equation is the Klein-Gordon, or generalized wave equation,

$$\Box + m^2 \varphi = 0,$$

(1.2.2)

$$\Box = \nabla^\mu \nabla_\mu, \quad \Box \chi = \frac{1}{\sqrt{(h)}} \partial_t \left[ \sqrt{(h)} \partial_t \chi \right] - \Delta \chi, \quad \Delta \chi = \frac{1}{\sqrt{(h)}} \partial_i \left[ \sqrt{(h)} h^{ik} \partial_k \chi \right], \quad (h) = \det(h_{ik}).$$

(1.2.3)

The conjugate momentum is

$$\pi = \dot{\varphi} \equiv \partial_t \varphi.$$

(1.2.4)

The Hamiltonian is

$$H_t = \frac{1}{2} \int_S ds \sqrt{(h_t)} \left\{ \pi^2 + h^{ik} \partial_i \varphi \partial_k \varphi + m^2 \varphi^2 \right\}.$$

(1.2.5)

1.3 Canonical commutation relations

The canonical commutation relations for the quantum field and conjugated momentum operators $\phi$, $\pi$ are of the form

$$[\phi(s,t), \phi(s',t)] = 0,$$

(1.3.1)

$$[\pi(s,t), \pi(s',t)] = 0,$$

(1.3.2)

$$[\phi(s,t), \pi(s',t)] = i \delta_t(s,s'),$$

(1.3.3)

where the delta function $\delta_t(s,s')$ is defined by

$$\int_S ds \sqrt{(h_t)} \delta_t(s,s') \chi(s) = \chi(s'), \quad \int_S ds' \sqrt{(h_t)} \delta_t(s,s') \chi(s') = \chi(s).$$

(1.3.4)
2 The conventional theory and its inconsistency

2.1 The standard quantization

In the conventional theory, the (symplectic [4]) inner product of solutions to the wave equation (1.2.2),

\[ (\varphi_1, \varphi_2)_\Omega = i \int_S ds \sqrt{\hbar} \left\{ \varphi_2^* (s, t) \frac{\partial \varphi_1 (s, t)}{\partial t} - \varphi_1 (s, t) \frac{\partial \varphi_2^* (s, t)}{\partial t} \right\}, \tag{2.1.1} \]

is introduced, the related norm being \((\varphi, \varphi)_\Omega\).

Let \(\{f_j\}\) be a complete set of positive norm solutions to the wave equation (1.2.2); then \(\{f_j^*\}\) will be a complete set of negative norm solutions, and \(\{f_j, f_j^*\}\) form a complete set of solutions. A scalar quantum field \(\phi\) is represented as follows:

\[ \phi(s, t) = \sum_j \{f_j(s, t)a_j + f_j^*(s, t)a_j^\dagger\}, \tag{2.1.2} \]

where

\[ [a_j, a_{j'}] = 0, \quad [a_j^\dagger, a_{j'}^\dagger] = 0, \quad [a_j, a_{j'}^\dagger] = \delta_{jj'}, \tag{2.1.3} \]

and

\[ \frac{da_j}{dt} = 0. \tag{2.1.4} \]

In view of eqs.(1.2.4),(2.1.4), the conjugate momentum is

\[ \pi(s, t) = \dot{\phi}(s, t) = \sum_j \{\dot{f}_j(s, t)a_j + \dot{f}_j^*(s, t)a_j^\dagger\}. \tag{2.1.5} \]

The commutators for \(\phi\) and \(\pi\) take the following form:

\[ [\phi(s, t), \phi(s', t)] = \sum_j \{f_j(s, t)f_j^*(s', t) - f_j(s', t)f_j^*(s, t)\}, \tag{2.1.6} \]

\[ [\pi(s, t), \pi(s', t)] = \sum_j \{\dot{f}_j(s, t)f_j^*(s', t) - \dot{f}_j(s', t)f_j^*(s, t)\}, \tag{2.1.7} \]

\[ [\phi(s, t), \pi(s', t)] = \sum_j \{f_j(s, t)\dot{f}_j^*(s', t) - \dot{f}_j(s', t)f_j^*(s, t)\}. \tag{2.1.8} \]

2.2 The Wald quantization

Let us consider the quantization advanced by Wald [4]. For every solution \(\varphi\) to the wave equation (1.2.2), by eq.([4].3.2.27), the relation

\[ \varphi(f) = \Omega(Ef, \varphi) \tag{2.2.1} \]

holds, where \(\varphi(f)\) is a smeared field, \(Ef\) is a solution to the wave equation, and \(\Omega\) is the symplectic structure given by eq.([4].4.2.6), which is equivalent to the inner product (2.1.1). The smeared Heisenberg field operator ([4].4.2.9) is

\[ \phi(f) = ib^j f_j = i\{a(K(Ef)) - a^\dagger(K(Ef))\}, \tag{2.2.2} \]
where $K(Ef)$ is a vector of the Hilbert space. By eq.([4].3.2.30), the commutation relation
\[ [\phi(f), \phi(g)] = -i\Omega(Ef, Eg) \] (2.2.3)
holds.

We have the following relations:
\[ \phi(f^*) = [\phi(f)]^\dagger = -ib_f, \] (2.2.4)
\[ [b_f, b_g] = i\Omega(Ef^*, Eg^*), \quad [b_f^\dagger, b_g^\dagger] = i\Omega(Ef, Eg), \quad [b_f, b_g^\dagger] = -i\Omega(Ef^*, Eg). \] (2.2.5)

Let, for a complete set \{Ef_j, Ef_j^*\} of the solutions to the wave equation, the relations
\[ \Omega(Ef_j, Ef_j^*) = 0, \quad \Omega(Ef_j^*, Ef_j') = 0, \quad \Omega(Ef_j^*, Ef_j') = i\delta_{jj'}, \quad \Omega(Ef_j, Ef_j^*) = -i\delta_{jj'} \] (2.2.6)
hold; then we obtain
\[ [b_j, b_{j'}] = 0, \quad [b_j^\dagger, b_{j'}^\dagger] = 0, \quad [b_j, b_j^\dagger] = \delta_{jj'}. \] (2.2.7)

We put
\[ \phi = \sum_j \{ (Ef_j)b_j + (Ef_j^*)b_j^\dagger \}, \] (2.2.8)
then
\[ \phi(f_j) = ib_j^\dagger, \quad \phi(f_j^*) = -ib_j, \] (2.2.9)
which corresponds to eqs.(2.2.2),(2.2.4). The representation (2.2.8) is equivalent to the standard representation (2.1.2).

### 2.3 The problem of commutation relations

In view of eq.(2.1.6), the canonical commutation relation (1.3.1) implies that the equality
\[ \sum_j f_j(s, t)f_j^*(s', t) = \sum_j f_j(s', t)f_j^*(s, t) \] (2.3.1)
must hold. Let us write the equality as
\[ \sum_j F_j(s, s') = \sum_j F_j(s', s), \quad F_j(s, s') \neq F_j(s', s). \] (2.3.2)

Generally, we have
\[ F_j(s', s) = F_{p(j)}(s, s') \] (2.3.3)
where $p$ is a permutation, such that
\[ p \circ p = I, \quad p^{-1} = p. \] (2.3.4)

Thus we obtain
\[ f_j(s', t)f_j^*(s, t) = f_{p(j)}(s, t)f_{p(j)}^*(s', t), \] (2.3.5)
whence
\[ \frac{f_j^*(s, t)}{f_{p(j)}(s, t)} = \frac{f_j^*(s', t)}{f_{p(j)}(s', t)} = z_j(t) = z_{p(j)}(t). \] (2.3.6)
For \( s' = s \) we obtain
\[
|f_p(j)|^2 = |f_j|^2, \quad |z_j| = 1, \quad z_j = e^{i2\alpha_j(t)},
\]
so that
\[
f_j(s, t) = e^{-\alpha_j(t)}f_j^0(s, t), \quad f_j^\ast(s, t) = e^{i\alpha_j(t)}f_j^0(s, t), \quad f_j^0 = f_p(j).
\]
In view of eqs.(2.1.8),(1.3.3),
\[
\frac{d\alpha_j}{dt} \neq 0.
\]
Similarly, from eqs.(2.1.7),(1.3.2) we obtain
\[
e^{-i\beta_j}f_j^\ast = -e^{i\beta_j}f_j^0, \quad \beta_j = \beta_j(t),
\]
so that
\[
e^{-i\beta_j} \frac{\partial}{\partial t}\left[e^{i\alpha_j}f_j^0\right] = -e^{i\beta_j} \frac{\partial}{\partial t}\left[e^{-i\alpha_j}f_j^0\right],
\]
whence generally
\[
\beta_j = \alpha_j, \quad \frac{\partial f_j^0}{\partial t} = 0,
\]
and
\[
f_j(s, t) = e^{-i\alpha_j(t)}f_j^0(s).
\]
But if
\[
\frac{\partial h}{\partial t} \neq 0,
\]
solutions to the wave equation (1.2.2) are not of the form
\[
\varphi(s, t) = u(t)v(s).
\]
Thus in the generic case of a nonstationary metric, the canonical commutation relations do not hold. The reason is that the wave equation is local with respect to the metric: For a given operator \( \phi(s, t) \), it is possible to obtain an arbitrary operator \( \phi(s', t) \) by choosing an appropriate metric \( h_t \), which results in the violation of the relation (1.3.1).

Note that the commutation relations (2.1.3) are worthwhile if and only if they imply the canonical commutation relations (1.3.1)-(1.3.3), which is not the case in the conventional theory.

The violation of eq.(1.3.1) leads to disastrous effects. In view of the uncertainty relation
\[
\Delta \Psi \phi(s, t) \Delta \Psi \phi(s', t) \geq \frac{1}{2} |\langle \Psi, [\phi(s, t), \phi(s', t)] \Psi \rangle|,
\]
measuring \( \phi(s', t) \) results in
\[
\Delta \phi(s, t) = \infty,
\]
i.e., in a prodigious quantum nonlocality. This nonlocality makes it possible to synchronize clocks at points \( s \) and \( s' \) in an absolute way or, what is the same, to introduce an absolute notion of simultaneity.

Let \( \phi(s, t) \) be measured quasicontinuously,
\[
\text{for } t \leq t_0 \quad \Psi = \Psi_s, \quad \phi(s, t)\Psi_s = \xi_s(t)\Psi_s,
\]
and \( \phi(s', t) \) be measured at \( t_0 \). Then the value of \( \xi_s \) changes by a jump at \( t_0 \). Note that the effect is absent for \( \partial h/\partial t = 0 \) but does not vanish in the limit \( \partial h/\partial t \to 0 \).

We conclude that the conventional theory should be denounced.
3 Hamiltonian version of the canonical formalism

We shall be based on the Hamiltonian version of the canonical formalism, which is the most reliable [5].

3.1 The Schrödinger picture

In view of eq. (1.2.5), we adopt the Schrödinger Hamiltonian

\[ H_{St} = \frac{1}{2} \int_S ds \sqrt{(h_t)} \left\{ \pi_S^2 + h^k \partial_i \phi_S \partial_k \phi_S + m^2 \phi_S^2 \right\} = \frac{1}{2} \int_S ds \sqrt{(h_t)} \left\{ \pi_S^2 - \phi_S \Delta \phi_S + m^2 \phi_S^2 \right\} \] (3.1.1)

where \( \phi_S \) and \( \pi_S \) are the Schrödinger operators for the field and conjugate momentum respectively.

The standard scalar product is defined by

\[(\chi_1, \chi_2) = (\chi_1, \chi_2)_t = \int_S ds \sqrt{(h_t)} \chi_1^*(s) \chi_2(s). \] (3.1.2)

Let \( \{u_j(s,t)\} \) be a complete set on \( S \) for every \( t \in T \), such that

\[ (u_j, u_{j'})_t = \delta_{jj'}, \] (3.1.3)

\[ u_j^* = u_{p(j)}, \quad u_j = u_{p(j)}^* = u_{(p \circ p)(j)}, \quad p \circ p = I, \quad p^{-1} = p, \quad \sum_p = \sum_j. \] (3.1.4)

We put

\[ \phi_S(s, t) = \frac{1}{\sqrt{2}} \sum_j \frac{1}{\sqrt{\omega_j(t)}} \{ u_j(s, t) a_{jS} + u_j^*(s, t) a_{jS}^\dagger \}, \] (3.1.5)

\[ \pi_S(s, t) = \frac{i}{\sqrt{2}} \sum_j \{ -u_j(s, t) a_{jS} + u_j^*(s, t) a_{jS}^\dagger \}, \] (3.1.6)

with

\[ \omega_{p(j)} = \omega_j, \] (3.1.7)

and

\[ \frac{da_{jS}}{dt} = 0, \quad [a_{jS}, a_{j'S}] = 0, \quad [a_{jS}^\dagger, a_{j'S}^\dagger] = 0, \quad [a_{jS}, a_{j'S}^\dagger] = \delta_{jj'}. \] (3.1.8)

We find in any picture

\[ [\phi(s, t), \phi(s', t)] = [\phi_S(s, t), \phi_S(s', t)] \]
\[ = \frac{1}{2} \sum_j \frac{\omega_j}{\omega_j} \{ u_j(s, t) u_j^*(s', t) - u_j^*(s, t) u_j(s', t) \} \]
\[ = \frac{1}{2} \sum_j \frac{1}{\omega_j} \{ u_j(s, t) u_j^*(s', t) - u_{p(j)}(s, t) u_{p(j)}^*(s', t) \} \]
\[ = \frac{1}{2} \sum_j \frac{1}{\omega_j} u_j(s, t) u_j^*(s', t) - \frac{1}{2} \sum_p \omega_{p(j)} \frac{1}{\omega_{p(j)}} u_{p(j)}(s, t) u_{p(j)}^*(s', t) = 0, \] (3.1.9)

\[ [\pi(s, t), \pi(s', t)] = -\frac{1}{2} \sum_j \omega_j \{ u_j(s, t) u_j^*(s', t) - u_j^*(s, t) u_j(s', t) \} = 0, \] (3.1.10)
We have seen that the relations (3.1.8) imply the relations (1.3.1)-(1.3.3). It easy to see that the reverse is true as well. We introduce operators
\[
\phi_j = (u_j, \phi), \quad \phi_j^\dagger = (\phi, u_j), \quad \pi_j = (u_j, \pi), \quad \pi_j^\dagger = (\pi, u_j).
\] (3.1.12)

It follows from the relations (1.3.1)-(1.3.3) that
\[
[\phi_j, \phi_j'] = 0, \quad [\phi_j, \phi_j^\dagger] = 0, \quad [\pi_j, \pi_j'] = 0, \quad [\pi_j, \pi_j^\dagger] = 0,
\]
\[
[\phi_j, \pi_j'] = i(u_j, u_j^*), \quad [\phi_j, \pi_j^\dagger] = i(u_j, u_j^*), \quad [\pi_j, \pi_j^\dagger] = i(u_j, u_j^*).
\] (3.1.13)

We find from eqs.(3.1.5),(3.1.6)
\[
a_j = \frac{1}{\sqrt{2}} \left\{ \sqrt{\omega_j} \phi_j + \frac{i}{\sqrt{\omega_j}} \pi_j \right\}, \quad a_j^\dagger = \frac{1}{\sqrt{2}} \left\{ \sqrt{\omega_j} \phi_j^\dagger - \frac{i}{\sqrt{\omega_j}} \pi_j^\dagger \right\}.
\] (3.1.14)

Now eqs.(3.1.14),(3.1.13) result in the commutation relations (3.1.8).

Thus the commutation relations (3.1.8) and (1.3.1)-(1.3.3) are equivalent to each other.

We have for the Schrödinger field and momentum
\[
[\phi_S(s_1, t_1), \phi_S(s_2, t_2)] = \frac{1}{2} \sum_j \frac{1}{\sqrt{\omega_j(t_1)\omega_j(t_2)}} \{u_j(s_1, t_1)u_j^*(s_2, t_2) - u_j^*(s_1, t_1)u_j(s_2, t_2)\}
\]
\[
= \frac{1}{2} \sum_j \frac{1}{\sqrt{\omega_j(t_1)\omega_j(t_2)}} u_j(s_1, t_1)u_j^*(s_2, t_2) - \frac{1}{2} \sum_j \frac{1}{\sqrt{\omega_j(t_1)\omega_j(t_2)}} u_p(j)(s_1, t_1)u_p(j)^*(s_2, t_2) = 0,
\] (3.1.15)
\[
[\pi_S(s_1, t_1), \pi_S(s_2, t_2)] = 0.
\] (3.1.16)

Let \( u_j \) be determined by
\[
\Delta u_j = -k_j^2 u_j, \quad k_j^2 = k_j^2(t).
\] (3.1.17)

We obtain
\[
H_{St} = \frac{1}{2} \sum_j \{[-\sqrt{\omega_j\omega_p(j)} + \frac{1}{\sqrt{\omega_j\omega_p(j)}}(m^2 + k_j^2)](a_p(j)s a_j s + a_p(j)^\dagger s a_j^\dagger s)
\]
\[
+ [\omega_j + \frac{1}{\omega_j}(m^2 + k_j^2)](a_j s a_j^\dagger s + a_j^\dagger s a_j s)\}.
\] (3.1.18)

We put
\[
\omega_j = (m^2 + k_j^2)1/2 = \omega_j(t), \quad \omega_p(j) = \omega_j,
\] (3.1.19)

then
\[
H_{St} = \frac{1}{2} \sum_j \omega_j(t)(a_j s a_j^\dagger s + a_j^\dagger s a_j s).
\] (3.1.20)

Normal ordering produces
\[
H_{St} = \sum_j \omega_j(t)a_j^\dagger s a_j s.
\] (3.1.21)

The Schrödinger equation
\[
\frac{d\Psi_{St}}{dt} = -iH_{St}\Psi_{St}
\] (3.1.22)
yields
\[ \Psi_{St} = U(t, t_0) \Psi_{St_0}, \tag{3.1.23} \]
where, with regard to
\[ [H_{St_1}, H_{St_2}] = 0, \tag{3.1.24} \]
\[ U(t, t_0) = \exp \left\{ -i \int_{t_0}^{t} H_{St} dt' \right\} = \prod_{j} \exp \left\{ -i \alpha_j(t, t_0) a_j^\dagger a_j \right\}, \tag{3.1.25} \]
\[ \alpha_j(t, t_0) = \int_{t_0}^{t} \omega_j(t') dt'. \tag{3.1.26} \]

3.2 The Heisenberg picture

In the Heisenberg picture, we have
\[ \Psi_H = U(t_0, t) \Psi_{St} = \Psi_{St_0}, \tag{3.2.1} \]
\[ A_{Ht} = U(t_0, t) A_{St} U(t, t_0), \tag{3.2.2} \]
so that
\[ a_{jHt} = e^{-i\alpha_j(t, t_0)} a_j S, \quad a^\dagger_{jHt} = e^{i\alpha_j(t, t_0)} a_j^\dagger S, \tag{3.2.3} \]
and
\[ \phi_H(s, t) = \frac{1}{\sqrt{2}} \sum_j \frac{1}{\sqrt{\omega_j(t)}} \left\{ u_j(s, t) a_{jHt} + u_j^*(s, t) a^\dagger_{jHt} \right\} \]
\[ = \frac{1}{\sqrt{2}} \sum_j \frac{1}{\sqrt{\omega_j(t)}} \left\{ \tilde{u}_j(s, t) a_j S + \tilde{u}_j^*(s, t) a_j^\dagger S \right\}, \tag{3.2.4} \]
\[ \tilde{u}_j(s, t) = e^{-i\alpha_j(t, t_0)} u_j(s, t). \tag{3.2.5} \]

The Hamiltonian
\[ H_{Ht} = H_{St} = H_t = \sum_j \omega_j(t) a_j^\dagger S a_j S. \tag{3.2.6} \]

The equations of motion are
\[ \frac{\partial A_{Ht}}{\partial t} = \left( \frac{\partial A_{St}}{\partial t} \right)_H + i[H_t, A_{Ht}], \tag{3.2.7} \]

We have
\[ [\phi_H(s_1, t_1), \phi_H(s_2, t_2)] = \frac{1}{2} \sum_j \frac{1}{\sqrt{\omega_j(t_1)\omega_j(t_2)}} \left\{ \tilde{u}_j(s_1, t_1) \tilde{u}_j^*(s_2, t_2) - \tilde{u}_j^*(s_1, t_1) \tilde{u}_j(s_2, t_2) \right\} \]
\[ = i \sum_j \frac{1}{\sqrt{\omega_j(t_1)\omega_j(t_2)}} u_j(s_1, t_1) u_j^*(s_2, t_2) \sin[\alpha_j(t_2, t_1)]. \tag{3.2.8} \]
3.3 The energy-momentum tensor

Normal ordering on the standard energy-momentum tensor \([1]\) produces

\[ T_{\mu\nu} = \left\{ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + \frac{1}{2} m^2 g_{\mu\nu} \phi^2 \right\}, \quad (3.3.1) \]

whence in the comoving reference frame we have

\[ T_{00} = \frac{1}{2} \left\{ \pi^2 + h^{ik} \partial_i \phi \partial_k \phi + m^2 \phi^2 \right\}, \quad (3.3.2) \]

\[ H_t = \int_S ds \sqrt{(h_t)} T_{00}, \quad (3.3.3) \]

\[ T_{ik} = \left\{ \partial_i \phi \partial_k \phi + \frac{1}{2} h_{ik}[\pi^2 - h^{lm} \partial_l \phi \partial_m \phi - m^2 \phi^2] \right\}; \quad (3.3.4) \]

and

\[ (\Psi, T_{ik} \Psi) = (\Psi, \partial_i \phi \partial_k \phi : \Psi) + h_{ik}[\pi^2 : -T_{00}], \quad (3.3.5) \]

4 Quantum-gravitational nonlocality. Covariance and geometry

4.1 The violation of the wave equation

With eqs.\((3.2.4),(3.2.5)\) in mind, we find

\[ (\Box + m^2) \left[ \frac{\tilde{u}_j}{\sqrt{\omega_j}} \right] = \frac{1}{\sqrt{(h)}} \frac{\partial \sqrt{(h)}}{\partial t} \left[ \frac{\tilde{u}_j}{\sqrt{\omega_j}} \right] + \frac{\partial}{\partial t} \left\{ \frac{\partial}{\partial t} \left[ \frac{u_j}{\sqrt{\omega_j}} \right] e^{-i\alpha_j} \right\} - i \frac{\partial}{\partial t} \left[ \sqrt{\omega_j} u_j \right] e^{-i\alpha_j}. \quad (4.1.1) \]

Let for some \((t, s)\)

\[ \frac{d\omega_j}{dt} = 0, \quad \frac{du_j}{dt} = 0 \quad (4.1.2) \]

hold, then

\[ (\Box + m^2) \left[ \frac{\tilde{u}_j}{\sqrt{\omega_j}} \right] = -i \frac{1}{\sqrt{(h)}} \frac{\partial \sqrt{(h)}}{\partial t} \sqrt{\omega_j} \tilde{u}_j. \quad (4.1.3) \]

Thus the wave equation is violated in the generic case of a nonstationary metric.

Note that in general relativity, the relation \((3.1.19)\) for \(\omega_j\) results from the geodesic equation,

\[ \ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0, \quad (4.1.4) \]

rather than from the wave equation.

The wave equation being rejected, the equations of motion are those in the Schrödinger and Heisenberg pictures respectively.
4.2 Quantum-gravitational nonlocality and commutation relations

A local change in the metric $h$ results in changing the Laplacian $\Delta$ and, by the same token, solutions to the equation (3.1.17), i.e., $k_j^2$, $u_j$, $\omega_j$, and $u_j/\sqrt{\omega_j}$. We call this phenomenon quantum-gravitational nonlocality.

Generally, quantum-gravitational nonlocality means that

$$A_{St}(s) = A_S[s; h_t], \quad H_t = H[h_t], \quad (4.2.1)$$

i.e., that a Schrödinger operator at time $t$ depends on the metric $h_t$ in the whole 3-space $S$.

The canonical commutation relation (1.3.1) cannot hold if the dependence of $\phi$ on the metric is local: In that case, for a given operator $\phi(s, t)$ the operator $\phi(s', t)$ might be arbitrary. Thus in order that the canonical commutation relations hold, quantum-gravitational nonlocality should take place.

A strictly local measurement of $\phi$ affects metric infinitesimally only and cannot be used for synchronizing clocks.

The degree of quantum-gravitational nonlocality may be given by the quantity

$$\beta = \frac{\partial}{\partial t} \left[ \frac{u}{\sqrt{\omega}} \right] / \left[ \frac{u}{\sqrt{\omega}} \right] \omega = \frac{\partial u}{\partial t}/u\omega - \frac{1}{2} \frac{d\omega}{dt}/\omega^2. \quad (4.2.2)$$

We have

$$\frac{d\omega}{dt} = \frac{k}{\omega} \frac{dk}{dt}, \quad (4.2.3)$$

so that

$$\beta = \frac{\partial u}{\partial t}/u\omega - \frac{k}{2\omega^3} \frac{dk}{dt}. \quad (4.2.4)$$

4.3 The principle of covariance and the geometric principle

Nonlocality is incompatible with the local principle of covariance. More general than the latter is the geometric principle [6]: Spacetime structure and dynamical equations should be phrased in geometric form. The principle of covariance is a local version of the geometric principle.

5 Applications to cosmology

5.1 The energy-momentum tensor consisted with metric

Let in eq.(3.3.5)

$$\langle \Psi_i : \partial_i \phi \partial_k \phi : \Psi \rangle \propto h_{ik} \quad (5.1.1)$$

hold, i.e.,

$$\langle \Psi_i : \partial_i \phi \partial_k \phi : \Psi \rangle = Ch_{ik}h^{lm}(\Psi_i : \partial_l \phi \partial_m \phi : \Psi). \quad (5.1.2)$$

Since

$$h^{ik}h_{ik} = 3, \quad (5.1.3)$$

we find

$$C = \frac{1}{3}. \quad (5.1.4)$$
and by eqs. (3.3.5), (3.3.2)

$$(\Psi, T_{ik}\Psi) = \frac{1}{3} h_{ik}(\Psi, \{2 : \pi^2 : -T_{00} - m^2 : \phi^2 :\} \Psi).$$  \hfill (5.1.5)

### 5.2 A homogeneous state

Let $\Psi$ be a homogeneous state, so that

$$(\Psi, \{2 : \pi^2 : -T_{00} - m^2 : \phi^2 :\} \Psi) = \frac{1}{V} \int_S ds \sqrt{(h)}(\Psi, \{2 : \pi^2 : -T_{00} - m^2 : \phi^2 :\} \Psi),$$

$$V = V_t = \int_S ds \sqrt{(h_t)}.$$  \hfill (5.2.1)

We have

$$\int_S ds \sqrt{(h)} T_{00} = \sum_j \omega_j N_j \quad N_j = N_j H = a_j^\dagger a_j,$$

$$\int_S ds \sqrt{(h)} : \pi^2 := \sum_j \omega_j N_j + \{aa + a^\dagger a^\dagger\},$$

$$\int_S ds \sqrt{(h)} : \phi^2 := \sum_j \frac{1}{\omega_j} N_j + \{aa + a^\dagger a^\dagger\}.$$  \hfill (5.2.2) \hfill (5.2.3) \hfill (5.2.4)

Let

$$N_j \Psi = n_j \Psi \quad \text{for all } j,$$

then

$$(\Psi, T_{ik}\Psi) = h_{ik} \frac{1}{3V} \sum_j \frac{\omega_j^2 - m^2}{\omega_j} n_j.$$  \hfill (5.2.5) \hfill (5.2.6)

Thus the pressure is

$$p = \frac{1}{3V} \sum_j \frac{\omega_j^2 - m^2}{\omega_j} n_j,$$

$$\rho = \frac{E}{V} = \frac{1}{V} \sum_j \omega_j n_j,$$

whereas the energy density is

$$\rho = \frac{E}{V} = \frac{1}{V} \sum_j \omega_j n_j.$$  \hfill (5.2.7) \hfill (5.2.8)

### 5.3 The Robertson-Walker spacetime

For the Robertson-Walker spacetime, the metric is of the form

$$h(s, t) = R^2(t) \kappa(s),$$

or

$$h_{ik} = R^2(t) \kappa_{ik},$$

so that we have

$$(h) = (\kappa) R^6, \quad (\kappa) = \det(\kappa_{ik}), \quad \sqrt{(h)} = R^3 \sqrt{(\kappa)}, \quad h^{ik} = \frac{\kappa^{ik}}{R^2},$$

and

$$\Delta = \frac{1}{R^2} \Delta_{\kappa}, \quad \Delta_{\kappa} \chi = \frac{1}{\sqrt{(\kappa)}} \partial_i \left[ \sqrt{(\kappa)} \kappa^{ik} \partial_k \chi \right].$$  \hfill (5.3.1) \hfill (5.3.2) \hfill (5.3.3)
The equation (3.1.17) results in
\[ \frac{1}{R^2(t)} \Delta \kappa u_j = -k_j^2 u_j, \]
so that, in view of eq.(3.1.3),
\[ \Delta \kappa u_j = -\gamma_j^2 u_j, \quad \gamma_j^2 = \text{const}, \quad k_j^2(t) = \frac{\gamma_j^2}{R^2(t)}, \quad u_j(s,t) = \frac{1}{R^{3/2}(t)} u_j^0(s), \]
and
\[ \omega_j = \left[ m^2 + \frac{\gamma_j^2}{R^2(t)} \right]^{1/2}, \]
the last relation being a familiar result of cosmology.

In eq.(4.2.4) we obtain
\[ u = u_0(s) \frac{R^3}{2 R(t)}, \quad k = \frac{\gamma}{R(t)}, \]
so that
\[ |\beta| = 3 \frac{dR/dt}{2\omega} - \frac{1}{2} \frac{(\gamma/R)^2 dR/dt}{\omega^3} = 3H - \frac{1}{2} \frac{k^2}{\omega^3} H = \left(3 - \frac{k^2}{\omega^2}\right) \frac{H}{2\omega} < \frac{3H}{2\omega}, \]
where \( H \) is the Hubble constant.

For \( H \approx \frac{1}{3} \times 10^{-17} \text{c}^{-1} \) and \( \omega \sim 10^{15} \text{c}^{-1}, \quad \beta < 10^{-32}. \)

With eqs.(5.2.7),(5.2.8) in mind, we have
\[ k_j^2 = \frac{b_j^2}{V^{2/3}}, \quad b_j^2 = \text{const}, \quad \omega_j = \left( m^2 + \frac{b_j^2}{V^{2/3}} \right)^{1/2}, \]
so that we find
\[ \frac{dE}{dV} = \frac{d(\rho V)}{dV} = \sum_j n_j \frac{d\omega_j}{dV} = -\frac{1}{3V} \sum_j n_j \frac{k_j^2}{\omega_j} = -p, \]
i.e.,
\[ dE = -pdV, \]
which is a standard relation.

### 5.4 Universe dynamics

In this and the next subsections, we follow the papers [6,7].

The \( S \)-projected Einstein equation yields
\[ G_{ik} = 8\pi \kappa_g (\Psi, T_{ik} \Psi) \Rightarrow 2\ddot{R}R + \dot{R}^2 + 1 = -8\pi \kappa_g \rho R^2 \]
where \( \kappa_g \) is the gravitational constant; eq.(5.3.12) amounts to
\[ \frac{d(\rho R^3)}{dR} = -3pR^2. \]
We obtain from eqs.(5.4.1),(5.4.2)
\[
\frac{d}{dR} \left( R \dot{R}^2 + R - \frac{8 \pi \kappa g}{3} \rho R^3 \right) = 0,
\]
whence
\[
R \ddot{R}^2 + R - \frac{8 \pi \kappa g}{3} \rho R^3 = L = \text{const.}
\]

The length $L$, which is an integral of motion, is called cosmic length. In accordance with this, the model considered is called the cosmic length universe.

The Friedmann universe corresponds to a particular value of the cosmic length,
\[
L_{\text{Friedmann}} = 0.
\]
In this sense, the Friedmann universe is the zero-length universe.

The value $L = 0$ results from the equation
\[
G_{0\mu} = 8 \pi \kappa g (\Psi, T_{0\mu} \Psi),
\]
which is violated by quantum jumps inherent in the generic case of interacting quantum fields.

### 5.5 Lifting the problem of missing dark matter

The most important problem facing modern cosmology is that of the missing dark matter [8]. Most of the mass of galaxies and an even larger fraction of the mass of clusters of galaxies is dark. The problem is that even more dark matter is required to account for the rate of expansion of the universe.

More specifically, for the Friedmann universe, the equation
\[
\Omega_0 = 2 q_0
\]
holds, where $\Omega$ is the density parameter,
\[
\Omega = \frac{\rho}{\rho_c},
\]
$\rho_c$ is the critical value of $\rho$, $q$ is the deceleration parameter,
\[
q = -\frac{\ddot{R} R}{R^2},
\]
and subscript 0 indicates present-day values. In particular, if $q_0 > 1/2$, the universe is closed and $\rho_0 > \rho_c$. But observational data give $\Omega_0 < 2 q_0$. Eq.(5.5.1) reduces to
\[
\Omega_0 = 1 + \frac{1}{R_0^2 H_0^2}.
\]

From eq.(5.4.4) we obtain
\[
\Omega_0 = 1 + \frac{1 - L/R_0}{R_0^2 H_0^2}
\]
in place of eq.(5.5.4). For
\[ p_0 \ll \frac{1}{3} \rho_0, \]
which is fulfilled, eq.(5.5.5) reduces to
\[ \Omega_0 = 2q_0 - \frac{L/R_0}{R_0^2 H_0^2} \]
in place of eq.(5.5.1).
Eq.(5.5.7) lifts the problem.

6 An application to black holes

6.1 The Lemaître metric
In the case of a black hole, the metric in the comoving reference frame is the Lemaître metric:
\[ h = \frac{1}{[(3/2r_s)(R-t)]^{2/3}} dR^2 + \frac{3}{2} \left[ \frac{3}{2}(R-t) \right]^{4/3} (d\theta^2 + \sin^2 \theta d\varphi^2), \]
where \( r_s \) is the Schwarzschild radius. The Schwarzschild coordinate is
\[ r = \left( \frac{3}{2} \right)^{2/3} r_s^{1/3}(R-t)^{2/3}. \]

6.2 Quantum field in the comoving reference frame
With the equation (3.1.17) in mind, we find
\[ \Delta \chi \equiv \Delta \bar{\chi} = \frac{1}{r^2} \partial_r [r^2 \partial_r \chi] + \frac{1}{r^2 \sin \theta} \partial_{\theta} [\sin \theta \partial_{\theta} \chi] + \frac{1}{r^2 \sin^2 \theta} \partial_{\varphi}^2 \chi = \Delta \bar{\chi} \]
where
\[ \bar{R} = (R, \theta, \varphi), \quad \bar{r} = (r, \theta, \varphi). \]
Thus eq.(3.1.17) reduces to
\[ \Delta \bar{r} u_j = -k_j^2 u_j, \]
whence
\[ u_j = u_j(r, \theta, \varphi) \]
with \( r \) given by eq.(6.1.2), and
\[ \frac{dk_j^2}{dt} = 0, \quad \omega_j = \left[ m^2 + k_j^2 \right]^{1/2}, \quad \frac{d\omega_j}{dt} = 0. \]
So in the comoving reference frame
\[ \omega_j = \text{const}, \quad H = \sum_j \omega_j a_j^\dagger a_j, \quad \frac{dH}{dt} = 0. \]
In eq.(4.2.4) we obtain
\[ \frac{d\omega}{dt} = 0, \]  
so that
\[ \beta = \frac{\partial u}{\partial t} / u\omega. \]  
We find from eq.(6.1.2)
\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} \left( \frac{r_s}{r} \right)^{1/2}. \]  
By [9], in view of \( \sqrt{h} \sim r^{3/2} \),
\[ \frac{\partial u}{\partial r} \sim \left\{ \left( k^2 + \frac{1}{r^2} \right)^{1/2} + \frac{1}{4r} \right\} |u|, \]  
so that
\[ |\beta| \sim \frac{1}{\omega} \left\{ \left( k^2 + \frac{1}{r^2} \right)^{1/2} + \frac{1}{4r} \right\} \left( \frac{r_s}{r} \right)^{1/2}. \]  
In particular,
\[ \text{for } r \gg \lambda = \frac{2\pi}{k} \quad |\beta| \sim \left[ \frac{\omega^2 - m^2 r_s}{\omega^2} \right]^{1/2}. \]  

7 No particle creation

7.1 Conservation of occupation numbers

We have in the comoving reference frame
\[ H_t = \sum_j \omega_j(t)N_j, \quad N_j = a_j^\dagger a_j, \quad [H_t, N_j] = 0, \]  
so that occupation numbers are conserved:
\[ \frac{dN_j}{dt} = \frac{dN_j}{dt} = 0, \]  
which implies that there is no particle creation in the case of a free quantum field.

In particular, neither the expanding universe nor black holes create particles.

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References


