Lectures on D-branes

Constantin P. Bachas

Centre de Physique Théorique, Ecole Polytechnique
91128 Palaiseau, FRANCE
bachas@cpht.polytechnique.fr

ABSTRACT

This is an introduction to the physics of D-branes. Topics covered include Polchinski's original calculation, a critical assessment of some duality checks, D-brane scattering, and effective worldvolume actions. Based on lectures given in 1997 at the Isaac Newton Institute, Cambridge, at the Trieste Spring School on String Theory, and at the 31rst International Symposium Ahrenshoop in Buckow.

June 1998

1Address after Sept. 1: Laboratoire de Physique Théorique, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris, FRANCE, email : bachas@physique.ens.fr
Lectures on D-branes

Constantin Bachas

1 Foreword

Referring in his ‘Republic’ to stereography – the study of solid forms – Plato was saying: ... for even now, neglected and curtailed as it is, not only by the many but even by professed students, who can suggest no use for it, nevertheless in the face of all these obstacles it makes progress on account of its elegance, and it would not be astonishing if it were unravelled. Two and a half millenia later, much of this could have been said for string theory. The subject has progressed over the years by leaps and bounds, despite periods of neglect and (understandable) criticism for lack of direct experimental input. To be sure, the construction and key ingredients of the theory – gravity, gauge invariance, chirality – have a firm empirical basis, yet what has often catalyzed progress is the power and elegance of the underlying ideas, which look (at least a posteriori) inevitable. And whether the ultimate structure will be unravelled or not, there is already a name waiting for ‘it’: $\mathcal{M}$ theory.

Few of the features of the theory, uncovered so far, exemplify this power and elegance better than D-branes. Their definition as allowed endpoints for open strings, generalizes the notion of quarks on which the QCD string can terminate. In contrast to the quarks of QCD, D-branes are however intrinsic excitations of the fundamental theory: their existence is required for consistency, and their properties – mass, charges, dynamics – are unambiguously determined in terms of the Regge slope $\alpha'$ and the asymptotic values of the dynamical moduli. They resemble in these respects conventional field-theory solitons, from which however they differ in important ways. D-particles, for instance, can probe distances much smaller than the size of the fundamental-string quanta, at weak coupling. In any case, D-branes, fundamental strings and smooth solitons fill together the multiplets of the various (conjectured) dualities, which connect all string theories to each other. D-branes have, in this sense, played a crucial role in delivering the important message of the ‘second string revolution’, that the way to reconcile quantum mechanics and Einstein gravity may be so constrained as to be ‘unique’.

Besides filling duality multiplets, D-branes have however also opened a window into the microscopic structure of quantum gravity. The D-brane model of black holes may prove as important for understanding black-hole thermodynamics, as has the Ising model proven in the past for understanding

second-order phase transitions. Technically, the D-brane concept is so power-
full because of the surprising relations it has revealed between supersym-
metric gauge theories and geometry. These relations follow from the fact that
Riemann surfaces with boundaries admit dual interpretations as field-theory
diagrams along various open- or closed-string channels. Thus, in particular,
the counting of microscopic BPS states of a black hole, an ultraviolet problem
of quantum gravity, can be mapped to the more familiar problem of study-
ing the moduli space of supersymmetric gauge theories. Conversely, ‘brane
engineering’ has been a useful tool for discussing Seiberg dualities and other
infrared properties of supersymmetric gauge theories, while low-energy super-
gavity, corrected by classical string effects, may offer a new line of attack on
the old problem of solving gauge theories in the planar (‘t Hooft) limit.

Most of these exciting developments will not be discussed in the present
lectures. The material included here covers only some of the earlier papers on
D-branes, and is a modest expansion of a previous ‘half lecture’ by the au-
thor (Bachas 1997a). The main difference from other existing reviews of the
same subject (Polchinski et al 1996, Polchinski 1996, Douglas 1996, Thor-
lacius 1998, Taylor 1998) is in the emphasis and presentation style. The aim
is to provide the reader (i) with a basis, from which to move on to reviews of
related and/or more advanced topics, and (ii) with an extensive (though far
from complete) guide to the literature. I will be assuming a working knowl-
edge of perturbative string theory at the level of Green, Schwarz and Witten
(1987) (see also Ooguri and Yin 1996, Kiritsis 1997, Dijkgraaf 1997, and vol-
ume one of Polchinski 1998, for recent reviews), and some familiarity with the
main ideas of string duality, for which there exist many nice and complimen-
tary lectures (Townsend 1996b and 1997, Aspinwall 1996, Schwarz 1997a and

A list of pedagogical reviews for further reading includes: Bigatti and
brane engineering of gauge theories, Maldacena (1996) and Youm (1997) for
(1997, 1998), Youm (1997) and Gauntlett (1997) for reviews of branes from the
complimentary, supergravity viewpoint. I am not aware of any extensive re-
views of type-I compactifications, of D-branes in general curved backgrounds,
and of semiclassical calculations using D-brane instantons. Some short lec-
tures on these subjects, which the reader may consult for further references,
include Sagnotti (1997), Bianchi (1997), Douglas (1997), Green (1997), Gut-
Last but not least, dualities in rigid supersymmetric field theories – a subject
intimately tied to D-branes – are reviewed by Intriligator and Seiberg (1995),
2 Ramond-Ramond fields

With the exception of the heterotic string, all other consistent string theories contain in their spectrum antisymmetric tensor fields coming from the Ramond-Ramond sector. This is the case for the type-IIA and type-IIB superstrings, as well as for the type-I theory whose closed-string states are a subset of those of the type-IIB. One of the key properties of D-branes is that they are the elementary charges of Ramond-Ramond fields, so let us begin the discussion by recalling some basic facts about these fields.

2.1 Chiral bispinors

The states of a closed-string theory are given by the tensor product of a left- and a right-moving worldsheet sector. For type-II theory in the covariant (NSR) formulation, each sector contains at the massless level a ten-dimensional vector and a ten-dimensional Weyl-Majorana spinor. This is depicted figuratively as follows:

\[
\begin{pmatrix}
|\mu\rangle \oplus |a\rangle \\
|\nu\rangle \oplus |b\rangle
\end{pmatrix}_{\text{left}} \otimes \begin{pmatrix}
|\nu\rangle \oplus |b\rangle \\
|\mu\rangle \oplus |a\rangle
\end{pmatrix}_{\text{right}},
\]

where \(\mu, \nu = 0, \ldots, 9\) and \(a, b = 1, \ldots, 16\) are, respectively, vector and spinor indices. Bosonic fields thus include a two-index tensor, which can be decomposed into a symmetric traceless, a trace, and an antisymmetric part: these are the usual fluctuations of the graviton \((G_{\mu\nu})\), dilaton \((\Phi)\) and Neveu-Schwarz antisymmetric tensor \((B_{\mu\nu})\). In addition, massless bosonic fields include a Ramond-Ramond bispinor \(H_{ab}\), defined as the polarization in the corresponding vertex operator

\[
V_{RR} \sim \int d^2\xi \ e^{i\rho \cdot X_\rho} \overline{S}^T \Gamma^0 H(p) S.
\]

In this expression \(S^a\) and \(\overline{S}^b\) are the covariant left- and right-moving fermion emission operators – a product of the corresponding spin-field and ghost operators (Friedan et al 1986), \(p^\mu\) is the ten-dimensional momentum, and \(\Gamma^0\) the ten-dimensional gamma matrix.

The bispinor field \(H\) can be decomposed in a complete basis of all gamma-matrix antisymmetric products

\[
H_{ab} = \sum_{n=0}^{10} \frac{i^n}{n!} H_{\mu_1 \ldots \mu_n} (\Gamma^{\mu_1 \ldots \mu_n})_{ab}.
\]

Here \(\Gamma^{\mu_1 \ldots \mu_n} \equiv \frac{1}{n!} \Gamma^{[\mu_1 \ldots \mu_n]}\), where square brackets denote the alternating sum over all permutations of the enclosed indices, and the \(n = 0\) term stands by convention for the identity in spinor space. I use the following conventions: the ten-dimensional gamma matrices are purely imaginary and obey
the algebra \( \{\Gamma^\mu,\Gamma^\nu\} = -2\eta^{\mu\nu} \) with metric signature \((-+...+)\). The chirality operator is \( \Gamma_{11} = \Gamma^0 \Gamma^1...\Gamma^9 \), Majorana spinors are real, and the Levi-Civita tensor \( \epsilon^{01...9} = 1 \).

In view of the decomposition (2.2), the Ramond-Ramond massless fields are a collection of antisymmetric Lorentz tensors. These tensors are not independent because the bispinor field is subject to definite chirality projections,

\[
H = \Gamma_{11} \quad H = \pm H \quad \Gamma_{11} .
\]

(2.3)

The choice of sign distinguishes between the type-IIA and type-IIB models. For the type-IIA theory \( S \) and \( \mathcal{S} \) have opposite chirality, so one should choose the sign plus. In the type-IIB case, on the other hand, the two spinors have the same chirality and one should choose the sign minus. To express the chirality constraints in terms of the antisymmetric tensor fields we use the gamma-matrix identities

\[
\Gamma^{\mu_1...\mu_n} \Gamma_{11} = (-)^n \Gamma_{11} \Gamma^{\mu_1...\mu_n} = \frac{\epsilon^{\mu_1...\mu_{10}}}{(10-n)!} \Gamma_{\mu_{10}...\mu_{n+1}}
\]

(2.4)

It follows easily that only even-\( n \) (odd-\( n \)) terms are allowed in the type-IIA (type-IIB) case. Furthermore the antisymmetric tensors obey the duality relations

\[
H^{\mu_1...\mu_n} = \frac{\epsilon^{\mu_1...\mu_{10}}}{(10-n)!} H_{\mu_{10}...\mu_{n+1}} , \quad \text{or equivalently} \quad H^{(n)} = \ast H^{(10-n)} .
\]

(2.5)

As a check note that the type-IIA theory has independent tensors with \( n = 0, 2 \) and 4 indices, while the type-IIB theory has \( n = 1, 3 \) and a self-dual \( n = 5 \) tensor. The number of independent tensor components adds up in both cases to \( 16 \times 16 = 256 \):

\[
\text{IIA :} \quad 1 + \frac{10 \times 9}{2!} + \frac{10 \times 9 \times 8 \times 7}{4!} = 256 ,
\]

\[
\text{IIB :} \quad 10 + \frac{10 \times 9 \times 8}{3!} + \frac{10 \times 9 \times 8 \times 7 \times 6}{2 \times 5!} = 256 .
\]

This is precisely the number of components of a bispinor.

Finally let us consider the type-I theory, which can be thought of as an orientifold projection of type-IIB (Sagnotti 1988, Hořava 1989a). This projection involves an interchange of left- and right-movers on the worldsheet. The surviving closed-string states must be symmetric in the Neveu-Schwarz sector and antisymmetric in the Ramond-Ramond sector, consistently with supersymmetry and with the fact that the graviton should survive. This implies the extra condition on the bispinor field

\[
(\Gamma^0 H)^T = -\Gamma^0 H .
\]

(2.6)

Using \( (\Gamma^\mu)^T = -\Gamma^0 \Gamma^\mu \Gamma^0 \) we conclude, after some straightforward algebra, that the only Ramond-Ramond fields surviving the extra projection are \( H^{(3)} \) and its dual, \( H^{(7)} \).
2.2 Supergraviton multiplets

The mass-shell or super-Virasoro conditions for the vertex operator $V_{RR}$ imply that the bispinor field obeys two massless Dirac equations,

$$\not{p} H = H \not{p} = 0 .$$  \hspace{1cm} (2.7)

To convert these to equations for the tensors we need the gamma identities

$$\Gamma^\mu \Gamma^{\nu_1 \ldots \nu_n} = \Gamma^{\mu \nu_1 \ldots \nu_n} - \frac{1}{(n-1)!} \eta^{\mu \nu_1} \Gamma^{\nu_2 \ldots \nu_n}$$  \hspace{1cm} (2.8)

and the decomposition (2.2) of a bispinor. After some straightforward algebra one finds

$$p^{[\mu} H^{\nu_1 \ldots \nu_n]} = p_\mu H^{\mu \nu_2 \ldots \nu_n} = 0 .$$  \hspace{1cm} (2.9)

These are the Bianchi identity and free massless equation for an antisymmetric tensor field strength in momentum space, which we may write in more economic form as

$$d H^{(n)} = d^\ast H^{(n)} = 0 .$$  \hspace{1cm} (2.10)

The polarizations of covariant Ramond-Ramond emission vertices are therefore field-strength tensors rather than gauge potentials.

Solving the Bianchi identity locally allows us to express the $n$-form field strength as the exterior derivative of a $(n-1)$-form potential

$$H_{\mu_1 \ldots \mu_n} = \frac{1}{(n-1)!} \partial_{[\mu_1} C_{\mu_2 \ldots \mu_n]} , \text{ or } H^{(n)} = d C^{(n-1)} .$$  \hspace{1cm} (2.11)

Thus the type-IIA theory has a vector ($C_\mu$) and a three-index tensor potential ($C_{\mu \nu \rho}$), in addition to a constant non-propagating zero-form field strength ($H^{(0)}$), while the type-IIB theory has a zero-form ($C$), a two-form ($C_{\mu \nu}$) and a four-form potential ($C_{\mu \nu \rho \sigma}$), the latter with self-dual field strength. Only the two-form potential survives the type-I orientifold projection. These facts are summarized in table 1. A $(p+1)$-form ‘electric’ potential can of course be traded for a $(7-p)$-form ‘magnetic’ potential, obtained by solving the Bianchi identity of the dual field strength.

From the point of view of low-energy supergravity all Ramond-Ramond fields belong to the ten-dimensional graviton multiplet. For N=2 supersymmetry this contains 128 bosonic helicity states, while for N=1 supersymmetry it only contains 64. For both the type-IIA and type-IIB theories, half of these states come from the Ramond-Ramond sector, as can be checked by counting the transverse physical components of the gauge potentials:

IIA : \hspace{1cm} $$8 + \frac{8 \times 7 \times 6}{3!} = 64 ,$$

IIB : \hspace{1cm} $$1 \times \frac{8 \times 7}{2!} + \frac{8 \times 7 \times 6 \times 5}{2 \times 4!} = 64 .$$
<table>
<thead>
<tr>
<th>Neveu-Schwarz</th>
<th>Ramond-Ramond</th>
</tr>
</thead>
<tbody>
<tr>
<td>type-IIA $G_{\mu\nu}, \Phi, B_{\mu\nu}$</td>
<td>$C_\mu, C_{\mu\nu\rho}; H^{(0)}$</td>
</tr>
<tr>
<td>type-IIB $G_{\mu\nu}, \Phi, B_{\mu\nu}$</td>
<td>$C, C_{\mu\nu}, C_{\mu\nu\rho\tau}$</td>
</tr>
<tr>
<td>type-I $G_{\mu\nu}, \Phi$</td>
<td>$C_{\mu\nu}$</td>
</tr>
<tr>
<td>heterotic $G_{\mu\nu}, \Phi, B_{\mu\nu}$</td>
<td></td>
</tr>
</tbody>
</table>

String origin of massless fields completing the N=1 or N=2 super-graviton multiplet of the various theories in ten dimensions.

This counting is simpler in the light-cone Green-Schwarz formulation, where the Ramond-Ramond fields correspond to a chiral SO(8) bispinor.

### 2.3 Dualities and RR charges

A $(p+1)$-form potential couples naturally to a $p$-brane, i.e. an excitation extending over $p$ spatial dimensions. Let $Y^\mu(\zeta^\alpha)$ be the worldvolume of the brane ($\alpha = 0, ..., p$), and let

$$\hat{C}^{(p+1)} \equiv C_{\mu_1...\mu_{p+1}}(Y) \partial_0 Y^{\mu_1}...\partial_p Y^{\mu_{p+1}}$$

be the pull-back of the $(p+1)$-form on this worldvolume. The natural (`electric’) coupling is given by the integral

$$I_{WZ} = \rho_{(p)} \int d^{p+1}\zeta \hat{C}^{(p+1)},$$

with $\rho_{(p)}$ the charge-density of the brane. Familiar examples are the coupling of a point-particle (`0-brane’) to a vector potential, and of a string (`1-brane’) to a two-index antisymmetric tensor. Since the dual of a $(p+1)$-form potential in ten dimensions is a $(7-p)$-form potential, there exists also a natural (`magnetic’) coupling to a $(6-p)$-brane. The sources for the field equation and Bianchi identity of a $(p+1)$-form are thus $p$-branes and $(6-p)$-branes.

Now within type-II perturbation theory there are no such elementary RR sources. Indeed, if a closed-string state were a source for a RR $(p+1)$-form,
The web of dualities relating the ten-dimensional superstring theories and $\mathcal{M}$ theory, as described in the text.

then the trilinear coupling

$$\langle \text{closed} | C^{(p+1)} | \text{closed} \rangle$$

would not vanish. This is impossible because the coupling involves an odd number of left-moving (and of right-moving) fermion emission vertices, so that the corresponding correlator vanishes automatically on any closed Riemann surface. What this arguments shows, in particular, is that fundamental closed strings do not couple ‘electrically’ to the Ramond-Ramond two-form. It is significant, as we will see, that in the presence of worldsheet boundaries this simple argument will fail.

Most non-perturbative dualities require, on the other hand, the existence of such elementary RR charges. The web of string dualities in nine or higher dimensions, discussed in more detail in this volume by Sen (see also the other reviews listed in the introduction), has been drawn in figure 1. The web holds together the five ten-dimensional superstring theories, and the eleven-dimensional $\mathcal{M}$ theory, whose low-energy limit is eleven-dimensional supergravity (Cremmer et al 1978), and which has a (fundamental ?) supermembrane (Bergshoeff et al 1987). The black one-way arrows denote compactifications of $\mathcal{M}$ theory on the circle $S^1$, and on the interval $S^1/Z_2$. In the small-radius limit these are respectively described by type-IIA string theory (Townsend 1995, Witten 1995), and by the $E_8 \times E_8$ heterotic model (Horava and Witten 1996a, 1996b). The two-way black arrows identify the strong-coupling limit of one theory with the weak-coupling limit of another. The type-I and heterotic SO(32) theories are related in this manner (Witten 1995, Polchinski and Witten 1996), while the type-IIB theory is self-dual (Hull and Townsend 1995). Finally, the two-way white arrows stand for perturbative T-dualities, after compactification on an extra circle (for a review of T-duality see Giveon et al 1994).
Consider first the type-IIA theory, whose massless fields are given by dimensional reduction from eleven dimensions. The bosonic components of the eleven-dimensional multiplet are the graviton and a antisymmetric three-form, and they decompose in ten dimensions as follows:

\[
G_{MN} \rightarrow G_{\mu\nu}, C_\mu, \Phi; \quad A_{MNR} \rightarrow C_{\mu\nu\rho}, B_{\mu\nu},
\]

(2.14)

where \( M, N, R = 0, \ldots, 10 \). The eleven-dimensional supergravity has, however, also Kaluza-Klein excitations which couple to the off-diagonal metric components \( C_\mu \). Since this is a RR field in type-IIA theory, duality requires the existence of non-perturbative 0-brane charges. In what concerns type-IIB string theory, its conjectured self-duality exchanges the two-forms \( B_{\mu\nu} \) and \( C_{\mu\nu} \). Since fundamental strings are sources for the Neveu-Schwarz \( B_{\mu\nu} \), this duality requires the existence of non-perturbative 1-branes coupling to the Ramond-Ramond \( C_{\mu\nu} \) (Schwarz 1995).

Higher \( p \)-branes fit similarly in the conjectured web of dualities. This can be seen more easily after compactification to lower dimensions, where dualities typically mix the various fields coming from the Ramond-Ramond and Neveu-Schwarz sectors. For example, type-IIA theory compactified to six dimensions on a K3 surface is conjectured to be dual to the heterotic string compactified on a four-torus (Duff and Minasian 1995, Hull and Townsend 1995, Duff 1995, Witten 1995). The latter has extended gauge symmetry at special points of the Narain moduli space. On the type-IIA side the maximal abelian gauge symmetry has gauge fields that descend from the Ramond-Ramond three-index tensor. These can be enhanced to a non-abelian group only if there exist charged 2-branes wrapping around shrinking 2-cycles of the K3 surface (Bershadsky et al 1996b). A similar phenomenon occurs for Calabi-Yau compactifications of type-IIB theory to four dimensions. The low-energy Lagrangian of Ramond-Ramond fields has a logarithmic singularity at special (conifold) points in the Calabi-Yau moduli space. This can be understood as due to nearly-massless 3-branes, wrapping around shrinking 3-cycles of the compact manifold, and which have been effectively integrated out (Strominger 1995). Strominger’s observation was important for two reasons: (i) it provided the first example of a brane that becomes massless and can eventually condense (Ferrara et al 1995, Kachru and Vafa 1995), and (ii) in this context the existence of RR-charged branes is not only a prediction of conjectured dualities – they have to exist because without them string theory would be singular and hence inconsistent.

3 D-brane tension and charge

The only fundamental quanta of string perturbation theory are elementary strings, so all other \( p \)-branes must arise as (non-perturbative) solitons. The effective low-energy supergravities exhibit, indeed, corresponding classical
solutions (for reviews see Duff, Khuri and Lu 1995, Stelle 1997 and 1998, Youm 1997), but these are often singular and require the introduction of a source. One way to handle the corrections at the string scale is to look for (super)conformally-invariant σ-models, a lesson sunk-in from the study of string compactifications. Callan et al (1991a, 1991b) found such solitonic five-branes in both the type-II and the heterotic theories. Their branes involved only Neveu-Schwarz backgrounds – being (‘magnetic’) sources, in particular, for the two-index tensor $B_{\mu\nu}$. Branes with Ramond-Ramond backgrounds looked, however, hopelessly intractable : the corresponding σ-model would have to involve the vertex (2.1), which is made out of ghosts and spin fields and cannot, furthermore, be written in terms of two-dimensional superfields. Amazingly enough, these Ramond-Ramond charged p-branes turn out to admit a much simpler, exact and universal description as allowed endpoints for open strings, or D(irichlet)-branes (Polchinski 1995).

3.1 Open-string endpoints as defects

The bosonic part of the Polyakov action for a free fundamental string in flat space-time and in the conformal gauge reads

$$I_F = \int_{\Sigma} \frac{d^2\xi}{4\pi\alpha'} \partial_a X^\mu \partial^a X_\mu ,$$

(3.1)

with $\Sigma$ some generic surface with boundary. For its variation

$$\delta I_F = - \int_{\Sigma} \frac{d^2\xi}{2\pi\alpha'} \delta X^\mu \partial_a \partial^a X_\mu + \int_{\partial\Sigma} \frac{d\xi^a}{2\pi\alpha'} \delta X^\mu \epsilon_{ab} \partial^b X_\mu$$

(3.2)

to vanish, the $X^\mu$ must be harmonic functions on the worldsheet, and either of the following two conditions must hold on the boundary $\partial\Sigma$,

$$\partial_\perp X^\mu = 0 \quad (\text{Neumann}),$$

or

$$\delta X^\mu = 0 \quad (\text{Dirichlet}).$$

(3.3)

Neumann conditions respect Poincaré invariance and are hence momentum-conserving. Dirichlet conditions, on the other hand, describe space-time defects. They have been studied in the past in various guises, for instance as sources for partonic behaviour in string theory (Green 1991b and references therein), as heavy-quark endpoints (Lüscher et al 1980, Alvarez 1981), and as backgrounds for open-string compactification (Pradisi and Sagnotti 1989, Hořava 1989b, Dai et al 1989). Their status of intrinsic non-perturbative excitations was not, however, fully appreciated in these earlier studies.

---

I use the label $a, b \cdots$ both for space-time spinors and for the (Euclidean) worldsheet coordinates of a fundamental string – the context should, hopefully, help to avoid confusion.
A static defect extending over $p$ flat spatial dimensions is described by the boundary conditions
\[ \partial_{\perp} X^\alpha = 0, \cdots, p = X^m = p+1, \cdots, 9 = 0, \] (3.4)
which force open strings to move on a $(p + 1)$-dimensional (worldvolume) hyperplane spanning the dimensions $\alpha = 0, \cdots, p$. Since open strings do not propagate in the bulk in type-II theory, their presence is intimately tied to the existence of the defect, which we will refer to as a $D_p$-brane. Consider complex radial-time coordinates for the open string – these map a strip worldsheet onto the upper-half plane,
\[ z = e^{\xi^0 + i\xi^1} \quad (0 < \xi^0 < \infty, \ 0 < \xi^1 < \pi). \] (3.5)
The boundary conditions for the bosonic target-space coordinates then take the form
\[ \partial X^\alpha = \partial X^\alpha \bigg|_{\text{Im} z = 0} \quad \text{and} \quad \partial X^m = -\partial X^m \bigg|_{\text{Im} z = 0}. \] (3.6)
Worldsheet supersymmetry imposes, on the other hand, the following boundary conditions on the worldsheet supercurrents (Green et al 1987): $J_F = \epsilon J_F$, where $\epsilon = +1$ in the Ramond sector, and $\epsilon = \text{sign}(\text{Im} z)$ in the Neveu-Schwarz sector. As a result the fermionic coordinates must obey
\[ \psi^\alpha = \epsilon \bar{\psi}^\alpha \bigg|_{\text{Im} z = 0} \quad \text{and} \quad \psi^m = -\epsilon \bar{\psi}^m \bigg|_{\text{Im} z = 0}. \] (3.7)
To determine the boundary conditions on spin fields, notice that their operator-product expansions with the fermions read (Friedan et al 1986)
\[ \psi^\mu(z)S(w) \sim (z - w)^{-1/2} \Gamma^\mu S(w), \] (3.8)
with a similar expression for right movers. Consistency with (3.7) imposes therefore the conditions,
\[ S = \Pi_{(p)} S \bigg|_{\text{Im} z = 0}, \] (3.9)
where
\[ \Pi_{(p)} = (i\Gamma_{11}\Gamma^{p+1})(i\Gamma_{11}\Gamma^{p+2}) \cdots (i\Gamma_{11}\Gamma^9) \] (3.10)
is a real operator which commutes with all $\Gamma^\alpha$ and anticommutes with all $\Gamma^m$. Since $\Pi_{(p)}$ flips the spinor chirality for $p$ even, only even-dimensional $D_p$-branes are allowed in type-IIA theory. For the same reason type-IIB and type-I theories allow only for odd-dimensional $Dp$-branes. In the type-I case we furthermore demand that (3.9) be symmetric under the interchange $S \leftrightarrow \bar{S}$. This implies $\Pi_{(p)}^2 = 1$, which is true only for $p = 1, 5$ and 9. All these facts are summarized in table 2.
The Dp-branes of the various string theories are (with the exception of the D9-brane) in one-to-one correspondence with the ‘electric’ Ramond-Ramond potentials of table 1, and their ‘magnetic’ duals. The two heterotic theories have no Ramond-Ramond fields and no Dp-branes.

The case \( p = 9 \) is degenerate, since it implies that open strings can propagate in the bulk of space-time. This is only consistent in type-I theory, i.e. when there are 32 D9-branes and an orientifold projection. The other Dp-branes listed in the table are in one-to-one correspondence with the ‘electric’ Ramond-Ramond potentials of table 1, and their ‘magnetic’ duals. We will indeed verify that they couple to these potentials as elementary sources. The effective action of a Dp-brane, with tension \( T(p) \) and charge density under the corresponding Ramond-Ramond \((p+1)\)-form \( \rho(p) \), reads

\[
I_{Dp} = \int d^{p+1} \zeta \left( T(p) e^{-\Phi} \sqrt{-\det \hat{G}_{\alpha\beta} + \rho(p) \hat{C}^{(p+1)}} \right),
\]

where

\[
\hat{G}_{\alpha\beta} = G_{\mu\nu} \partial_{\alpha} Y^\mu \partial_{\beta} Y^\nu
\]

is the induced worldvolume metric. The cases \( p = -1, 7, 8 \) are somewhat special. The D(-1)-brane sits at a particular space-time point and must be interpreted as a (Euclidean) instanton with action

\[
I_{D(-1)} = T_{(-1)} e^{-\Phi} + i \rho_{(-1)} C^{(0)} \bigg|_{\text{position}}.
\]

Its ‘magnetic’ dual, in a sense to be made explicit below, is the D7-brane. Finally the D8-brane is a domain wall coupling to the non-propagating nine-form, i.e. separating regions with different values of \( H^{(0)} \) (Polchinski and Witten 1996, Bergshoeff et al 1996).

The values of \( T(p) \) and \( \rho(p) \) could be extracted in principle from one-point functions on the disk. Following Polchinski (1995) we will prefer to extract them from the interaction energy between two static identical D-branes. This

<table>
<thead>
<tr>
<th>String Theory</th>
<th>Dp-branes</th>
</tr>
</thead>
<tbody>
<tr>
<td>type-IIA</td>
<td>( p = 0, 2, 4, 6, 8 )</td>
</tr>
<tr>
<td>type-IIB</td>
<td>( p = -1, 1, 3, 5, 7, 9 )</td>
</tr>
<tr>
<td>type-I</td>
<td>( p = 1, 5, 9 )</td>
</tr>
</tbody>
</table>
approach will spare us the technicalities of normalizing vertex operators correctly, and will furthermore extend naturally to the study of dynamical D-brane interactions (Bachas 1996).

### 3.2 Static force: field-theory calculation

Viewed as solitons of ten-dimensional supergravity, two D-branes interact by exchanging gravitons, dilatons and antisymmetric tensors. This is a good approximation, provided their separation $r$ is large compared to the fundamental string scale. The supergravity Lagrangian for the exchanged bosonic fields reads (see Green et al 1987)

$$I_{\text{IIA,B}} = -\frac{1}{2\kappa_{(10)}^2} \int d^{10}x \sqrt{-G} \left[ e^{-2\Phi} \left( R - 4(d\Phi)^2 + \frac{1}{12}(dB)^2 \right) + \sum \frac{1}{2n!} H^{(n)}_2 \right]$$

(3.14)

where $n = 0, 2, 4$ for type-IIA theory, $n = 1, 3$ for type-IIB, while for the self-dual field-strength $H^{(5)}$ there is no covariant action we may write down. Since this is a tree-Lagrangian of closed-string modes, it is multiplied by the usual factor $e^{-2\Phi}$ corresponding to spherical worldsheet topology. The D-brane Lagrangian (3.11), on the other hand, is multiplied by a factor $e^{-\Phi}$, corresponding to the topology of the disk. The disk is indeed the lowest-genus diagram with a worldsheet boundary which can feel the presence of the D-brane. These dilaton pre-factors have been absorbed in the terms involving Ramond-Ramond fields through a rescaling

$$C^{(p+1)} \rightarrow e^{\Phi} C^{(p+1)} .$$

(3.15)

A careful analysis shows indeed that it is the field strengths of the rescaled potentials which satisfy the usual Bianchi identity and Maxwell equation when the dilaton varies (Callan et al 1988, Li 1996b, Polyakov 1996).

To decouple the propagators of the graviton and dilaton, we pass to the Einstein metric

$$g_{\mu\nu} = e^{-\Phi/2} G_{\mu\nu} ,$$

(3.16)

in terms of which the effective actions take the form

$$I_{\text{IIA,B}} = -\frac{1}{2\kappa_{(10)}^2} \int d^{10}x \sqrt{-\hat{g}} \left[ R + \frac{1}{2}(d\Phi)^2 + \frac{1}{12}e^{-\Phi}(dB)^2 \right.
\left. + \sum \frac{1}{2(p+2)!} e^{(3-p)\Phi/2}(dC^{(p+1)})^2 \right]$$

(3.17)

and

$$I_{D_p} = \int d^{p+1}\zeta \left( T_{(p)} e^{(p-3)\Phi/4} \sqrt{-\det \hat{g}_{\alpha\beta}} + \rho_{(p)} \hat{C}^{(p+1)} \right) .$$

(3.18)
To leading order in the gravitational coupling the interaction energy comes from the exchange of a single graviton, dilaton or Ramond-Ramond field, and reads

$$\mathcal{E}(r) \delta T = -2\kappa_{(10)}^2 \int d^{10}x \int d^{10}\tilde{x} \left[ j_\Phi \Delta \tilde{j}_\Phi - j_C \Delta \tilde{j}_C + T_{\mu\nu} \Delta^{\mu\nu,\rho\tau} \tilde{T}_{\rho\tau} \right]$$  \hspace{1cm} (3.19)

Here $j_\Phi$, $j_C$ and $T_{\mu\nu}$ are the sources for the dilaton, Ramond-Ramond form and graviton obtained by linearizing the worldvolume action for one of the branes, while the tilde quantities refer to the other brane. $\Delta$ and $\Delta^{\mu\nu,\rho\tau}$ are the scalar and the graviton propagators in ten dimensions, evaluated at the argument $(x - \tilde{x})$, and $\delta T$ the total interaction time. To simplify notation, and since only one component of $C^{(p+1)}$ couples to a static planar $Dp$-brane, we have dropped the obvious tensor structure of the antisymmetric field.

The sources for a static planar defect take the form

$$j_\Phi = \frac{p - 3}{4} T_{(p)} \delta(x^+)$$  \hspace{1cm} (3.20)

$$j_C = \rho_{(p)} \delta(x^+)$$

$$T_{\mu\nu} = \frac{1}{2} T_{(p)} \delta(x^+) \times \begin{cases} \eta_{\mu\nu} & \text{if } \mu, \nu \leq p \\ 0 & \text{otherwise} \end{cases}$$

where the $\delta$-function localizes the defect in transverse space. The tilde sources are taken identical, except that they are localized at distance $r$ away in the transverse plane. The graviton propagator in the De Donder gauge and in $d$ dimensions reads (Veltman 1975)

$$\Delta^{\mu\nu,\rho\tau}_{(d)} = \left( \eta^{\mu\rho} \eta^{\nu\tau} + \eta^{\mu\tau} \eta^{\nu\rho} - \frac{2}{d-2} \eta^{\mu\rho} \eta^{\nu\tau} \right) \Delta_{(d)} \right),$$  \hspace{1cm} (3.21)

where

$$\Delta_{(d)}(x) = \int \frac{d^d p}{(2\pi)^d} e^{ipx}. \hspace{1cm} (3.22)$$

Putting all this together and doing some straightforward algebra we obtain

$$\mathcal{E}(r) = 2V_{(p)} \kappa_{(10)}^2 \left[ \rho_{(p)}^2 - T_{(p)}^2 \right] \Delta_{(9-p)}^{E}(r),$$  \hspace{1cm} (3.23)

where $V_{(p)}$ is the (regularized) p-brane volume and $\Delta_{(9-p)}^{E}(r)$ is the (Euclidean) scalar propagator in $(9 - p)$ transverse dimensions. The net force is as should be expected the difference between Ramond-Ramond repulsion and gravitational plus dilaton attraction.

### 3.3 Static force: string calculation

The exchange of all closed-string modes, including the massless graviton, dilaton and $(p + 1)$-form, is given by the cylinder diagram of figure 2. Viewed as
Two D-branes interacting through the exchange of a closed string. The diagram has a dual interpretation as Casimir force due to vacuum fluctuations of open strings.

an annulus, this same diagram also admits a dual and, from the field-theory point of view, surprising interpretation: the two D-branes interact by modifying the vacuum fluctuations of (stretched) open strings, in the same way that two superconducting plates attract by modifying the vacuum fluctuations of the photon field. It is this simple-minded duality which may, as we will see below, revolutionize our thinking about space-time.

The one-loop vacuum energy of oriented open strings reads

\[
\mathcal{E}(r) = -\frac{V(p)}{2} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \int_0^\infty \frac{dt}{t} \text{Str} e^{-\pi t(k^2 + M^2)/2} = 
\]

\[
= -2 \times \frac{V(p)}{2} \int_0^\infty \frac{dt}{t} \frac{e^{-\pi t/2}}{(2\pi t)^{(p+1)/2}} Z(t),
\]

where

\[
Z(t) = -\frac{1}{2} \sum_{s=2,3,4} (-)^s \frac{\theta_2^4 \left( \frac{0}{2} \right)}{\eta^{12} \left( \frac{t}{2} \right)}
\]

is the usual spin structure sum obtained by supertracing over open-string oscillator states (see Green et al 1987), and we have set \( \alpha' = 1/2 \). Strings stretching between the two D-branes have at the \( N \)th oscillator level a mass
$M^2 = (r/\pi)^2 + 2N$, so that their vacuum fluctuations are modified when we separate the D-branes. The vacuum energy of open strings with both endpoints on the same defect is, on the other hand, $r$-independent and has been omitted. Notice also the (important) factor of 2 in front of the second line: it accounts for the two possible orientations of the stretched string.

The first remark concerning the above expression, is that it vanishes by the well-known $\theta$-function identity. Comparing with eq. (3.23) we conclude that

$$T_{(p)} = \rho_{(p)} \ , (3.26)$$

so that Ramond-Ramond repulsion cancels exactly the gravitational and dilaton attraction. As will be discussed in detail later on, this cancellation of the static force is a consequence of space-time supersymmetry. It is similar to the cancellation of Coulomb repulsion and Higgs-scalar attraction between 't Hooft-Polyakov monopoles in N=4 supersymmetric Yang-Mills (see for example Harvey 1996).

To extract the actual value of $T_{(p)}$ we must separate in the diagram the exchange of RR and NS-NS closed-string states. These are characterized by worldsheet fermions which are periodic, respectively antiperiodic around the cylinder, so that they correspond to the $s = 4$, respectively $s = 2, 3$ open-string spin structures. In the large-separation limit ($r \to \infty$) we may furthermore expand the integrand near $t \sim 0$:

$$Z(t) \simeq (8 - 8) \times \left(\frac{t}{2}\right)^4 + o(e^{-1/t}) \ , (3.27)$$

where we have here used the standard $\theta$-function asymptotics. Using also the integral representation

$$\Delta^E_{(d)}(r) = \frac{\pi}{2} \int_0^\infty dl \ (2\pi^2 l)^{-d/2} e^{-r^2/2 \pi l} \ , (3.28)$$

and restoring correct mass units we obtain

$$\mathcal{E}(r) = V_{(p)} \ (1 - 1) \ 2\pi (4\pi^2 \alpha')^{3-p} \Delta^E_{(3-p)}(r) + o(e^{-r/\sqrt{\alpha'}}) \ . (3.29)$$

Comparing with the field-theory calculation we can finally extract the tension and charge-density of type-II Dp-branes,

$$T_{(p)}^2 = \rho_{(p)}^2 = \frac{\pi}{\kappa^2_{(10)}} (4\pi^2 \alpha')^{3-p} \ . (3.30)$$

These are determined unambiguously, as should be expected for intrinsic excitations of a fundamental theory. Notice that in the type-I theory the above interaction energy should be multiplied by one half, because the stretched open strings are unoriented. The tensions and charge densities of type-I D-branes are, therefore, smaller than those of their type-IIB counterparts by a factor of $\frac{1}{\sqrt{2}}$. 
4 Consistency and duality checks

String dualities and non-perturbative consistency impose a number of relations among the tensions and charge densities of D-branes, which we will now discuss. We will verify, in particular, that the values (3.30) are consistent with T-duality, with Dirac charge quantization, as well as with the existence of an eleventh dimension. From the string-theoretic point of view, the T-duality relations are the least surprising, since the symmetry is automatically built into the genus expansion. Verifying these relations is simply a check of the annulus calculation of the previous section. That the results obey also the Dirac conditions is more rewarding, since these test the non-perturbative consistency of the theory. What is, however, most astonishing is the fact that the annulus calculation ‘knows’ about the existence of the eleventh dimension.

4.1 Charge quantization

Dirac’s quantization condition for electric and magnetic charge (Dirac 1931) has an analog for extended objects in higher dimensions (Nepomechie 1985, Teitelboim 1986a,b). Consider a Dp-brane sitting at the origin, and integrate the field equation of the Ramond-Ramond form over the transverse space. Using Stokes’ theorem one finds

\[ \int_{S(8-p)} \ast H^{(p+2)} = 2\kappa_{(10)}^2 \rho_{(p)} \]

(4.1)

where \( S(8-p) \) is a (hyper)sphere, surrounding the defect, in transverse space. This equation is the analog of Gauss’ law. Now Poincaré duality tells us that

\[ \ast H^{(p+2)} = H^{(8-p)} \simeq dC^{(7-p)} , \]

(4.2)

where the potential \( C^{(7-p)} \) is not globally defined because the Dp-brane is a source in the Bianchi identity for \( H^{(8-p)} \). Following Dirac we may define a smooth potential everywhere, except along a singular (hyper)string which drills a hole in \( S(8-p) \). The hole is topologically equivalent to the interior of a hypersphere \( S(7-p) \). These facts are easier to visualize in three-dimensional space, where a point defect creates a string singularity which drills a disk out of a two-sphere, while a string defect creates a sheet singularity which drills a segment out of a circle, as in figure 3.

The Dirac singularity is dangerous because a Bohm-Aharonov experiment involving \((6-p)\)-branes might detect it. Indeed, the wave-function of a \((6-p)\)-brane transported around the singularity picks a phase

\[ \text{Phase} = \rho_{(6-p)} \int_{S(7-p)} C^{(7-p)} = \rho_{(6-p)} \int_{S(8-p)} H^{(8-p)} . \]

(4.3)

\(^4\)Schwinger (1968) and Zwanziger (1968) extended Dirac’s argument to dyons. The generalization of their argument to higher dimensions involves a subtle sign discussed recently by Deser et al (1997, 1998).
A 1-brane creates a 3-index “electric” field $H^{(3)}$. Electric flux in $d=4$ space-time dimensions is given by an integral of the dual vector over a circle $S_{(1)}$. The ‘magnetic’ potential is a scalar field, coupling to point-like (Euclidean) instantons, and jumping discontinuously across the depicted sheet singularity.

For the (hyper)string to be unphysical, this phase must be an integer multiple of $2\pi$. Putting together equations (4.1-4.3) we thus find the condition

$$\text{Phase} = 2\kappa^2 \rho_{(p)} \rho_{(6-p)} = 2\pi n.$$  

(4.4)

The charge densities (3.30) satisfy this condition with $n = 1$. D-branes are therefore the minimal Ramond-Ramond charges allowed in the theory, so one may conjecture that there are no others.

Dirac’s argument is strictly-speaking valid only for $0 \leq p \leq 6$. In order to extend it to the pair $p = -1, 7$, note that a D7-brane creates a (hyper)-sheet singularity across which the Ramond-Ramond scalar, $C^{(0)}$, jumps discontinuously by an amount $2\kappa^2 \rho_{(7)}$. Dirac quantization ensures that the exponential of the (Euclidean) instanton action (3.13) has no discontinuity across the sheet, whose presence cannot therefore be detected by non-perturbative physics. It is the four-dimensional analog of this special case that is, as a matter of fact, illustrated in figure 3.

A final comment concerns the type-I theory, where the extra factor of $\frac{1}{\sqrt{2}}$ in the charge densities seems to violate the quantization condition. The puzzle

\footnote{The D3-brane is actually also special, since it couples to a self-dual four-form.}
is resolved by the observation (Witten 1996b) that the dynamical five-brane excitation consists of a *pair* of coincident D5-branes, so that

\[ \rho^{I}_{(1)} = \sqrt{\frac{\pi}{2\kappa_{(10)}^2}} (4\pi^2\alpha') \quad \text{and} \quad \rho^{I}_{(5)} = 2 \times \sqrt{\frac{\pi}{2\kappa_{(10)}^2}} (4\pi^2\alpha')^{-1}. \]  

(4.5)

This is consistent with heterotic/type-I duality, as well as with the fact that the orientifold projection removes the collective coordinates of a single, isolated D5-brane (Gimon and Polchinski 1996).

### 4.2 T-duality

T-duality is a discrete gauge symmetry of string theory, that transforms both the background fields and the perturbative (string) excitations around them (see Giveon *et al* 1994). The simplest context in which it occurs is compactification of type-II theory on a circle. The general expression for the compact (ninth) coordinate of a closed string is

\[ z\partial X^9 = \frac{i}{2} \left( \frac{n_9\alpha'}{R_g} + m_9 R_g \right) + i \sqrt{\frac{\alpha'}{2}} \sum_{k \neq 0} \tilde{a}_k z^{-k} \]

\[ \bar{z}\bar{\partial} X^9 = \frac{i}{2} \left( \frac{n_9\alpha'}{R_g} - m_9 R_g \right) + i \sqrt{\frac{\alpha'}{2}} \sum_{k \neq 0} \tilde{a}_k \bar{z}^{-k} \]  

(4.6)

Here \( n_9 \) and \( m_9 \) are the quantum numbers corresponding to momentum and winding, and \( z = e^{\xi^0 + i\xi^1} \) with \( 0 \leq \xi^1 < 2\pi \). A T-duality transformation inverts the radius of the circle, interchanges winding with momentum numbers, and flips the sign of right-moving oscillators:

\[ R'_g = \frac{\alpha'}{R_g}, \quad (n'_9, m'_9) = (m_9, n_9) \quad \text{and} \quad \tilde{a}^9_k = -\tilde{a}^9_k. \]  

(4.7)

It also shifts the expectation value of the dilaton, so as to leave the nine-dimensional Planck scale unchanged,

\[ \frac{R'_g}{\kappa_{(10)}^2} = \frac{R_g}{\kappa_{(10)}^2}. \]  

(4.8)

The transformation (4.7) can be thought of as a (hybrid) parity operation restricted to the antiholomorphic worldsheet sector:

\[ \bar{\partial} X^9' = -\bar{\partial} X^9. \]  

(4.9)

Since the parity operator in spinor space is \( i\Gamma^9\Gamma_{11} \), bispinor fields will transform accordingly as follows:

\[ H' = iH \Gamma^9\Gamma_{11}. \]  

(4.10)
Using the gamma-matrix identities of section 2, we may rewrite this relation in component form,

\[ H'_{\mu_1 \ldots \mu_n} = H_{9 \mu_1 \ldots \mu_n} \quad \text{and} \quad H'_{9 \mu_1 \ldots \mu_n} = -H_{\mu_1 \ldots \mu_n}, \]

for any \( \mu_i \neq 9 \). T-duality exchanges therefore even-\( n \) with odd-\( n \) antisymmetric field strengths, and hence also type-IIA with type-IIB backgrounds. Consistency requires that it also transform even-\( p \) to odd-\( p \) D-branes and vice versa.

To see how this comes about let us consider a D\((p+1)\)-brane wrapping around the ninth dimension. We concentrate on the ninth coordinate of an open string living on this D-brane. It can be expressed as the sum of the holomorphic and anti-holomorphic pieces (4.6), with an extra factor two multiplying the zero modes because the open string is parametrized by \( \xi^1 \in [0, \pi] \). The Neumann boundary condition \( \partial X^9 = \bar{\partial} X^9 \) at real \( z \), forces furthermore the identifications

\[ a_k = \tilde{a}_k, \quad \text{and} \quad m_9 = 0. \] (4.12)

This is consistent with the fact that open strings can move freely along the ninth dimension on the D-brane, but cannot wind.

Now a T-duality transformation flips the sign of the antiholomorphic piece, changing the Neumann to a Dirichlet condition, \(^6\)

\[ a'_k = -\tilde{a}'_k, \quad \text{and} \quad n'_9 = 0. \] (4.13)

The wrapped D\((p+1)\)-brane is thus transformed, in the dual theory, to a D\( p \)-brane localized in the ninth dimension (Hořava 1989b, Dai et al 1989, Green 1991a). Open strings cannot move along this dimension anymore, but since their endpoints are fixed on the defect they can now wind. The inverse transformation is also true: a D\( p \)-brane, originally transverse to the ninth dimension, transforms to a wrapped D\((p+1)\)-brane in the dual theory. All this is compatible with the transformation (4.11) of Ramond-Ramond fields, to which the various D-branes couple. Furthermore, since a gauge transformation should not change the (nine-dimensional) tension of the defect, we must have

\[ 2\pi R_9 T_{(p+1)} = T'_{(p)}. \] (4.14)

Using the formulae (4.7-4.8) one can check that the D-brane tensions indeed verify this T-duality constraint. Conversely, T-duality plus the minimal Dirac quantization condition fix unambiguously the expression (3.30) for the D-brane tensions.

\(^6\)For general curved backgrounds with abelian isometries this has been discussed by Alvarez et al 1996, and by Dorn and Otto 1996.
4.3 Evidence for d=11

The third and most striking set of relations are those derived from the conjectured duality between type-IIA string theory and \( \mathcal{M} \) theory compactified on a circle (Witten 1995, Townsend 1995). The eleven-dimensional supergravity couples consistently to a supermembrane (Bergshoeff et al 1987), and has furthermore a (‘magnetic’) five-brane (Güven 1992) with a non-singular geometry (Gibbons et al 1995). After compactification on the circle there exist also Kaluza-Klein modes, as well as a Kaluza-Klein monopole given by the Taub-NUT×\( \mathbb{R}^7 \) space (Sorkin 1983, Gross and Perry 1983). The correspondence between these excitations and the various branes on the type-IIA side is shown in table 3. The missing entry in this table is the eleven-dimensional counterpart of the D8-brane, which has not yet been identified (for a recent attempt see Bergshoeff et al 1997). The problem is that massive type-IIA supergravity (Romans 1986), which prevails on one side of the wall (Polchinski and Witten 1996, Bergshoeff et al 1996), seems to have no ancestor in eleven dimensions (Bautier et al 1997, Howe et al 1998).

Setting aside the D8-brane, let us consider the tensions of the remaining excitations listed in table 3. The tensions are expressed in terms of \( \kappa_{(10)} \) and the Regge slope on the type-IIA side, and in terms of \( \kappa_{(11)} \) and the compactification radius on the \( \mathcal{M} \)-theory side. To compare sides we must identify the ten-dimensional Planck scales,

\[
\frac{\kappa_{(10)}^2}{\kappa_{(11)}^2} = \frac{2\pi R_{11}}{\kappa_{(11)}^2}. \tag{4.15}
\]

Equating the fundamental string tension (\( T_F \)) with the tension of a wrapped membrane fixes also \( \alpha' \) in terms of eleven-dimensional parameters. This leaves us with five consistency checks of the conjectured duality, which are indeed explicitly verified.

How much of this truly tests the eleven-dimensional origin of string theory? To answer the question we must first understand how the entries on the \( \mathcal{M} \)-theory side of table 3 are obtained. Because of the scale invariance of the supergravity equations, the tensions of the classical membrane and fivebrane solutions are a priori arbitrary. Assuming minimal Dirac quantization, and the BPS equality of mass and charge, fixes the product

\[
2\kappa_{(11)}^2 T_2^M T_5^M = 2\pi. \tag{4.16}
\]

An argument fixing each of the tensions separately was first given by Duff, Liu and Minasian (1995) and further developed by de Alwis (1996,1997) and Witten (1997a).\(^7\) It uses the Chern-Simons term of the eleven-dimensional Lagrangian,

\[
I_{11d} = -\frac{1}{2\kappa_{(11)}^2} \int d^{11}x \left[ \sqrt{-G} \left( R + \frac{1}{48} (dA)^2 \right) + \frac{1}{6} A \wedge dA \wedge dA \right], \tag{4.17}
\]

\(^7\)See also Lu (1997), Brax and Mourad (1997, 1998) and Conrad (1997).
### BPS Excitations of Type-IIA String Theory and M Theory Compactified on a Circle

<table>
<thead>
<tr>
<th>tension</th>
<th>type-IIA</th>
<th>$\mathcal{M}$ on $S^1$</th>
<th>tension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\sqrt{\pi}}{\kappa_{(10)}} (2\pi \sqrt{\alpha'})^3$</td>
<td>D0-brane</td>
<td>K-K excitation</td>
<td>$\frac{1}{R_{11}}$</td>
</tr>
<tr>
<td>$T_F = (2\pi \alpha')^{-1}$</td>
<td>string</td>
<td>wrapped membrane</td>
<td>$2\pi R_{11} \left(\frac{2\pi^2}{\kappa_{(11)}^2}\right)^{1/3}$</td>
</tr>
<tr>
<td>$\frac{\sqrt{\pi}}{\kappa_{(10)}} (2\pi \sqrt{\alpha'})^{-1}$</td>
<td>D2-brane</td>
<td>membrane</td>
<td>$T_2^M = \left(\frac{2\pi^2}{\kappa_{(11)}^2}\right)^{1/3}$</td>
</tr>
<tr>
<td>$\frac{\sqrt{\pi}}{\kappa_{(10)}} (2\pi \sqrt{\alpha'})^{-1}$</td>
<td>D4-brane</td>
<td>wrapped five-brane</td>
<td>$R_{11} \left(\frac{2\pi^2}{\kappa_{(11)}^2}\right)^{2/3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{\kappa_{(10)}^2} (2\pi \alpha')$</td>
<td>NS-five-brane</td>
<td>five-brane</td>
<td>$\frac{1}{2\pi} \left(\frac{2\pi^2}{\kappa_{(11)}^2}\right)^{2/3}$</td>
</tr>
<tr>
<td>$\frac{\sqrt{\pi}}{\kappa_{(10)}} (2\pi \sqrt{\alpha'})^{-3}$</td>
<td>D6-brane</td>
<td>K-K monopole</td>
<td>$\frac{2\pi^2 R_{11}^2}{\kappa_{(11)}^2}$</td>
</tr>
<tr>
<td>$\frac{\sqrt{\pi}}{\kappa_{(10)}} (2\pi \sqrt{\alpha'})^{-5}$</td>
<td>D8-brane</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Correspondence of BPS excitations of type-IIA string theory, and of $\mathcal{M}$ theory compactified on a circle. Equating tensions and the ten-dimensional Planck scale on both sides gives seven relations for two unknown parameters. Supersymmetry and consistency imply three Dirac quantization conditions, leaving us with two independent checks of the conjectured duality.
where \( A \equiv \frac{1}{3!} A_{MN R} \, dx^M \wedge dx^N \wedge dx^R \) is the three-index antisymmetric form encountered already in section 2. In a nutshell, the coefficient of this Chern-Simons term is fixed by supersymmetry (Cremmer et al 1978), but in the presence of electric and magnetic sources it is also subject to an independent quantization condition.  

Let me describe the argument in the simpler context of five-dimensional Maxwell theory with a (abelian) Chern-Simons term,

\[
I_{5d}^{\text{MCS}} = -\frac{1}{2\kappa_5^2} \int d^5 x \left( \frac{1}{4} F^2 + \frac{k}{6} A \wedge F \wedge F \right) .
\]

(4.18)

Assume that the theory has both elementary electric charges \( q \) (coupling through \( I_{WZ} = q \int A_\mu dx^\mu \)), and dual minimally-charged magnetic strings. If we compactify the fourth spatial dimension on a circle of radius \( L \), the effective four-dimensional action reads

\[
I_{4d}^{\text{MCS}} = -\frac{1}{2\kappa_4^2} \int d^4 x \left( \frac{1}{4} F^2 + \frac{1}{2} (da)^2 + \frac{k}{2} a F \wedge F \right) ,
\]

(4.19)

where \( \kappa_4^2 = 2\pi L \kappa_5^2 \) and \( a = A_4 \). The scalar field \( a \) must be periodically identified, since under a large gauge transformation

\[
a \rightarrow a + \frac{1}{q} L .
\]

(4.20)

Such a shift changes, however, the \( \theta \)-term of the four-dimensional Lagrangian, and is potentially observable through the Witten effect, namely as a shift in the electric charge of a magnetic monopole (Witten 1979). This latter is a magnetic string wrapping around the compact fourth dimension. To avoid an immediate contradiction we must require that the induced charge be an integer multiple of \( q \), so that it can be screened by elementary charges bound to the monopole.

In order to quantify this requirement, consider the \( \theta \)-term resulting from the shift (4.20). In the background of a monopole field it will give rise to an interaction (Coleman 1981)

\[
-\frac{k}{2\kappa_4^2 q L} \int d^4 x \, F_{r_0}^* F_{r_0}^{(\text{monopole})} = \frac{2\pi^2 k}{\kappa_5^2 q^2} \int dt A_0 ,
\]

(4.21)

where we have here integrated by parts and used the monopole equation \( \partial_r F_{r_0}^{(\text{monopole})} = (2\pi/q) \delta^{(3)}(\vec{r}) \). The interaction (4.21) describes precisely the

\[\text{The quantization of the abelian Chern-Simons term in the presence of a magnetic source was first discussed in 2+1 dimensions (Henneaux and Teitelboim 1986, Polychronakos 1987).}\]
Witten effect, i.e. the fact that the magnetic monopole has acquired a non-vanishing electric charge. Demanding that the induced charge be an integer multiple of \( q \) leads, finally, to the quantization condition

\[
q^3 = \frac{2\pi^2 k}{\kappa^2(5)n}.
\] (4.22)

This is the sought-for relation between the coefficient of the (abelian) Chern-Simons term and the elementary electric charge of the theory.

Let us apply now the same reasoning to \( \mathcal{M} \)-theory. Compactifying to eight dimensions on a three-torus gives an effective eight-dimensional theory with both electric and magnetic membranes. The latter are the wrapped five-branes of \( \mathcal{M} \)-theory, which may acquire an electric charge through a generalized Witten effect. Demanding that a large gauge transformation induce a charge that can be screened by elementary membranes leads to the quantization condition

\[
(T_2^M)^3 = \frac{2\pi^2}{\kappa^2(11)n}.
\] (4.23)

This relates the electric charge density or membrane tension, \( T_2^M \), to the coefficient, \( k = 1 \), of the Chern-Simons term. The membrane tension predicted by duality corresponds to the maximal allowed case \( n = 1 \).

We can finally return to our original question: How much evidence for the existence of an eleventh dimension in string theory does the ‘gedanken data’ of table 3 contain? Note first that Dirac quantization relates the six tensions pairwise. Furthermore, since the maximal non-chiral 10d supergravity is unique, it must contain a \( B \wedge H^{(4)} \wedge H^{(4)} \) term obtained from the 11d Chern-Simons term by dimensional reduction. An argument similar to the one described above can then be used to fix the product \( T_2(2)T_F \) of D2-brane and fundamental-string tensions. Thus, supersymmetry and consistency determine (modulo integer ambiguities) all but two of the tensions of table 3, without any reference either to the ultraviolet definition of the theory or to the existence of an eleventh dimension. We are therefore left with a single truly independent check of the conjectured duality, which we can take to be the relation

\[
T_{(0)}T_F = 2\pi T_{(2)}.
\] (4.24)

This is a trivial geometric identity in \( \mathcal{M} \)-theory, which had no a priori reason to be satisfied from the ten-dimensional viewpoint.

The sceptic reader may find that a single test constitutes little evidence for the duality conjecture. \(^9\) The above discussion, however, underscores what might be the main lesson of the ‘second string revolution’: the ultimate theory

\(^9\)To be sure, the existence of threshold bound states of D-particles – the Kaluza-Klein modes of the supergraviton – constitutes further, a priori independent, evidence for the duality conjecture (Yi 1997, Sethi and Stern 1998, Porrati and Rozenberg 1998).
may be unique precisely because reconciling quantum mechanics and gravity is such a constraining enterprise.

5 D-brane interactions

D-branes in supersymmetric configurations exert no net static force on each other, because (unbroken) supersymmetry ensures that the Casimir energy of open strings is zero. Setting the branes in relative motion (or rotating them) breaks generically all the supersymmetries, and leads to velocity- or orientation-dependent forces. We will now extend Pochinski’s calculation to study such D-brane interactions. Some surprising new insights come from the close relationship between brane dynamics and supersymmetric gauge theory—a theme that will be recurrent in this and in the subsequent sections. Two results of particular importance, because they lie at the heart of the M(atrix)-model conjecture of Banks et al. (1997), are the dynamical appearance of the eleven-dimensional Planck length, and the simple scaling with distance of the leading low-velocity interaction of D-particles. Since space-time supersymmetry plays a key role in our discussion, we will first describe in some more detail the general BPS configurations of D-branes.

5.1 BPS configurations

A planar static D-brane is a BPS defect that leaves half of the space-time supersymmetries unbroken. This follows from the equality $T_{(p)} = \rho_{(p)}$, and the (rigid) supersymmetry algebra, appropriately extended to take into account $p$-brane charges (de Azcarraga et al. 1989, see Townsend 1997 for a detailed discussion). Alternatively, we can draw this conclusion from a worldsheet point of view. On a closed-string worldsheet the thirty-two space-time supercharges are given by contour integrals of the fermion-emission operators,

$$Q = \oint \frac{dz}{z} S \quad \text{and} \quad \bar{Q} = -\oint \frac{d\bar{z}}{\bar{z}} \bar{S}. \quad (5.1)$$

Holomorphicity allows us to deform the integration contours, picking (eventually) extra contributions only from points where vertex operators have been inserted. This leads to supersymmetric Ward identities for the perturbative closed-string S-matrix in flat ten-dimensional space-time.

Now in the background of a $Dp$-brane we must also define the action of the (unbroken) supercharges on the open strings. The corresponding integrals, at fixed radial time $\xi$, run over a semi-circle as in figure 4. Moving the integration to a later time, is allowed only if the contributions of the worldsheet boundary vanish. This is the case for the sixteen linear combinations

$$Q + \Pi_{(p)}\bar{Q} = \int_0^\pi d\xi^1 \left( S + \Pi_{(p)} \bar{S} \right), \quad (5.2)$$
The semi-circles are two snapshots of an open string at fixed radial time \( \xi^0 = \log|z| \). A charge is conserved when its time variation can be expressed as a holomorphic plus antiholomorphic contour integral around the shaded region in the upper complex plane. This means that the contributions of the linear segments on the worldsheet boundary should vanish.

for which the holomorphic and antiholomorphic pieces add up to zero on the real axis by virtue of the boundary conditions (3.9). The remaining sixteen supersymmetries are broken spontaneously by the D\( p \)-brane, and cannot thus be realized linearly within the perturbative string expansion.

Consider next a background with two planar static D-branes, to which are associated two operators, \( \Pi_{(p)} \) and \( \tilde{\Pi}_{(\tilde{p})} \). These operators depend on the orientation, but not on the position, of the branes. More explicitly, we can put equation (3.10) in covariant form

\[
\Pi_{(p)} = -\frac{i^{p+1}}{(p+1)!} \omega_{\mu_0 \cdots \mu_p}^{(p)} \Gamma^{\mu_0 \cdots \mu_p} \tag{5.3}
\]

where

\[
\omega^{(p)} \equiv \frac{1}{(p+1)!} \omega_{\mu_0 \cdots \mu_p}^{(p)} dY^{\mu_0} \wedge \cdots \wedge dY^{\mu_p} = \sqrt{-\hat{g}} \, d\zeta^0 \wedge \cdots \wedge d\zeta^p \tag{5.4}
\]

is the (oriented) volume form of the D\( p \)-brane, and we have done some simple \( \Gamma \)-matrix rearrangements. There is of course a similar expression for the tilde brane. In the background of these two D-branes, the linearly-realized supercharges are a subset of (5.2), namely

\[
\mathcal{P}(Q + \Pi_{(p)} \bar{Q}) = \int_0^\pi d\xi^1 \, \mathcal{P}(S + \Pi_{(p)} \bar{S}) \tag{5.5}
\]
with $\mathcal{P}$ an appropriate projection operator. Demanding that the corresponding contour integrals cancel out on a worldsheet boundary that is stuck on the tilde brane leads to the condition

$$\mathcal{P} \tilde{\Pi}(\tilde{p}) = \mathcal{P} \Pi(p) ,$$

which admits a non-vanishing solution if and only if

$$\det \left( 1 - \tilde{\Pi}(\tilde{p}) \Pi^{-1}(p) \right) = 0 .$$

The number of unbroken supersymmetries is the number of zero eigenvalues of the above matrix. Every extra D-brane and/or orientifold imposes of course one extra condition, which has to be satisfied simultaneously.

A trivial solution to these BPS equations is given by two (or more) identical, parallel D$_p$-branes at arbitrary separation $r$. This background preserves sixteen supersymmetries and has, of course, a $r$-independent vacuum energy, consistently with the cancellation of forces found by Polchinski. Flipping the orientation of one brane sends $\Pi(p) \rightarrow -\Pi(p)$, thus breaking all space-time supersymmetries. The resulting configuration describes a brane and an anti-brane, attracting both gravitationally, and through Ramond-Ramond exchange. In the force calculation of section 3.3, this amounts to reversing the sign of the $s = 4$ spin structure, i.e. of the GSO projection for the stretched open string. The surviving Neveu-Schwarz ground state becomes, in this case, tachyonic at a critical separation $r_{cr} = \sqrt{2\pi^2\alpha'}$, beyond which the attractive force between the brane and the anti-brane diverges (Banks and Susskind 1995, Arvis 1983).

Other solutions to the BPS conditions can be found with two orthogonal D-branes. For such a configuration

$$\tilde{\Pi}(\tilde{p}) \Pi^{-1}(p) = \pm \prod_{m \in p \cup \tilde{p}} \Gamma^m ,$$

where $p \cup \tilde{p}$ denotes the set of dimensions spanned by one or other of the branes but not both, and the overall sign depends on the choice of orientations. The eigenvalues of the above operator depend only on the even number ($d_\perp$) of dimensions in $p \cup \tilde{p}$. For $d_\perp = 4n + 2$ the eigenvalues are all purely imaginary, and supersymmetry is completely broken. For $d_\perp = 4$ or 8, on the other hand, half of the eigenvalues are $+1$, so eight of the supersymmetries are linearly-realized in the background. Examples of $d_\perp = 4$ configurations (for early discussions see Bershadsky et al 1996a, Sen, 1996, Douglas 1995) include a D4-brane and a D-particle, a D5-brane and a parallel D-string, or two completely transverse D2-branes. Examples of $d_\perp = 8$ configurations are a D8-brane and a D-particle, or two completely transverse D4-branes. As the reader can verify easily, all configurations with the same value of $d_\perp$ can be (at least formally) related by the T-duality transformations of section 4.2.
It will be useful later on to know the spectrum of an open string stretching between two orthogonal D-branes. Such a string has $d_\perp$ coordinates obeying mixed (DN) boundary conditions: Neumann at one endpoint and Dirichlet at the other. A bosonic DN coordinate has a half-integer mode expansion, while its fermionic partner is integer modded in the Neveu-Schwarz sector and half-integer modded in the Ramond sector. Using the standard expressions for the subtraction constants of integer or half-integer modded fields, we find the mass formula (in units $2\alpha' = 1$),

$$M^2 = \left( \frac{r}{\pi} \right)^2 + 2N_{osc} + \left\{ \begin{array}{cl} d_\perp/4 - 1 & \text{NS} \\ 0 & \text{R} \end{array} \right.,$$

with $N_{osc}$ the sum of the oscillator frequencies. Furthermore, Neveu-Schwarz and Ramond states are spinors of $SO(d_\perp)$ and $SO(1, 9 - d_\perp)$ – the two maximal Lorentz subgroups that such a brane configuration could leave unbroken. Note in particular that for $d_\perp = 4$ the massless states have the content of a six-dimensional hypermultiplet, while for $d_\perp = 8$ the only massless state is a two-dimensional (anti)chiral fermion, which is a singlet of the unbroken chiral $(8,0)$ supersymmetry (Banks, Seiberg and Silverstein 1997, Rey 1997).

There exist also BPS configurations with D-branes at arbitrary angles (Berkooz et al. 1996). A solution, for instance, of equation (5.6) with eight unbroken supersymmetries is given by $P = \frac{1}{2}(1 - \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9)$, and

$$\Pi_{(\rho)}^{-1} = -(\cos\theta \Gamma^6 + \sin\theta \Gamma^8)(\cos\theta \Gamma^7 + \sin\theta \Gamma^9) \Gamma^6 \Gamma^7.$$  

(5.10)

It describes two identical D-branes, one of which spans the dimensions (67) and is transverse to the dimensions (89), whereas the second has undergone a (relative) unitary rotation in the $\mathbb{C}^2$ plane ($x^6 + ix^7, x^8 + ix^9$). The case $d_\perp = 4$ discussed above corresponds to the special angle $\theta = \frac{\pi}{2}$. A more exotic example with six unbroken supersymmetries can be obtained by a rotation that preserves a quaternionic structure (Gauntlett et al. 1997). For a review of BPS configurations of intersecting and/or overlapping branes see Gauntlett (1997).

### 5.2 D-brane scattering

The velocity-dependent forces between D-branes can be analyzed by calculating the semi-classical phase shift for two moving external sources. I will here follow the original calculation (Bachas 1996) for two identical Dp-branes in the Neveu-Ramond-Schwarz formulation. The same results can be obtained in the light-cone boundary-state formalism (Callan and Klebanov 1996, Green and Gutperle 1996, Billo et al. 1997), and can be furthermore extended to different D-branes (Lifschytz 1996), non-vanishing worldvolume fields (Lifschytz 1997, Lifschytz and Mathur 1997, Matusis 1997), orbifold backgrounds (Husain et al. 1997), type-I theory (Danielsson and Ferretti 1997), and to study spin-dependent interactions (Morales et al. 1997, 1998).
Two D-particles scattering with relative velocity $v$ and impact parameter $b$. The broken lines depict a virtual pair of oriented strings being stretched by the relative motion. The imaginary part of the phase shift gives the probability that these virtual strings materialize.

Consider two identical parallel D$p$-branes, one of which is at rest, while the second is moving with velocity $v$ and impact parameter $b$, as shown in figure 5. It is convenient to define the boost parameter corresponding to this relative motion,

$$v \equiv \tanh(\pi \epsilon) .$$

(5.11)

Thinking of the D-brane interaction as Casimir force leads us to study the spectrum of an open string stretched between these two external sources. If the motion is along the ninth dimension, the coordinates $X^1, \cdots, 8$ retain their conventional mode expansions. The mode expansion of the light-cone coordinates $X^\pm = (X^0 \mp X^9)/\sqrt{2}$, on the other hand, is modified to

$$X^\pm = -i \sqrt{\frac{\alpha'}{2}} \sum_{k=\infty}^\infty \left[ \frac{a_k^\pm}{k \pm i \epsilon} z^{-k \mp i \epsilon} + \frac{a_k^\mp}{k \pm i \epsilon} \bar{z}^{-k \mp i \epsilon} \right].$$

(5.12)

It is indeed easy to verify that $X^0$ and $X^9$ obey, respectively, Neumann and Dirichlet conditions at $\xi^1 = 0$, so that one end-point of the open string is fixed on the static D$p$-brane. Furthermore, $X^\pm(\xi^0, \pi) = e^{\pm \pi \epsilon} X^\pm(\xi^0, 0)$, so that the other endpoint is boosted with velocity $v$, consistently with the fact that it is fixed on the moving D$p$-brane. The mode expansions of the fermionic
light-cone coordinates can be derived similarly with the result

\[ \psi^\pm = \sqrt{\alpha'} \sum_k \psi^\pm_k z^{-k+\pm i\epsilon}, \quad \bar{\psi}^\pm = \sqrt{\alpha'} \sum_k \psi^\mp_k \bar{z}^{-k-\mp i\epsilon}, \quad (5.13) \]

where \( k \in \mathbb{Z} \) in the Ramond sector of the open string, while \( k \in \mathbb{Z} + \frac{1}{2} \) in the Neveu-Schwarz sector.

The relevant feature in the above expressions is the shift of all oscillator frequencies by an amount \( \pm i\epsilon \). Similar expansions arise in the twisted sectors of an orbifold, with \( i\epsilon \) replaced by a (real) rotation angle. Using the standard formulae for the partition functions of free massless fields with twisted boundary conditions we find (here \( 2\alpha' = 1 \))

\[ \delta(b, v) = -2 \times \frac{V(p)}{2} \int_0^\infty \frac{dt}{t} (2\pi^2 t)^{-p/2} e^{-t^2/2\pi} Z(\epsilon, t), \quad (5.14) \]

where

\[ Z(\epsilon, t) = -\frac{1}{2} \sum_{s=2,3,4} (-)^s \frac{\theta_s(\frac{u}{2} | \frac{v}{2}) \theta_s^3(0 | \frac{u}{2})}{\theta_1(\frac{u}{2} | \frac{v}{2}) \eta^9(\frac{u}{2})}. \quad (5.15) \]

Expressions (5.14–5.15) generalize Polchinski’s calculation to the case of moving D-branes. As a check of normalizations notice that in the leading \( v \to 0_+ \) approximation the above result reduces correctly to the quasi-static phase shift

\[ \delta(b, v) \simeq \int_{-\infty}^\infty d\tau \mathcal{E}(\sqrt{b^2 + \nu^2}\tau^2) + \cdots, \quad (5.16) \]

with \( \mathcal{E}(\tau) \) the static interaction energy, eq. (3.24). This follows from the fact that for small (first) argument the function \( \theta_1 \) vanishes linearly, with \( \theta_1'(0|\tau) = 2\pi \eta^3(\tau) \). Of course, since the D-branes feel no static force, this leading quasi-static phase shift is zero.

The supergravity, \( b \to \infty \), limit of the phase shift can be obtained from the \( t \to 0 \) corner of the integration region. It is to this end convenient to first put the partition function, using Jacobi’s identity, in the simpler form (Green and Gutperle 1996),

\[ Z(\epsilon, t) = \frac{\theta^4_1(\frac{u}{2} | \frac{v}{2})}{\theta_1(\frac{u}{2} | \frac{v}{2}) \eta^9(\frac{u}{2})}. \quad (5.17) \]

With the help of the modular transformations

\[ \theta_1(\frac{-\nu}{\tau} | -\frac{1}{\tau}) = \sqrt{i\tau} e^{i\pi\nu^2/\tau} \theta_1(\nu|\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau), \quad (5.18) \]

as well as of the product representations (here \( q = e^{2i\pi\tau} \))

\[ \frac{\theta_1(\nu|\tau)}{\eta(\tau)} = 2q^{1/2} \sin(\pi\nu) \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i\nu})(1 - q^n e^{-2\pi i\nu}) \quad (5.19) \]
and
\[ \eta(\tau) = q^{\frac{1}{8\pi}} \prod_{n=1}^{\infty} (1 - q^n), \]  
(5.20)
we can easily extract the \( t \to 0 \) behaviour of the partition function. This leads to the following asymptotic behaviour for the phase shift, in the limit \( b \to \infty \):
\[ \delta \simeq -\frac{V(p)}{(2\pi \sqrt{\alpha'})^p} \Gamma \left( 3 - \frac{p}{2} \right) \left( \frac{4\pi \alpha'}{b^2} \right)^{3-p/2} \frac{\sinh^4(\pi \epsilon/2)}{\sinh(\pi \epsilon)} + \cdots , \]  
(5.21)
where the corrections come from the exchange of massive closed strings, and hence fall off exponentially with distance. It is a simple (but tedious) exercise to recover the above result by repeating the calculation of section 3.2, with one of the two external sources boosted to a moving frame. Alternatively, this result can be compared to the classical action for geodesic motion in the appropriate supergravity background (Balasubramanian and Larsen 1997).

5.3 The size of D-particles

That the string calculation should reproduce the supergravity result at sufficiently large impact parameter is reassuring, but hardly surprising. A more interesting question to address is what happens if we try to probe a D-brane at short, possibly substringy scales. In order to answer this question let us note that the partition function \( Z(\epsilon, t) \) has poles along the integration axis, at \( \epsilon t/2 = k\pi \) with \( k \) any odd positive integer. These correspond to zeroes of the trigonometric sine in the product representation of \( \theta_1 \). As a result the phase shift acquires an imaginary (absorptive) part, equal to the sum over the positions of the poles of \( \pi \) times the residue of the integrand. A straightforward calculation gives (Bachas 1996)
\[ \text{Im} \delta = \sum_{\text{multiplets}} \frac{\text{dim}(s)}{2} \sum_{k \text{ odd}} \exp \left[ -\frac{2\pi \alpha' k}{\epsilon} \left( \frac{b^2}{(2\pi \alpha')^2} + M(s)^2 \right) \right], \]  
(5.22)
where the sum runs over all supermultiplets of dimension \( \text{dim}(s) \) and oscillator-mass \( M(s) \) in the open-string spectrum, and we have restricted our attention to D-particles, i.e. we have set \( p = 0 \). This result has a simple interpretation: as the two D-particles move away from each other, they transfer continuously their energy to any open strings that happen to stretch between them (see figure 5). A virtual pair of open strings can thus materialize from the vacuum and stop completely, or slow down the motion. \(^{10}\) The phenomenon is T-dual to the more familiar pair production in a background electric field, whose

\(^{10}\) Since the D-branes are extremely heavy at weak string coupling, the back reaction is a higher-order effect. For a discussion of D-brane recoil see Berenstein et al (1996), and Kogan et al (1996).
rate in open string theory has been calculated earlier by Bachas and Porrati (1992). It is worth stressing that this imaginary part cannot arise from the exchange of any finite number of closed-string states.

The onset of this dissipation puts a lower limit on the distance scales probed by the scattering,

\[ b \gtrsim \sqrt{\epsilon/T_F} , \]  

(5.23)

where \( T_F = (2\pi \alpha')^{-1} \) is the fundamental string tension, and the symbol \( \gtrsim \) stands for ‘sufficiently larger than’. In the ultrarelativistic regime \( v \to 1 \), so that \( \epsilon \simeq -\frac{1}{2\pi} \log(1 - v^2) \gg 1 \). The D-particle behaves in this limit as a black absorptive disk, of area much bigger than string scale and growing logarithmically with energy. This typical Regge behaviour characterizes also the high-energy scattering of fundamental strings (see for example Amati et al 1987, 1989). To probe substringy distances we must consider the opposite regime of low velocities, \( \epsilon \simeq v/\pi \ll 1 \). The stringy halo is not excited, in this regime, all the way down to impact parameters \( b \gtrsim \sqrt{v/\pi T_F} \). Quantum mechanical uncertainty \( (\Delta x \Delta p \gg 1) \) puts, on the other hand, an independent lower limit

\[ b T_{(0)} \nu \gtrsim 1 . \]  

(5.24)

Saturating both bounds simultaneously gives

\[ b_{\text{min}}^3 \sim \frac{1}{T_F T_{(0)}} \sim \frac{1}{T_{(2)}} , \]  

(5.25)

where we have used here the tension formula (4.24). We thus conclude that the dynamical size of D-particles is comparable to the (inverse cubic root of the) membrane tension, i.e. to the eleven-dimensional Planck scale of M-theory! Since this is smaller than string length at weak string-coupling, perturbative string theory does not capture all the degrees of freedom at short scales.\(^{11}\)

The fact that D-branes are much smaller than the fundamental strings at weak string coupling was conjectured early on by Shenker (1995). The appearance of the eleven-dimensional Planck scale in the matrix quantum mechanics of D-particles was first noticed by Kabat and Pouliot (1996) and by Danielsson and Ferretti (1996). A systematic analysis of the above kinematic regime, and of the validity of the approximations, was carried out by Douglas et al (1996). Needless to say that this small dynamical scale of D-particles cannot be seen by using fundamental-string probes – one cannot probe a needle with a jelly pudding, only with a second needle! This is confirmed by explicit calculations of closed-string scattering off target D-branes (see Hashimoto and Klebanov 1997, Thorlacius 1998 and references therein).

\(^{11}\)One should contrast this with the example of magnetic monopoles in N=2, d=4 Yang-Mills theory, whose size is comparable to the Compton wavelength of the fundamental quanta. Thus, even though monopoles are very heavy at weak coupling, the high-energy behaviour of the theory is still correctly captured by the (super)gauge bosons.
One other striking feature of the low-velocity dynamics of D-particles follows from an analysis of the real part of the phase shift. Expanding expressions (5.14, 5.17) for $\epsilon \simeq v \to 0$, and for any impact parameter, we find

$$\delta(b, v) \simeq -\epsilon^3 \left( \frac{2\pi^2\alpha'}{b^2} \right)^3 + o(\epsilon^7). \quad (5.26)$$

Notice that $\delta$ must flip sign under time reversal and is hence an odd function of velocity, and that the interaction time blows up as $1/|v|$ because two slow particles stay longer in the vicinity of each other. The generic form of the low-velocity expansion therefore is: $\delta(v, b) = \delta_0(b)/v + \delta_1(b)v + \delta_2(b)v^3 + \cdots$. Comparing with eq. (5.26) we conclude that, not only the static, but also the $o(v^2)$ force between two D-particles is zero. Since the $o(v^2)$ scattering of heavy solitons can be described by geodesic motion in the moduli space of zero modes (Manton 1982), what we learn is that the moduli space of D-particles is (at least to this order of the genus expansion) completely flat. Furthermore, as first recognized clearly by Douglas et al (1997), the leading $o(v^4)$ interaction has the same power-law dependence on impact parameter in the supergravity regime ($b \gg \sqrt{\alpha'}$) as at substringy scales ($b \ll \sqrt{\alpha'}$). Both of these facts are a result of space-time supersymmetry. As will become, indeed, clear in the following section, our phase-shift calculation could be rephrased as a one-loop calculation of the effective quantum action for a vector multiplet, in a theory with sixteen unbroken supercharges. Velocity is related by supersymmetry to the field strength, so that the $o(v^{2k})$ force between D-branes can be read off the $2k$-derivative terms in the quantum action. Using helicity-supertrace formulae it can be shown that, at one-loop order, the two-derivative terms are not corrected, while the only contributions to the four-derivative terms come from short (BPS) supersymmetry multiplets (Bachas and Kiritsis 1997). Since all excited states of an open string are non-BPS, this explains why the leading $o(v^4)$ interaction has a trivial dependence on the string scale.

This result implies that the (matrix) quantum mechanics of D-particles, obtained by truncating the open-string theory to its lightest modes, captures correctly the leading $o(v^4)$ supergravity interactions. It was, furthermore, shown recently that these interactions are not modified by higher-loop and non-perturbative corrections (Paban et al 1998, see also Becker and Becker 1997, Dine and Seiberg 1997, Dine et al 1998). This is an important ingredient of the conjecture by Banks et al (1997), which will not be pursued here any further. We will instead go back now and discuss the classical worldvolume actions of D-branes.

12This would not be true if the scattering branes carried electric and magnetic charge.
6 Worldvolume actions

Although the full dynamics of a soliton cannot be separated from the field theory in which it belongs, its low-energy dynamics can be approximated by quantum mechanics in the moduli-space of zero modes. For an extended \( p \)-brane defect, the zero modes give rise to massless worldvolume fields, and the quantum mechanics becomes a \((p + 1)\)-dimensional field theory. Similar considerations apply to a \( D_p \)-brane, whose long-wavelength dynamics we will analyze in this section. Two striking features of this analysis are (i) the natural emergence of a noncommutative space-time, and (ii) the power of the combined constraints of T-duality and Lorentz invariance.

6.1 Noncommutative geometry

The perturbative excitations of a static planar \( D_p \)-brane are described by an open string theory, interacting with the closed strings in the bulk. The Dirichlet boundary conditions do not modify the usual spectrum of the open string, but force its center-of-mass momentum to lie along the \( D \)-brane. The low-energy excitations make up, therefore, a vector supermultiplet dimensionally-reduced from ten down to \((p + 1)\) dimensions,

\[
A^\mu(\zeta^\beta) \to A^\alpha = 0, \ldots, p(\zeta^\beta), \quad Y^m = p + 1, \ldots, 9(\zeta^\beta). \tag{6.1}
\]

The worldvolume scalars \( Y^m(\zeta^\beta) \) are the transverse space-time coordinates of the \( D_p \)-brane, i.e. the Goldstone modes of broken translation invariance. There are no physical degrees of freedom in the longitudinal coordinates, which in the natural ‘static’ gauge are used to parametrize the worldvolume, \( Y^\alpha = \zeta^\alpha \). The extra physical bosonic excitations of the open string correspond instead to a worldvolume-vector gauge field – a feature that characterizes all \( D \)-branes, and which was overlooked in the earlier supersymmetric ‘brane scans’ (see Duff 1997).

That much can be in fact deduced from an analysis of the low-energy supergravity solutions. String theory becomes, however, important when one considers multiple, closely-spaced \( D \)-branes. In addition to the massless vector multiplets describing the dynamics of each defect, there are now extra potentially-light fields corresponding to the (ground states of) open strings stretching between different \( D \)-branes. In the simplest case of \( n \) parallel identical \( D_p \)-branes, the ensuing low-energy field theory is a dimensionally-reduced maximally-supersymmetric Yang-Mills theory as above, but with a non-abelian gauge group

\[
U(n) \simeq U(1)_{CM} \times SU(n)_{\text{relative}}. \tag{6.2}
\]

This is, indeed, the low-energy limit of an oriented open-string theory with a Chan-Paton index \( i = 1, \cdots n \), labelling the \( n \) possible string endpoints.
A D-brane sandwich, and the four types of oriented open strings giving rise to massless states in the coincidence limit. Each multiplet $A_{ij}$ contains a worldvolume vector and $9 - p$ worldvolume scalars. These latter are the non-commuting transverse coordinates of the D-branes.

The special case $n = 2$ is illustrated in figure 6. The scalar fields of the $U(1)$ vector multiplet, $Y_{\text{CM}}^m = (Y_{11}^m + Y_{22}^m)/2$, are the transverse center-of-mass coordinates of the ‘sandwich’, while the relative motion is described by (matrix-valued) coordinates in the Lie algebra of $SU(2)$. The scalar potential, $V \sim \text{tr} \ [Y^m, Y^r]^2$, is flat for any mutually commuting expectation values $< Y^m >$. These correspond precisely to the arbitrary positions of the two D-branes, consistently with the fact that the net static force vanishes. At non-zero separation, the complex vector multiplet $A_{12} + i A_{21}$ acquires a mass, and the $SU(2)$ gauge theory is in a spontaneously-broken, Coulomb phase. It is intriguing that the permutation of the brane positions is an element of the Weyl subgroup of $SU(2)$ – quantum indistinguishability of the excitations is thus part of the local, gauge symmetry in this picture. The non-abelian nature of the D-brane coordinates, first recognized clearly by Witten (1996a), puts in a precise context earlier more general ideas about the possible role of noncommutative geometry in physics (see for instance Connes 1994, Madore 1995).
In the type-I theory, the above $U(n)$ vector multiplet is truncated by the orientifold projection. The projection must antisymmetrize the Chan-Paton indices of the worldvolume vector and symmetrize those of the worldvolume scalars, or vice versa. The reason is that the corresponding vertex operators,

$$V^a = \int d\xi^a \partial_b X^a e^{ip \cdot X}, \quad V^m = \frac{1}{2\pi \alpha'} \int d\xi^a \epsilon^{ab} \partial_b X^m e^{ip \cdot X},$$  \hspace{0.5cm} (6.3)

have opposite parity under worldsheet orientation reversal. Tadpole cancellation forces an antisymmetric projection for the 9-branes, giving the standard $SO(32)$ gauge group. Consistency of the operator-product expansions then requires (Gimon and Polchinski 1996) a $SO(n)$ gauge group for the D-strings and a $USp(n)$ group for the D5-branes, with the worldvolume scalars in the symmetric, respectively antisymmetric, $n \otimes n$ representations. A single D-string, in particular, has no worldvolume gauge fields, consistently with the fact that it is dual to the heterotic string (Polchinski and Witten 1996). Likewise a single D5-brane has no transverse coordinates – the minimal dynamical excitation, dual to the heterotic five-brane, is a pair of D5-branes with a worldvolume $USp(2) \simeq SU(2)$ gauge field (Witten 1996b).

Figure 6 summarizes in itself many of the new insights brought by D-branes. The light states of stretched open strings, generically invisible in the effective supergravity, are the important degrees of freedom in various settings. They are responsible, in particular, for the thermodynamic properties of near-extremal black holes (Strominger and Vafa 1996, Callan and Maldacena 1996), and for the richness of the D-particle spectrum which lies at the heart of the (M)atrix-model conjecture (Banks et al 1997). Furthermore, the realization of supersymmetric gauge field theories as worldvolume theories has led to an improved understanding of the former through brane constructions (Hanany and Witten 1997, Banks et al 1996, Elitzur et al 1997a,1997b), while more recently the connection with supergravity has raised new hopes of solving certain large-$n$ superconformal gauge theories in the planar, ’t Hooft limit (Maldacena 1997, Gubser et al 1998, Witten 1998).

### 6.2 Dirac-Born-Infeld and Wess-Zumino terms

The effective action of a D-brane, used in the force calculation of section 3, can be generalized to take into account the dynamics of the worldvolume gauge field, and the coupling to arbitrary supergravity backgrounds. The action for a single D-brane can be written as

$$I_{Dp} = \int d^{p+1}\zeta \left( \mathcal{L}_{DBI} + \mathcal{L}_{WZ} + \cdots \right),$$  \hspace{0.5cm} (6.4)

where the Dirac-Born-Infeld and Wess-Zumino (or Chern-Simons) lagrangians are given by

$$\mathcal{L}_{DBI} = T(p) e^{-\Phi} \sqrt{-\det \left( \hat{G}_{\alpha\beta} + \hat{B}_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta} \right)},$$  \hspace{0.5cm} (6.5)
and
\[ \mathcal{L}_{WZ} = T(p) \hat{C} \wedge e^{2\pi\alpha'F} \wedge G \mid_{(p+1)-\text{form}}. \] (6.6)

Here \( \hat{B}_{\alpha\beta} \) is the pull-back of the Neveu-Schwarz two-form,
\[ \hat{C} \equiv \sum_n \frac{1}{n!} \hat{C}_{\alpha_1 \cdots \alpha_n} d\zeta^{\alpha_1} \wedge \cdots \wedge d\zeta^{\alpha_n} \] (6.7)
is the sum over all electric and magnetic RR-form potentials, pulled back to the worldvolume of the D-brane, and \( F = dA \) is the worldvolume field-strength two form, normalized so that the coupling on a boundary of the fundamental-string worldsheet is \( \oint A_\alpha dX^\alpha \). The geometric part of the Wess-Zumino action reads
\[ G = \sqrt{A(T)/A(N)} = 1 - \frac{1}{48} \left( p_1(T) - p_1(N) \right) + \cdots \] (6.8)
where \( T \) and \( N \) are the tangent and normal bundles of the brane, \( A \) is the appropriately-normalized ‘roof genus’,\(^{13}\) and \( p_1 \) is the first Pontryagin class (see for instance Milnor and Stasheff 1974, Eguchi et al 1980, or Nakahara 1990 for definitions). The next term in the expansion of (6.8) is an eight-form, whose presence in the D-brane action has not been explicitly checked. All multiplications in \( \mathcal{L}_{WZ} \), including those in the Taylor expansions of the square root and of the exponential, must be understood in the sense of forms – what one integrates in the end is the coefficient of the \( d\zeta^0 \wedge \cdots \wedge d\zeta^p \) term in the expansion. Strictly-speaking, since the RR potentials cannot be globally defined in the presence of D-branes, one must use the fact that \( e^{2\pi\alpha'F} \wedge G - 1 \) is an exact form, and integrate by parts to express all but Polchinski’s coupling in terms of the RR field-strengths. Note, finally, that since \( T \oplus N = \hat{S} \), the pullback of the space-time tangent bundle, we can use the multiplicative property of the roof genus,
\[ A(T) \wedge A(N) = A(\hat{S}) = 1 + \frac{1}{192\pi^2} \text{tr} (\hat{R} \wedge \hat{R}) + \cdots, \] (6.9)
to trade the dependence on either \( T \) or \( N \) for dependence on the (pulled back) target-space curvature two-form, \( \hat{R}_{\alpha\beta} d\zeta^\alpha \wedge d\zeta^\beta \).

The Dirac-Born-Infeld lagrangian is a generalization of the geometric volume of the brane trajectory, in the presence of Neveu-Schwarz antisymmetric tensor and worldvolume gauge fields (Leigh 1989). It was first derived in the context of type-I string theory in ten dimensions (Fradkin and Tseytlin 1985). The Wess-Zumino lagrangian, on the other hand, generalizes Polchinski’s coupling of Dp-branes to Ramond-Ramond \((p+1)\)-forms. The gauge-field

\(^{13}\)The conventional normalization amounts to choosing units \( 4\pi^2\alpha' = 1 \), in which all type-II D-branes have the same tension \( \sqrt{\pi}/\kappa(10) \). The roof genus is also frequently denoted \( \hat{A} \), but I here reserve the use of hats to denote pullbacks on the worldvolume.
dependence was derived by Li (1996a) and Douglas (1996), and the gravitational terms for trivial normal bundle by Bershadsky et al (1996) and Green et al (1997). The extension to non-trivial normal bundles was given in special cases by Witten (1997b) and Mourad (1997), and more generally by Cheung and Yin (1997) and Minasian and Moore (1997). Unlike $\mathcal{L}_{WZ}$, which is related as we will see to anomalies and is thus believed to be exact, the ‘kinetic’ action is known to receive corrections involving acceleration terms and derivatives of the field-strength background (Andreev and Tseytlin 1988, Kitazawa 1987). These reflect the non-local nature of the underlying open-string theory. The fermionic completion of the action (6.5-6.6), compatible with space-time supersymmetry and with worldvolume $\kappa$-symmetry, has been derived by several authors (Bergshoeff and Townsend 1997, Cederwall et al 1997a,1997b, Cederwall 1997, Aganagic et al 1997a,1997b, Abou Zeid and Hull 1997), but will not be discussed in this lecture.

The generalization of this action to multiple D-branes is non-trivial. The transverse fluctuations $Y^m$, the ‘tangent frame’ $\partial_\alpha Y^\mu$ used to pullback tensors to the worldvolume, and the field strength $F_{\alpha\beta}$, all take now their values in the Lie algebra of $U(n)$. The tree-level action, obtained from the disk diagram, must be given by a single trace in the fundamental representation of the gauge group, but the ordering of the various terms is a priori ambiguous. Things simplify considerably if the supergravity backgrounds do not depend on the coordinates $x^m$ that are transverse to the unperturbed D-branes. 14 T-duality reduces in this case the problem to that of finding the non-abelian extension of the gauge-field action only. This is straightforward for $\mathcal{L}_{WZ}$, in which we need only make the replacement

$$e^{2\pi\alpha'F} \rightarrow \text{tr}_n e^{2\pi\alpha'F} .$$

(6.10)

The proper non-abelian generalization of the Born-Infeld action, on the other hand, is not known. The leading quadratic term in the $\alpha'$-expansion of this action is unambiguous, thanks to the cyclic property of the trace. The ordering ambiguities in the next-to-leading, quartic, term are resolved by the well-known fact that the 4-point disk-amplitude has total symmetry under permutations of the external states (Green, Schwarz and Witten 1987). One natural generalization (Tseytlin 1997) is to evaluate all higher-order terms with the same symmetrized-trace prescription, but this fails to reproduce some known facts about the open-string effective action (Hashimoto and Taylor 1997, see also Brecher and Perry 1998, Brecher 1998). Since commutators of field strengths cannot be separated in invariant fashion from higher-derivative (‘acceleration’) terms, there is in fact no reason to expect that a non-abelian D-brane action in a simple, closed form will be found.

Having summarized the known facts about effective D-brane actions, we

---

14The more general case has been considered by Douglas (1997) and Douglas, Kato and Ooguri (1997).
will spend the remainder of this review justifying them, and exhibiting some of their salient features.

### 6.3 Type-I theory

We first consider the special case $p = 9$, which allows contact with the familiar action of the type-I string theory. The type-I background has 32 D9-branes plus an orientifold (Sagnotti 1988, Hořava 1989a), which truncates $U(32)$ to $SO(32)$ and projects out of the spectrum all antisymmetric-form fields other than $C^{(2)}$ and its dual $C^{(6)}$. Since the gauge fields live on the D9-branes, their action should be given entirely by $I_{D9}$, after appropriate truncation of the unnecessary fields. Note, in contrast, that the purely-gravitational part of the type-I, tree-level lagrangian has contributions from three distinct diagrams: sphere, disk and real projective plane. The gauge-field independent pieces in $I_{D9}$ — representing the contributions of the disk — cannot, therefore, be directly matched to the effective supergravity action.

Expanding out the Born-Infeld action in powers of the field strength, neglecting the (leading) cosmological term which is removed by the orientifold projection, and using the total symmetry of the 4-point function, we find

$$I_{BI} = T_{(9)} (\pi \alpha')^2 \int d^{10}x \ e^{-\Phi} \left[ \text{tr} \ (F_{\mu\nu}F^{\mu\nu}) - \frac{(\pi \alpha')^2}{12} \text{tr} \ (t_8 F^4) + \cdots \right] \quad (6.11)$$

where the dots stand for higher orders in $\alpha'$, the $F_{\mu\nu}$ are hermitean, and $t_8$ is the well-known eight-index tensor of string theory (without its $\epsilon$ piece)

$$t_8 F^4 = 16 F_{\mu\nu} F^{\nu\rho} F^{\lambda\mu} F_{\rho\lambda} + 8 F_{\mu\nu} F^{\nu\rho} F_{\rho\lambda} F^{\lambda\mu} - 4 F_{\mu\nu} F^{\nu\rho} F_{\rho\lambda} F^{\lambda\mu} - 2 F_{\mu\nu} F_{\rho\lambda} F^\lambda F^\rho + \cdots \quad (6.12)$$

The quartic piece can be written alternatively as a symmetrized trace,

$$\text{tr} \ (t_8 F^4) = 24 \text{Str} \left( F^4 - \frac{1}{4} (F^2)^2 \right), \quad (6.13)$$

with matrix multiplication of the Lorentz indices implied. Expanding out similarly the Wess-Zumino action of the D9-branes, which have of course a trivial normal bundle, we find

$$I_{GS} = T_{(9)} (\pi \alpha')^2 \int d^{10}x \left[ C^{(6)} \wedge X_4 + (\pi \alpha')^2 C^{(2)} \wedge X_8 \right], \quad (6.14)$$

where

$$X_4 = 2 \text{tr} \ F^2 + \cdots \quad X_8 = \frac{2}{3} \text{tr} \ F^4 + \frac{1}{12} \text{tr} \ F^2 \text{tr} \mathcal{R}^2 + \cdots . \quad (6.15)$$

Here the dots stand for purely-gravitational corrections to $X_4$ and $X_8$, multiplication is in the sense of forms, and we recall that $\mathcal{L}_{WZ}$ was defined only up to total derivatives.
The Green-Schwarz anomaly-cancellation mechanism: the classical interference of the two vertices in $I_{GS}$, through the exchange of a RR two-form, cancels the one-loop hexagon anomaly.

One can recognize in the above expressions many of the standard features of $SO(32)$ superstring theory. The terms in $I_{GS}$ are the Green-Schwarz couplings that cancel the hexagon anomaly, as shown in figure 7 (Green and Schwarz 1984, 1985a,b) They have been calculated directly from the disk diagram by Callan et al (1988). They are often expressed in terms of traces ("Tr") in the adjoint representation of $SO(32)$, via the relations

$$\text{Tr} F^2 = 30 \text{ Tr} F^2, \quad \text{Tr} F^4 = 24 \text{ Tr} F^4 + 3 (\text{tr} F^2)^2.$$  \hspace{1cm} (6.16)

This is less economical but more natural from the point of view of the effective supergravity. The anomalous Green-Schwarz couplings are, furthermore, related, through space-time supersymmetry, to the two leading terms which we have exhibited in the expansion of the Born-Infeld action (de Roo et al 1993, Tseytlin 1996a,b, see also Lerche 1988). Comparing the coefficients of the various terms is a non-trivial check of our normalizations. \footnote{I thank J. Conrad for discussions on this point.} For instance, the tensor structure multiplying $\text{tr} F^4$ has the correct supersymmetric form, $t_8 - \frac{1}{4} \epsilon_{10} C^{(2)}$ (Tseytlin 1996a). The two terms in the $X_8$ polynomial also have the right relative weights (Green et al 1987). Finally, we can put the quadratic Yang-Mills lagrangian in the standard form, $\text{tr} (F_{\mu \nu} F^{\mu \nu}) / 2 g_{(10)}^4$, with

$$\frac{g_{(10)}^4}{\kappa_{(10)}^2} = 2^{11} \pi^7 \alpha'$.  \hspace{1cm} (6.17)

This is, indeed, the relation between the type-I gauge and gravitational coupling constants (Sakai and Abe 1988, see also Shapiro and Thorn 1987, Dai and Polchinski 1989 for the bosonic case).
6.4 The power of T-duality

The effective D-brane actions (6.4–6.6) must be compatible with the T-duality transformations discussed in subsection 4.2. Indeed, being discrete gauge symmetries, T-dualities leave the entire open and closed string theory, and hence also its low-energy limit, unchanged (Dai et al 1989, Hořava 1989b, Green 1991a). Now T-duality transforms in general a $D_p$-brane to a $D_{p'}$-brane, with $p' \neq p$, and exchanges transverse brane coordinates with worldvolume gauge fields. Thus combined with reparametrization invariance, it puts stringent constraints on the dynamics of these latter (Bachas 1996, Douglas 1995, Bergshoeff et al 1996, Bergshoeff and de Roo 1996, Green et al 1996).

We have already explained how the simplest duality – inversion of the radius of a compactification circle – maps wrapped $D_p$-branes to transverse $D(p-1)$ branes, and vice versa. It also maps the corresponding component of the $p$-dimensional gauge field, to the extra transverse coordinate of the lower brane,

$$ A'_{0,\ldots,p-1} = A_{0,\ldots,p-1}, \quad Y'_{p} = 2\pi\alpha' A_{p}, \quad Y'_{p+1,\ldots,9} = Y_{p+1,\ldots,9}. \quad (6.18) $$

Here the $D_p$-brane spans the dimensions $(1,\ldots,p)$, $x^p = x^p + 2\pi R_p$ is the spatial coordinate that we dualize, and both branes are being parametrized in static gauge. All fields are functions of the common worldvolume coordinates $\zeta^0,\ldots,p-1$. In addition, the $D_p$-brane fields may depend on the coordinate $\zeta^p$, or equivalently on its conjugate momentum, in which case the $D(p-1)$-brane fields depend on the dual winding in the $p$th dimension. This dependence drops out in the limit $R_p = \alpha'/R'_p \to 0$, since momentum (or winding prime) modes become infinitely massive and decouple. Thus a non-compact transverse D-brane coordinate can be thought of as a gauge field in some vanishingly-small internal dimension.

Let us focus attention to the case $p = 1$. The effective action of a type-IIB D-string, wrapping around a tiny (first) dimension reads

$$ I = T(1) \int d^2 \zeta \left[ \sqrt{1 - \left(\partial_0 Y^m\right)^2} - (2\pi\alpha' F_{01})^2 + \hat{C}_{01} + 2\pi\alpha' F_{01} \hat{C} \right]. \quad (6.19) $$

We have here assumed for simplicity a flat space-time and vanishing $\Phi$ and $B_{\mu\nu}$ backgrounds. The T-duality transformation $R'_1 = \alpha'/R_1$ changes the D-string to a D-particle, the worldvolume electric field $F_{01} = \partial_0 A_1$ to a velocity, and the RR backgrounds as follows :

$$ C'_1 = C, \quad C'_\mu = C_{\mu1} \quad (\mu \neq 1). \quad (6.20) $$

The effective D-string action transforms therefore to

$$ I' = T'(0) \int d\zeta^0 \left[ \sqrt{1 - (\partial_\zeta Y^0)^2} + \hat{C}'_0 \right]. \quad (6.21) $$

---

\[16\] Extended supersymmetry also relates brane coordinates to gauge fields, thereby constraining the dynamics of the latter.
This is indeed the effective action of a D-particle coupling to the dual RR one-form background, as should be expected.

If one neglects acceleration terms, the action of a charged point particle in flat spacetime is fixed uniquely by Lorentz invariance, together with invariance under reparametrizations of the worldline. Running the logic backwards, we could thus start with the D-particle action (6.21) and invoke T-duality plus locality to determine the (abelianized) gauge dynamics on the worldvolume of D-strings, or higher D-branes. This is quite remarkable, since from simple kinematic constraints one is deriving apparently non-trivial information about open-string gauge dynamics. The anomaly-cancelling Green-Schwarz terms are, for example, related to the covariantization of Polchinski’s coupling (Douglas 1995), while the speed of light \((c = 1)\) is mapped, under T-duality, to the limiting electric field of the Born-Infeld action (Bachas 1996). Notice that, due to Regge behaviour, \(c\) appears as a dynamical rather than purely kinematic limit. The corresponding dissipation mechanism is the pair production calculated in subsection 5.3.

7 Topological aspects of brane dynamics

The effective gauge theories on the worldvolume of D-branes have a rich spectrum of both perturbative and non-perturbative excitations. These are worldvolume projections of the various branes from the bulk which, like fundamental strings, can terminate on the D-branes of interest, or form with them stable bound states. Much can be learned about these dynamics by simple topological considerations of the worldvolume fluxes and charges, and of their spacetime counterparts. We conclude this guided tour of D-branes with a brief discussion of such issues.

7.1 Branes inside branes

One immediate consequence of the Wess-Zumino action (6.6) is that the worldvolume gauge fields and the geometry can induce different RR charges on D-branes. We will illustrate this phenomenon with some concrete examples of \(D_p\)-branes wrapped around a compact cycle \(\Sigma_k\), such as a \(k\)-torus or a supersymmetric \(k\)-cycle of a Calabi-Yau space. To simplify the discussion we assume that the target space is a direct product of \(d\)-dimensional Minkowski space \((\mathbb{R}^d)\) times a compactification manifold \((\Sigma_{10-d})\), and that the brane worldvolume can be factorized accordingly, \(W_{p+1} = \Sigma_k \times W_{p-k+1}\) with \(\Sigma_k \subset \Sigma_{10-d}\) and \(W_{p-k+1} \subset \mathbb{R}^d\). We also assume antisymmetric-tensor backgrounds that are covariantly constant on the compactification manifold, as well as a vanishing dilaton field.

Our first example is a D2-brane wrapped around a two-cycle \(\Sigma_2\), on which
we turn on a (quantized) magnetic flux,
\[ \frac{1}{2\pi} \int_{\Sigma_2} F = k \quad (7.1) \]
This gives rise to a Wess-Zumino coupling
\[ I_{WZ} = T(2) \int_{W_3} \hat{C}^{(3)} + k T(0) \int_{W_1} \hat{C}^{(1)} \quad (7.2) \]
showing that the D2-brane has acquired \( k \) units of RR one-form charge. We are therefore describing a configuration of a D2-brane bound to \( k \) type-IIA D-particles, or equivalently of a (transverse) membrane boosted along the hidden eleventh dimension of M-theory.

To confirm this latter interpretation, notice that the Dirac-Born-Infeld action in 2+1 dimensions reads
\[ I_{DBI} = T(2) \int d^3 \zeta \sqrt{-\hat{G}} \sqrt{1 + 2\pi^2 \alpha' 2 F_{\alpha\beta} F^{\alpha\beta}} \quad (7.3) \]
with indices raised by the induced metric. Extremizing this action for a static D2-brane yields a magnetic field proportional to the volume form \( (\omega_{\Sigma_2}) \) of the 2-cycle,
\[ F = 2\pi k \frac{\omega_{\Sigma_2}}{\omega_{\Sigma_2}} \quad (7.4) \]
Using the relations between type-IIA and M-theory scales we can write the energy of this configuration (for zero RR backgrounds) as
\[ E = \sqrt{M^2 + \left( \frac{k}{R_{11}} \right)^2} \quad \text{with} \quad M = T(2) \int \omega_{\Sigma_2} \quad (7.5) \]
This is the expected energy of an excitation of mass \( M \), carrying \( k \) units of momentum in the eleventh dimension. The scalar dual to the vector field on the worldvolume can be in fact interpreted as the eleventh coordinate of the membrane (Townsend 1996a, Schmidhuber 1996),
\[ \frac{2\pi \alpha' F_{\alpha\beta}}{(1 + 2\pi^2 \alpha' 2 F^2)^{1/2}} = \sqrt{-\hat{G}} \epsilon_{\alpha\beta\gamma} \partial_{\gamma} Y^{11} \quad (7.6) \]
This duality transformation maps indeed the magnetic field (7.4) to a uniform motion along \( x^{11} \), as should be the case.

Our second example consists of \( n \) coincident D4-branes wrapping around a four-cycle \( \Sigma_4 \). A non-abelian \( k \)-instanton configuration,
\[ -\frac{1}{8\pi^2} \int_{\Sigma_4} \text{tr} F \wedge F = k \quad (7.7) \]
induces RR one-form charge equal to that carried by
\[ k + \frac{n}{48} \left( p_1(T) - p_1(N) \right) \] (7.8)


Consider, for example, the case of \( \Sigma_4 \) a four-torus, so that our configuration consists of \( n \) flat D4-branes and \( k \) (anti)D-particles. As explained in subsection 5.1, such a configuration leaves eight unbroken supersymmetries, and has the following content of low-lying open-string excitations: (i) a ten-dimensional \( U(n) \) vector multiplet reduced down to the worldvolume of the four-branes, (ii) a ten dimensional \( U(k) \) vector multiplet reduced likewise to the worldline of the particles, and (iii) a six-dimensional hypermultiplet, in the \((n, k)\) representation of the gauge group, and living also on the particle worldline. In terms of the unbroken supersymmetries, the adjoint fields decompose into vector plus hypermultiplets. The vacuum manifold of this effective theory has a Coulomb branch along which the gauge group breaks generically to \( U(1)^n \times U(1)^k \), and a Higgs branch along which only a single \( U(1) \) remains unbroken. These correspond, respectively, to D-particles separated from the D4-branes in the transverse space, or bound to them as finite-size instantons. The dimension around a generic point on the Higgs branch is given by the total number of scalar fields in the hypermultiplets, minus the number of gauge transformations and of D-flatness conditions (see for example Hassan and Wadia 1997)
\[
\text{dim} \mathcal{M}_k^n = 4(nk + n^2 + k^2) - (n^2 + k^2 - 1) - 3(n^2 + k^2 - 1) = 4(nk + 1).
\] (7.9)
This is indeed the dimension of the moduli space of \( k \) \( U(n) \) instantons on a flat torus. Similar constructions work for instantons on a K3 surface, for which the first Pontryagin class \( p_1(K3) = -48 \) (Bershadsky et al 1996b, Vafa 1996), as well as for instantons on a ALE space (Douglas and Moore 1996).

Our last example is a type-IIB D-string winding around a stable cycle \( \Sigma_4 \) of unit radius. Turning on a worldsheet electric field gives a coupling linear in the Neveu-Schwarz antisymmetric tensor,
\[
I_{D1} = \frac{1}{2\pi\alpha'} \int d^2\zeta \Pi_1 \hat{B}_{01} + \cdots
\] (7.10)
where \( \Pi_1 = \delta \mathcal{L}/\delta \partial_0 A_1 \) is the momentum conjugate to \( A_1 \). We have here gone to the \( A_0 = 0 \) gauge, used \( \zeta^4 \) to parametrize the stable cycle, and set for simplicity the RR backgrounds to zero. Since the Wilson line \( \oint d\zeta^1 A_1 \) is a periodic
variable with period $2\pi$, its conjugate momentum in the quantum theory is integer, $\int d\zeta \Pi_1 = 2\pi q$. The coupling (7.10) describes then precisely the gauge charge carried by $q$ fundamental winding strings, bound to the D-string under consideration (Witten 1996a, Callan and Klebanov 1996, Schmidhuber 1996). This is in accordance with the prediction of SL(2,$\mathbb{Z}$) duality, which requires the existence of subthreshold bound states of $p$ D-strings and $q$ fundamental strings, for all pairs $(p,q)$ of relatively prime integers (Schwarz 1995).

### 7.2 Branes ending on branes

The coupling of D-branes to $B_{\mu\nu}$ can be understood from a simple worldsheet argument. Under a gauge transformation $\delta B = d\Xi$, with $\Xi$ an arbitrary one-form, the action of a fundamental string changes by a boundary term

$$
\delta_\Xi I_F = \frac{1}{2\pi \alpha'} \int_{\partial \Sigma} d\xi^a \hat{\Xi}_a .
$$

(7.11)

To cancel this variation, we must assume that the gauge fields living on the worldvolumes of D-branes have also a universal transformation $\delta A_a = -\hat{\Xi}_a/2\pi \alpha'$. This explains the appearance of the gauge-invariant combination $\hat{B} + 2\pi \alpha' F$ in the Dirac-Born-Infeld action. Invariance of the Wess-Zumino action, on the other hand, requires that the (sum over all) RR potentials transform as

$$
\delta_\Xi C = C \land e^{d\Xi} .
$$

(7.12)

The RR antisymmetric tensors have of course their own independent gauge transformations,

$$
\delta_\Lambda C = d\Lambda ,
$$

(7.13)

with $\Lambda$ (a sum of) arbitrary forms. Redefining the RR potentials, $\tilde{C} \equiv C \land e^{-B}$, so as to make them invariant under the $\Xi$-transformations, modifies the $\Lambda$-transformations which would in this case mix the RR forms and the Neveu-Schwarz tensor.\(^{17}\)

This argument confirms what we have used from the very beginning, i.e. that a fundamental string may end on any D-brane, on whose worldvolume it couples as an elementary electric charge. The dynamics of such open strings can in fact be analyzed from the viewpoint of the worldvolume Born-Infeld action (Callan and Maldacena 1998, Gibbons 1998). We can, however, also generalize the argument to see what other branes can end on D-branes (Strominger 1996). Consider indeed the variation of the Wess-Zumino action of a $Dp$-brane under a gauge transformation of the RR $(p+1)$-form,

$$
\delta_\Lambda I_{Dp} = T_{(p)} \int_{\partial \Sigma_{p+1}} \tilde{\Lambda}^{(p)} ,
$$

(7.14)

\(^{17}\)I thank E. Kiritsis for this argument.
where \( \partial W_{p+1} \) is the boundary of the brane worldvolume. We may cancel this variation if the boundary happens to lie on the worldvolume of a \( D(p+2) \)-brane, on which it appears as the trajectory of a \( (p-1) \)-dimensional magnetic charge! Indeed, the anomalous Bianchi identity on the worldvolume of the \( D(p+2) \)-brane reads

\[
d \wedge F = 2\pi \delta^{(3)}(\partial W_{p+1}) ,
\]

where we have used the normalization of electric charge to one. It can be checked that the variation of the Wess-Zumino action of the higher brane will then cancel precisely (7.14).

We thus conclude that D-strings can terminate on D3-branes, on whose worldvolume they appear as magnetic monopoles, that D2-branes can terminate on D4-branes, and so on for higher \( p \). Notice that if the branes are orthogonal, the number of dimensions along one or other of the branes but not both is exactly four. These configurations leave therefore one quarter of unbroken supersymmetries. Various duality transformations map the above examples to other configurations of branes. Note in particular that lifted to M-theory, the D2-D4 brane configuration teaches us that the M-theory membrane can terminate on a M-theory fivebrane.

### 7.3 Branes created by crossing branes

The final point I want to discuss has to do with the role of Wess-Zumino terms in cancelling chiral anomalies. We already saw this in the context of type I theory, but the phenomenon is more general and can in fact be used to fix completely the form of the Wess-Zumino couplings (Green et al 1997). Consider for example two stacks of \( n \) and \( n' \) D5-branes, spanning worldvolumes \( \mathcal{W}_6 \) and \( \mathcal{W}_6' \), which have generically a two-dimensional intersection, \( \mathcal{I} = \mathcal{W}_6 \cap \mathcal{W}_6' \). Let us concentrate on the gauge part of the Wess-Zumino action of the first stack. Under a worldvolume gauge transformation, this has an anomalous variation given by

\[
\delta_\xi I_{D5} = T_{(5)} (2\pi \alpha')^2 \int_{\mathcal{W}_6} d\tilde{H}^{(3)} \wedge \text{tr}(\xi F) = \frac{n'}{2\pi} \int_{\mathcal{I}} \text{tr}(\xi F) ,
\]

with \( \xi \) in the Lie algebra of \( U(n) \). We have here used the standard descent formulae

\[
\text{tr}(F \wedge F) = d \omega_3(A) \quad \text{and} \quad \delta_\xi \omega_3(A) = d \text{tr}(\xi F) ,
\]

with \( \omega_3(A) \) the Chern-Simons three form, as well as the (anomalous) Bianchi identity

\[
d\tilde{H}^{(3)} = 2\kappa^2_{(10)} T_{(5)} n' \delta^{(2)}(\mathcal{I}) .
\]

This is the projection on \( \mathcal{W}_6 \) of the bulk Bianchi identity showing that the prime branes are magnetic sources for the RR two-form.
Anomalous creation of a stretched string when two orthogonal D4-branes cross.

Thus gauge invariance seems to be violated on the intersection, but the anomaly can be precisely cancelled by $n'$ chiral fermions in the fundamental representation of $U(n)$. But as we have already explained in section 5.1, string theory provides us precisely with the required fermions – the massless states of the fundamental strings stretching from one to the other stack of D5-branes, and transforming in the $(n, \bar{n}')$ representation of the $U(n) \times U(n')$ gauge group. Reversing the argument, since the embedding theory is non-anomalous, the presence of the Wess-Zumino couplings is necessary to cancel the apparent violation of charge conservation on the intersection, by inflow from the bulk of the D5-branes. The gravitational anomaly of the intersection fermions cancels similarly the anomalous variation of the gravitational Wess-Zumino action. To see how one must use the (Whitney sum) decompositions of the tangent bundles (since the branes fill all dimensions)

$$T_{W_b} = T_x \oplus N_{W_b'},$$

and similarly for the prime brane, together with the multiplicative property of the roof genus. It follows that the anomalous variations of the pullback normal bundles cancel between the two stacks of D5-branes, while those of the bundle tangent to the intersection add up and cancel against the anomalous fermion loop.

The anomalous inflow of charge, required to cancel the chiral anomaly on the intersection, has an interesting T-dual interpretation (Bachas et al 1997, Danielsson et al 1997, Bergman et al 1998). Consider indeed two (stacks of) D4-branes oriented so as to have a unique common transverse dimension, say $x_1$. The lowest-lying state of an open string stretching between the two stacks is a chiral fermion, but since it is completely localized in space the role of momentum is played by its (oriented) stretching. It thus satisfies the
T-dual Weyl equation,

\[ p_0 = T_F \delta x_1 , \]  

(7.20)

with \( \delta x_1 \) the transverse displacement of the two D4-branes. Now as the D4-branes move continuously past each other, the above energy level crosses continuously the zero axis. Thus in the second-quantized theory a string must be anomalously created or destroyed, as illustrated in figure 8.

Since this is a topological phenomenon, we should expect it to commute with any (sequence of) duality transformations. Consider in particular the following chain,

(IIA) \( D(2345) \otimes D(6789) \hookrightarrow F(1) \)

\[ \xrightarrow{T(6)} \]

(IIB) \( D(23456) \otimes D(789) \hookrightarrow F(1) \)

\[ \xrightarrow{S} \]

(IIB) \( \text{NS}(23456) \otimes D(789) \hookrightarrow D(1) \)

\[ \xrightarrow{T(56)} \]

(IIB) \( \text{NS}(23456) \otimes D(56789) \hookrightarrow D(156) \)

Here \( F, D \) and \( \text{NS} \) denote a fundamental string, a D-brane and a Neveu-Schwarz fivebrane, the dimensions which these branes span are indicated inside parentheses, and \( X \otimes Y \hookrightarrow Z \) means ‘\( Z \) is being created when \( X \) crosses \( Y \)’. The sequence of T- and S-duality transformations tells us that a D3-brane must be created when a NS-brane and a D5-brane, sharing two common dimensions, cross each other. From the fermionic character of the original stretched fundamental string, we also know that only a single stretched brane in a supersymmetric state is allowed (Bachas et al 1998). These two basic rules of brane engineering have been indeed postulated by Hanany and Witten (1997), in order to avoid immediate contradictions with the known behaviour of three-dimensional supersymmetric gauge models.

I stop here because time is up – not because the subject has been exhausted. The reader has hopefully acquired the tools, as well as the motivation, to move on to some of the exciting applications of D-branes to gauge theories and black-hole physics.
Acknowledgments

I thank David Olive, Pierre van Baal and Peter West for organizing a wonderful Newton-Institute workshop. These lectures were also presented at the 1997 Trieste Spring School on ‘String Theory, Gauge Theory and Quantum Gravity’, and in a shorter format at the ‘31st International Symposium Ahrenshoop’ in Buckow. I thank the organizers for the invitations to speak, and present to them my sincere apologies for failing to meet their (generously-elastic) proceedings deadlines. I am finally indebted to P. Bain, J. Conrad, E. Cremmer, G. Gibbons, M.B. Green, A. Greenspoon, M. Henneaux, A. Kehagias, E. Kiritsis, J.X. Lu, S. Mukhi, H. Partouche, B. Pioline, A. Polychnakos, S. Sethi and P. Vanhove for comments and discussions that helped improve this manuscript.

Note on conventions

Throughout the text I have used $X^\mu(\xi^a)$ for the coordinates of a fundamental string, $Y^\mu(\zeta^\alpha)$ for those of a D-brane, and $x^\mu$ for the space-time coordinates. I reserve the capital N to count supersymmetries, and the little $n$ for the number of D-branes. Hats denote pullbacks of supergravity fields from the bulk onto the worldvolumes of branes. $T_F = 1/2\pi\alpha'$ is the tension of the fundamental string, not to be confused with the worldsheet supercurrent which I have denoted $J_F$. I use ‘et al’ when referring to papers with three or more coauthors, and indicate in parenthesis the publication year when available, or the year of submission to the archives otherwise. All authors and all archive numbers can be found in the bibliography at the end.

References

and Interactions’, Vanderbilt University, and CERN-La Plata-Santiago de Compostela School of Physics, hep-th/9709180.


