Scale Invariance of Rich Cluster Abundance: A Possible Test for Models of Structure Formation

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ABSTRACT

We investigate the dependence of cluster abundance $n(> M, r_{cl})$, i.e., the number density of clusters with mass larger than $M$ within radius $r_{cl}$, on scale parameter $r_{cl}$. Using numerical simulations of clusters in the CDM cosmogonic theories, we notice that the abundance of rich clusters shows a simple scale invariance such that $n[> (r_{cl}/r_0)^{a}M, r_{cl}] = n[> M, r_0]$, in which the scaling index $a$ remains constant in a scale range where halo clustering is fully developed. The abundances of scale $r_{cl}$ clusters identified from IRAS are found basically to follow this scaling, and yield $a \sim 0.5$ in the range $1.5 < r_{cl} < 4h^{-1}$Mpc. The scaling gains further supports from independent measurements of the index $a$ using samples of X-ray and gravitational lensing mass estimates. We find that all the results agree within error limit as: $a \sim 0.5 - 0.7$ in the range of $1.5 < r_{cl} < 4h^{-1}$Mpc. These numbers are in good consistency with the predictions of OCDM ($\Omega_M = 0.3$) and LCDM ($\Omega_M + \Omega_\Lambda = 1$), while the standard CDM model has different behavior. The current result seems to favor models with a low mass density.

Subject headings: cosmology: theory - galaxies: clusters: general - large-scale structure of universe

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1. Introduction

It is generally believed that the gravitational clustering of scale free initial perturbations is self-similar (e.g. Peebles 1980). The statistical descriptions of large scale structures, such as correlation functions and number densities, show somewhat of a scaling behavior. Among various observed objects, clusters probably are the best for studying this gravitational scaling, because their distribution and clustering are dominated by gravitation and less “contaminated” by hydro processes. Therefore, observed clusters can directly be identified as massive halos of N-body simulation samples. A multi-scale identification of clusters will be useful to reveal the expected scaling.

Traditionally, clusters are identified from either observed or simulated distribution of galaxies or masses within a sphere of given radius $r_{cl}$, say, the frequently adopted Abell radius $r_{cl} = 1.5 h^{-1}$ Mpc where $h = H_0/100$ km s$^{-1}$ Mpc$^{-1}$. However, a priori choice of $r_{cl}$ is somewhat arbitrary, or dependent on observational limits. Clusters have already been identified and studied on very different radius $r_{cl}$. For example, a wide range of radius covering from 0.01 - 4.3 $h^{-1}$ Mpc has been used in the current determination of cluster masses (e.g. Wu & Fang 1996, 1997a and references therein). Therefore, a multi-scale identification of clusters is a necessary extension of the “standard” Abell radius $r_{cl} = 1.5 h^{-1}$ Mpc. It will provide information, such as the scale invariance of the mass functions of clusters, which can not be seen by one scale identification. This is the goal of this paper.

As we know, cluster study has been significant for understanding the nature and evolution of cosmic structures on scales of $\sim 1 - 10$ Mpc. In particular, cosmological parameters can be well constrained by the observed properties of clusters, such as the abundances and correlations (Bahcall & Cen 1992 (BC); Jing et al. 1993; Jing & Fang 1994; Viana & Liddle 1996; Carlberg et al. 1997a; Bahcall, Fan & Cen 1997), the substructures (Jing et al. 1995), the luminosity-temperature relations (Oukbir & Blanchard 1997) and the gravitational lensing phenomena (Wu & Fang 1996; Wu et al. 1997a). We believe that the multi-scale identification of clusters and the scaling invariance would be able to add more constraints on relevant parameters.

In §2, we summarize the theoretical background of the possible scaling of the multi-scale identified clusters. In §3, we present the result of identifying clusters on different scales in three popular structure formation models: 1) the standard cold dark matter model (SCDM), 2) low density, flat CDM model with a non-zero $\lambda$ (LCDM), and 3) open CDM model (OCDM). Using these samples, we show that the mass functions of rich clusters are scaling invariant. In §4, the predictions of scaling are compared with observations of clusters identified from IRAS sample. In §5, the index of the scaling is detected from samples of X-ray and gravitational lensing clusters with mass estimates. Finally, we describe our main findings in §6.
2. Theoretical background

It is generally believed that cosmic gravitational clustering after the decoupling can roughly be described by three régimes: 1. linear regime; 2. quasilinear régime which is dominated by scale-invariant radial infall; and 3. fully developed nonlinear régime dominated by virialized nonradial motion. Since the amplitudes of the cosmic temperature fluctuations revealed by COBE are as small as $\Delta T/T \sim 10^{-6}$ (Bennett et al. 1996), the gravitational clustering should remain in the linear régime, or at most in the quasilinear régime, on comoving scales larger than about $10 h^{-1}$ Mpc and at redshifts higher than 2. On the other hand, observations indicate that clusters of galaxies with size of about $1.5 h^{-1}$ Mpc are probably the largest fully developed or virialized objects. Thus, the quasilinear evolution should be substantial on the scales from about 2 to 10 $h^{-1}$ Mpc. Obviously, these facts are useful for discriminating among models of structure formation.

Because different régime generally has different behavior of scaling, an effective way of doing model discrimination would be to compare the model predicted scaling with observations.

In linear régime, the scaling is straightforward. If the primeval density perturbations are Gaussian and have power law spectrum $P(k) = |\delta(k)|^2 = Ak^n$, the variance of the initial mass fluctuation within a spherical region on length scale $R$ is given by $\langle M \rangle_R \propto R^{(3-n)/2}$. Therefore, there is a scaling of $R \rightarrow R'$, and $\langle M \rangle_{R'} \rightarrow (R'/R)^{(3-n)/2} \langle M \rangle_R$. Since $\langle M \rangle_R$ is a statistical average with respect to $N(M, R)dM$, the number density of radius $R$ halo with mass from $M$ to $M + dM$, the scaling of $\langle M \rangle_R$ indicates the scaling of the number density as

$$n[> (R'/R)^{(3-n)/2} M, R'] = n(> M, R)$$

(1)

where

$$n(> M, R) \equiv \int_M^\infty N(M, R)dM$$

(2)

which is the accumulative number density of radius $R$ halo with mass larger than $M$.

Since $R$ is the initial radius of the halos, not the radius $r_{cl}$ of the developed halos, the scaling Eq.(1) cannot be directly tested by observed halos. Only in the stage of linear régime, the scale of halos $r_{cl} \propto R$, we have then scaling relationship as

$$n[> (r_{cl}'/r_{cl})^{-\alpha} M, r_{cl}'] = n(> M, r_{cl})$$

(3)

where scaling index $\alpha_L = (3 - n)/2$, subscript $L$ is for linear régime. Namely, the scaling index $\alpha$ is a function of $n$, or it is model-dependent.

Beyond linear régime, the existence of scaling behavior of gravitational clustering can be shown by comparing the gravitational clustering of cosmic matter with the evolution of the profile of a growing interface, or surface roughing. The later is described by the so-called KPZ equation (Kardar, Parisi & Zhang 1986), which mainly consists of terms of nonlinear evolution and stochastic force, or stochastic initial perturbations. A major breakthrough in this approach has been to show the existence of universal scaling properties of the surface growth by the dynamical
It has been found that the equations describing the evolution of cosmic density perturbations are essentially KPZ-like (Barabási & Stanley 1995). The equations of cosmic gravitational clustering contain the similar nonlinear term and a stochastic initial perturbations. The gravitational potential of cosmic matter plays the similar role as the height $h(x)$ of the surface. Thus, the structure formation in the universe substantially is also a phenomena of structural “surface” growth: an initially flat or smooth 3-dimension surface described by the Robertson-Walker metric evolved into a roughen one. Consequently, the cosmic gravitational clustering should also show the scaling feature as the KPZ-like surface growths.

The scaling of a KPZ-like surface is given by $\langle (h^2 - \bar{h}^2) \rangle \propto L^\chi f(t/L^z)$, where $L$ is the length of the coarse-grained average; $t$ is time. The function $f(x)$ is $\sim x^{\chi/z}$ when $x$ is small, and $f(x) \sim$ const. when $x$ is large (Vicsek 1992). The scaling indexes $\chi$ and $z$ depend on the indexes of the spectrum of the stochastic force or the initial perturbations on the surface. Because $r_{cl}$ is, in fast, the scale of a coarse-grained average of mass field of the universe, we have

$$\langle M \rangle \propto r_{cl}^\alpha \left[ f(t/r_{cl}^\beta) \right]$$

(4)

Since the scaling indexes $\alpha$ and $\beta$ depend on the indexes of the power spectrum of the initial perturbations, they are model-dependent.

From eq.(4) one can generally conclude the existence of scaling during quasilinear and non-linear regimes. The scaling is characterized by 1.) for small scale $r_{cl}$ and/or longer (later) time $t$, the coarse-grained mass distributions have scaling $\langle M \rangle \propto r_{cl}^\alpha$; 2.) for larger $r_{cl}$ and/or earlier time $t$, the mass distribution will deviate from the $\alpha$ scaling. Because larger $r_{cl}$ and earlier time $t$ corresponds to quasilinear regime, the deviation from $\alpha$-scaling should be due to the difference between the scaling behaviors of the quasilinear and non-linear regimes. The indexes $\alpha$ and $\beta$ are initial-perturbation dependent. Thus, one can expect that the $\alpha$-scaling and the deviation from this scaling are useful for discriminating among models of structure formation.

These general conclusions are illustrated by some semi-analytical approaches. For instance, with the assumption of “stable clustering”, the index $\gamma$ of the two-point correlation function $\xi(r) \propto r^{-\gamma}$ at the fully developed non-linear regime is found to be $\gamma = 3(n + 3)/(n + 5)$ (Peebles 1965). This turns to $\alpha = 3/(n + 5)$ (Pandmanabhan & Engineer 1998). This means that the $\alpha$-scaling of highly virialized halos is indeed dependent on the initial density perturbations.

As for the deviation from the $\alpha$-scaling, one can refer to the power transfer via mode coupling in quasilinear regime. It has been found that for CDM-like spectrum, the power transfer of density perturbations is from large scales to small ones (e.g. Suto & Sasaki 1991). This is, the larger scale perturbations relatively have higher power in the quasilinear regime than fully developed non-linear regime. Therefore, the index $\alpha$ in quasilinear regime will generally be larger than that of non-linear regime. Thus, one can predict that the index $\alpha$ should show a “going up” on larger scales. This conclusion can also be illustrated by spherical in-fall model. Using scale-invariant spherical in-fall approximation, it has been shown that an initial scaling of $\alpha_L$ may lead to a scaling of $\alpha' = 3/(4 - \alpha_L)$. For $\alpha = (3 - n)/2$, we have $\alpha' = 6/(n + 5)$, which is larger than the $\alpha$.
given by “stable clustering”. Therefore, the scaling index in the quasilinear regime is larger than that of non-linear regime (Padmanabhan 1996).

Despite these semi-analytical results are consistent with the scenario of KPZ-like dynamical scaling in general, it is difficult to semi-analytically calculate the accurate relation between the scaling index and various initial perturbations. One cannot test models by the scaling index calculated from assumptions like the “stable clustering” or spherical in-fall. The studies of surface growth has shown that in the case of 3-D one cannot find the index of dynamical scaling analytically (Vicsek 1992). To find the number of the scaling, and to test models by this scaling, numerical study is necessary. This motivated us to investigate the dynamical scaling of the gravitational clustering numerically.

3. Mass functions of clusters and its scaling

3.1. Multi-scale identifications of clusters

In order to study the mass functions of clusters at different scales, we have performed N-body simulations with the P$^3$M code (Jing & Fang 1994; Jing et al. 1995; Wu et al. 1997b) for models of the SCDM, LCDM and OCDM. The cosmological parameters ($\Omega_M, \Omega_\Lambda, h, \sigma_8$) are taken to be $(1.0, 0.0, 0.5, 0.62), (0.3, 0.7, 0.75, 1.0), (0.3, 0.0, 0.75, 1.0)$ for the SCDM, LCDM and OCDM, respectively. The models with these parameters seem to provide a good approximation to many observational properties of the universe, especially the abundance of clusters (e.g. Jing & Fang 1994; Bahcall, Fan & Cen 1997).

The simulation parameters, including box size $L$, number of particles $N_p$, and the effective force resolution $\eta$, are chosen to be $(L, N_p, \eta) = (310 h^{-1} \text{ Mpc}, 64^3, 0.24 h^{-1} \text{ Mpc})$. We have run 8 realizations for each model. A particle has mass of $3.14 \times 10^{13} \Omega_M h^{-1} M_{\odot}$, which is small enough to resolve reliably the rich clusters with $M > 3.0 \times 10^{14} h^{-1} M_{\odot}$.

To effectively identify clusters with different comoving radius, we haven’t employed the traditional friends-of-friends algorithm, but instead developed an algorithm based on the discrete wavelet transformation (DWT) (see §A1 of Appendix). The details of the DWT identification of clusters will be reported separately (Xu, Fang & Deng, 1998). Briefly, we first describe the distribution of the particles by a 3-D matrix, and then do fast 3-D Daubechies 4 DWT and the reversed transformations to find the wavelet function coefficients (WFCs) and scaling function coefficients (SFCs) on various scales. For each scale, the cells with SFCs larger than those of the random sample by a given statistical significance are picked up as cluster candidates. Around each of the candidates, we place a $6^3$ grid with the size of cluster diameter and search for the accurate center. The cluster center is taken as the position with largest mass surrounded. The mass is measured by counting the particles within a sphere of radius $r_{cl}$, the volume of which is the same as the cells. Whenever two clusters are closer than $2r_{cl}$, the cluster with smaller $M$ is deleted from
the list. We iterate the above steps for particles which have not been listed as cluster members until no further clusters are found. We will call clusters identified by radius $r_{cl}$ as $r_{cl}$-clusters. Since the DWT technique treats the identification at different $r_{cl}$ in a uniform way, it is suitable to study the $r_{cl}$-dependence of clusters.

Fig. 1 shows the derived cluster mass function $n(> M, r_{cl})$, which is the number density of clusters with mass larger than $M$ within radius $r_{cl}$. For all the three models the mass functions of $r_{cl} = 1.5h^{-1}$Mpc clusters given by the DWT method are found to be in good agreement with those derived from friends-of-friends identification (Jing & Fang 1994). At higher $z$, these two methods, the friends-of-friends and DWT, also provide the same mass functions for $r_{cl} = 1.5h^{-1}$Mpc clusters. It turns out that the DWT technique of identifying clusters is indeed reliable. This algorithm can also be applied to real samples of galaxy distribution. Since the number densities of galaxies in real samples are much less than those of particles of N-body simulation samples, sampling error may lead to some false identification of poor clusters. However, for rich clusters, the effect of sampling error is small.

Since in our identified sample each cluster is characterized by two parameters $M$ and $r_{cl}$, it is inconvenient to define richness by mass alone. Instead, the relative richness can be defined by number density. This is, clusters with mass $M_1$ and scale $r_1$ are considered to have the same richness as clusters with $M_2$ and $r_2$ if $n(> M_1, r_1) = n(> M_2, r_2)$. In this paper, the number density $n$ is expressed as $1(d)^{-3}$, where $d$ is the mean separation of the clusters. Therefore, clusters with $n(> M, r_{cl}) \leq 1(50h^{-1}$ Mpc$)^{-3}$ correspond to rich clusters of $M > 5.5 \times 10^{14}h^{-1}$ $M_{\odot}$ on scale $r_{cl} = 1.5h^{-1}$ Mpc for SCDM at $z=0$.

3.2. The scaling of mass functions

It can be easily seen from Fig. 1 that all the mass functions on various scales have a similar shape. This is, if the $1.5 h^{-1}$ Mpc mass functions are shifted horizontally along $M$-axis, they can approximately coincide with the mass functions of $r_{cl} = 3, 6$ and $12 h^{-1}$ Mpc clusters, respectively. Especially, a good match is found in the range of abundances lower than, or richness higher than, $1(50 h^{-1}$ Mpc$)^{-3}$.

This similarity can indeed be described by a scale invariance of the abundance mentioned in §2

$$n(> (r_{cl}/r_0)^{\alpha}M, r_{cl}) \simeq n(> M, r_0)$$

(5)

where $\alpha$ is the index of the scaling. Eq.(5) indicates that the number density of halos (clusters) having masses $> M$ within radius $r_0$ is the same as that of halos having masses $> (r_{cl}/r_0)^{\alpha}M$ within $r_{cl}$. We have tested eq.(5) by calculating $\alpha$ for $n(> M, r_{cl})$ in the abundance range of $n(> M, r_{cl}) < 1(50h^{-1}$ Mpc$)^{-3}$, and found that $\alpha$ remains roughly to be a constant within $5 - 10\%$ in the radius range $r_{cl} \sim 1 - 6 h^{-1}$ Mpc when $r_0 = 1.5h^{-1}$ Mpc.
The scale-invariance is conveniently expressed by a mass-radius scaling which is the solution of the following equation:

\[ n[> M(r_{cl}), r_{cl}] = n[> M(1.5), 1.5] \]  

where 1.5 denotes \( r_{cl} = 1.5h^{-1} \) Mpc. If eq.(5) is correct, we will have

\[ M(r_{cl}) = \left( \frac{r_{cl}}{1.5} \right)^{\alpha} M(1.5) \]  

(In this paper, we denote M from scaling by \( M(r_{cl}) \), the mass within \( r \) by \( M(r) \).)

In Fig. 2, we plot \( \log[M(r_{cl})/M(1.5)] \) against \( \log r_{cl} \) from the solution of eq.(6) for the three models, in which the “richness” is taken to be \( n[> M(1.5), 1.5] = 1(50h^{-1} \text{ Mpc})^{-3} \). The loci of \( \log[M(r_{cl})/M(1.5)] \) vs. \( \log r_{cl} \) can be fairly well approximated by straight lines in the range of \( 1.5 < r_{cl} < 6h^{-1} \) Mpc. Especially, for the LCDM the relation of \( \log[M(r_{cl})/M(1.5)] - \log r_{cl} \) is quite straight in this range. For SCDM and OCDM they also do not deviate too much from straight lines. Therefore, the slopes of these lines, or \( \alpha \)'s, nearly remain constant in the radius range \( 1.5 < r_{cl} < 6h^{-1} \) Mpc and “richness” range \( n[> M(r_{cl}), r_{cl}] \leq 1(50h^{-1} \text{ Mpc})^{-3} \). The straight line fitting yields \( \alpha \approx 0.60 \) for LCDM and OCDM, while \( \alpha \approx 0.80 \) for SCDM.

Fig. 2 shows that all curves \( \log[M(r_{cl})/M(1.5)]-\log r_{cl} \) are going up around \( r_{cl} \sim 6h^{-1} \) Mpc. In other words, the index \( \alpha \) becomes larger in the range of \( r_{cl} > 6h^{-1} \) Mpc. This is qualitatively in good agreement with the behavior of the scaling predicted in last section. As the gravitation clustering on scales larger than \( 6h^{-1} \) Mpc probably still partially remains of infall evolution, the scaling index is affected by the quasilinear evolution, and higher than that on scales less than \( 6h^{-1} \)Mpc. Indeed, on such large scales, more and more \( (M, r_{cl}) \)-identified halos don’t have dense virialized cores in their centers, but are with complicated and irregular shapes, one of which is illustrated in the Fig. 1 of Jing and Fang (1994). We also find that the “going up” behavior of the scaling index \( \alpha \) is more significant at the quasilinear regime at earlier times. A similar result can also be seen in the comparison of universal profile with N-body simulation on larger scale (Diaferio & Geller, 1997).

3.3. The “universal” density profile

It has been proposed that the mass density profiles of regular clusters may universally be fitted by \( M(r) \propto \ln(1 + r/a) - r/(r + a) \), where \( a \) is proportional to the virialized radius of clusters. Both \( r \) and \( a \) are in unit of \( r_{200} \), the radius where the average interior density is 200 times higher than the critical density of the universe (Navarro, Frenk & White 1996). In the range of \( r \geq a \), the universal mass profile can be approximated as a power law

\[ M(r) \propto r^{\alpha_{pro}} \]  

Obviously, the index \( \alpha_{pro} \) depends on the fitting of developed virialized core with \( a \) (Carlberg et al. 1997b).
Eq.(8) looks very similar to eq.(7). However, we should not identify the index $\alpha_{\text{pro}}$ with $\alpha$, because $r_{\text{cl}}$ in eq.(7) is the scale used to identify clusters, while $r$ in eq.(8) is the radius of each cluster. $\alpha_{\text{pro}}$ will be the same as $\alpha$ only if clusters identified by different $r_{\text{cl}}$ have the same mass density profile. Fig. 3 shows, however, a systematic difference of mass density profile of clusters with different $r_{\text{cl}}$. Moreover, the larger the scale, the larger the systematic difference. This is because the clusters identified by large $r_{\text{cl}}$ are still partially in the quasilinear regime for which the halos are pre-virialized. Halos, which lack a virialized core in their centers, cannot be described by the universal density profile.

It should be pointed out that an original meaning of the “universal” is the independence of profile on the initial condition, i.e., $\alpha_{\text{pro}}$ is a universal number regardless the parameters of initial spectrum like the index $n$. Namely, all initial parameters are forgotten in the late time (non-linear regime) clustering. Theoretically, it is far from clear of the condition under which the late time profiles do not remember the initial condition. At least, the assumption of the stable clustering cannot co-exist with an initial-condition-independent profile (Pandmanabhan & Engineer 1998). Fig. 3 shows that the mean profiles $M(r)$ of clusters are significantly different for different models. Namely, the late time profiles do remember the initial condition.

4. Scaling invariance of IRAS clusters

4.1. Multi-scale identified clusters from IRAS

We can directly test the scale invariance via eqs.(6) and (7). To do this, we have applied the multi-scale DWT method to identify galaxy clusters in the redshift survey of the Infrared Astronomical Satellite (IRAS) galaxies with flux limit of $f_{60} \geq 1.2 Jy$ (Fisher et al. 1995b). The IRAS surveys are uniform and complete down to galactic latitudes $\pm 5^\circ$ from the galactic plane. The 1.2 Jy IRAS survey consists of 5313 galaxies, which cover 87.6% of sky, and with 12.34% of the sky belonging to the so-called “excluded zones”.

We fill up the “excluded zones” and holes with a randomly distributed galaxies, which are generated with the same mean number density, and the same radial (redshift) selection function as other areas. This treatment may lead to a little underestimate of the number density of clusters. However, in terms of the ratio between abundances of clusters on different scales the effect of “excluded zones” is small. Moreover, random data, like Poisson distributions, may give rise to false statistics on order higher than second, but this problem will not be met in the statistics of abundance.

As usual, to reduce the effect of radial selection function, we divide the sample into a series of shells with thickness $\sim 2500 \text{ km s}^{-1}$ in redshift space, and measure the number count functions $n(>N_g, r_{\text{cl}})$ in three shells of $v$: [2500-5000], [5000-7500] and [7500-10000] km s$^{-1}$. There are only few galaxies beyond 10000 km s$^{-1}$. They are not suitable for studying the scaling of the cluster.
abundance.

With these redshift shells \( z_i \), the density field \( \delta \rho(s, z_i) \) can be reconstructed by a 2-D DWT decomposition with bases \( \psi_{j,l}(s) \) as

\[
\delta \rho(s, z_i) = \sum_j \sum_{l=0}^{2^j-1} \tilde{w}_{j,l} \psi_{j,l}(s)
\]

where the WFCs \( \tilde{w}_{j,l} \) can be found from the galaxy distribution of the sample (A2 Appendix).

Since the wavelet bases \( \psi_{j,l} \) are orthogonal and complete, the density field \( \delta \rho(s, z_i) \) is the same as descriptions by other orthogonal-complete decomposition, say the bases of spherical harmonics and Bessel functions (Fisher et al. 1995a). It has been shown that the power spectrum detected by the DWT is the same as that by a Fourier decomposition (Pando & Fang 1998). The mass density given by the DWT is also the same as that of Bessel-spherical harmonics. We choose the DWT bases only because it is easy to detect the \( j \)-dependence, or scale-dependence.

We will use \( \delta \rho(s, z_i) \) to identify clusters on various scales \( j \) by the same way developed in §3.1. Similar to the N-body simulation, the cluster center is defined as the point around which a maximum number of galaxies are enclosed within a cylinder with length of 3000 km s\(^{-1}\) and radius equal to that of the cluster. Whenever two clusters are closer than \( 2 r_{cl} \), the cluster with smaller number of galaxies \( N_g \) is deleted from the list. We iterate the steps until no further clusters are found. The result gives a number count function \( n_{\text{IRAS}}(>N_g,r_{cl}) \), which is the number density of IRAS clusters consisting of \( >N_g \) galaxies within radius \( r_{cl} \).

4.2. \( N_g-M \) conversion

Mass is not directly measurable for IRAS clusters. To transfer the number count function \( n(>N_g,r_{cl}) \) to mass function \( n(>M,r_{cl}) \), we need a conversion between the number count and mass for the IRAS clusters. Because the total luminosity of a \( r_{1.5} \)-cluster is proportional to its richness (mass), and the mass-luminosity ratio for these cluster is independent of the total mass of the clusters (Bahcall & Cen 1993), it is reasonable to assume that the mass \( M \) of a cluster is proportional to the total number \( N_{\text{total}} \) of the galaxies in the given cluster, i.e.

\[
M = A N_{\text{total}}
\]

Obviously, \( A \) is only a number-mass conversion coefficient, not the mean mass of galaxies, as the cluster mass \( M \) is dominated by dark matter. The total number of galaxies is related to the counted galaxy by the selection function \( \phi(z) \) (Fisher et al. 1995b):

\[
N_{\text{total}} = N_g / \phi(z)
\]

We have then

\[
M = A * N_g / \phi(z)
\]
We calibrate the coefficient $A$ at a fixed mass, i.e., at a given abundance $n \approx 10^{-5} h^3$ Mpc$^{-3}$ through the equation:

$$n_{IRAS}(>N_g, 1.5) = n_{BC}(>M, 1.5) \quad (13)$$

where subscript $BC$ means the mass function of Bahcall and Cen (1992), and $n_{IRAS}(>N_g, 1.5)$ is from the shell of [5000-7500] kms$^{-1}$. We solve for $N_g$ and $M$. Using this pair of $N_g$ and $M$, we derive the coefficient $A$ from eq.(12).

With this $A$, one can transfer the number count function $n_{IRAS}(>N_g, r_{cl})$ into mass function $n_{IRAS}(>M, r_{cl})$ for the entire range of $N_g$. The result of $n_{IRAS}(>M, r_{cl})$ is plotted in Fig. 4, in which the horizontal errors of IRAS clusters are caused by the Poisson errors of number counting of galaxies in a given cluster, and the vertical errors are from the Poisson errors of counting the clusters.

The BC mass function is also shown in Fig.4. It has been known that SCDM model doesn’t fit BC’s MF while LCDM fits well(Jing & Fang 1994). IRAS results confirm this conclusion. One can find that the mass function of the IRAS clusters with $r_{cl} = 1.5 h^{-1}$ Mpc is basically the same as the mass function of BC sample, especially, for clusters with richness $M > 10^{14} h^{-1} M_\odot$. Namely, the masses of clusters identified from the 1.2 Jy IRAS samples are statistically the same as those of the clusters in the BC sample. This result is consistent with the following fact: the IRAS galaxies trace the local large scale structures. It has been found that many optically identified structures, including superclusters, voids, great attractor and Abell clusters, have been identified from density field constructed from IRAS data (Webster, Lahav & Fisher 1997). Considering that the clusters of BC sample consist of optical and X-ray clusters, and the IRAS galaxies are biased, containing more later type galaxies, Fig. 4 indicates that the early-type galaxies map about the same mass field as late-type, despite the early-type galaxies are clustered more strongly than late-type galaxies. This is because that in terms of second order of statistics the segregation between the early and late-type galaxies is almost linear, and scale-independent at relatively large scales. Using Stromlo-APM redshift survey it has been shown that for second order statistics the scale-dependence of the segregation between early and late-type galaxies is not large than 1 $\sigma$ in the range from 1 to 20 $h^{-1}$ Mpc (Loveday et al. 1995, Fang, Deng & Xia 1998).

One can further test the reasonableness of the calibration (12) by comparing different redshift shells. With eq. (12) and the selection function $\phi(z)$ (Fisher et al. 1995b), we found that the best values of $A$ for the three shells are, respectively, $10^{12.1\pm0.2}$, $10^{11.9\pm0.2}$, $10^{11.7\pm0.3}$ $h^{-1} M_\odot$. They are indeed the same within error limit. Thus, the mass functions $n_{IRAS}(>M, r_{cl})$ from the shells of [2500-5000] and [7500-10000] kms$^{-1}$ are the same as [5000-7500] km s$^{-1}$, and also the same as the BC sample. Therefore, the number of $A$ provides a consistent $N_g - M$ conversion for the entire sample of the 1.2 Jy IRAS galaxies. Using the conversion of Abell richness to cluster mass: $M/N_c = 0.6 \times 10^{13} h^{-1} M_\odot$ (Bahcall & Cen 1993), we have $A \equiv M/N_{total} = 0.8 \times 10^{12} h^{-1} M_\odot$ gives $N_c/N_{total} \approx 7$. It means every count of the 1.2 Jy IRAS galaxies (after selection-function correction) corresponds to about 7 times Abell count of optical galaxies in counting the mass of a cluster.
4.3. Scaling of IRAS clusters and models

Because $M/L$ reaches a constant asymptotic value beyond $r_{cl} \sim 1$ Mpc (Bahcall, Lubin & Dorman 1995), and there is no evidence of significant scale dependence of bias factor of IRAS galaxies from 1 to 10 $h^{-1}$ Mpc, the $N_g$-$M$ conversion eq.(12) should be applicable on scales $> 1.5h^{-1}$Mpc. Thus we can find mass functions $n(M, r_{cl})$ from number-count functions $n(N_g, r_{cl})$ of IRAS cluster with $r_{cl} > 1.5 h^{-1}$Mpc.

Using these IRAS $n(M, r_{cl})$, we test the scaling by eq.(6). The solutions of eq.(6) for both IRAS data and simulation sample are shown in Fig. 5. The theoretical curves in Fig. 5 are similar to Fig. 2, and the richness parameter is taken to be $1(50 h^{-1} \text{Mpc})^{-3}$.

Since only mass ratios $M(r_{cl})/M(1.5)$ of the solutions eq.(6) are plotted in Fig. 5, the result doesn’t depend on the value of A. The effect of linear bias of galaxies will also be canceled in this ratio. The errors of IRAS data in Fig. 5 are calculated from both horizontal and vertical errors of the mass function (Fig. 4). Since the mass function is very steep for rich clusters, i.e. $|d\ln n/d\ln M| \gg 1$, an uncertainty of $\ln n$ will transfer to a relatively small uncertainty of $\ln M$. Because the mean number density of 1.2 Jy IRAS galaxies is low, the major source of errors in Fig. 5 is from Poisson errors of the number of galaxies in clusters.

Despite the errors are large, Fig. 5 already shows that the IRAS data is basically consistent with the prediction: there is a scaling invariance in the range of $1.5 < r_{cl} < 4.5 h^{-1}$Mpc with index $\alpha \sim 0.5$, and the scaling index of scaling is “going up” on larger scales. The results of simulation samples show that the index of scaling is model-dependent. The three panels of Fig. 5 generally are in agreement with the two low mass models (LCDM and OCDM) within $1 \sigma$, but show a systematic disfavor of the SCDM at $\geq 1 \sigma$. The error bars in Fig. 5 are slightly overestimated by assuming that the Poisson errors of mass estimates between two scales are independent. They might be about $1/\sqrt{2}$ times smaller if the two Poisson errors are completely correlated (cloud in cloud scenario). So, in each of the three shells the SCDM is away from the data at the level of $\geq 1 \sim 1.4 \sigma$, or disfavored at a confidence level of $\geq 68\% - 82\%$. If all the shells are binned together, the confidence should certainly be higher than that of individual shell, because the Poisson errors will be smaller. However, it is difficult to estimate the influence of the selection function. Moreover, if spiral galaxies were underrepresented within about $1.5 h^{-1}$ Mpc, the true values of the scaling index should be lower than that shown in Fig. 5, i.e. it strengthens the conclusion of disfavoring the SCDM. Therefore, the number $68\% - 82\%$ can be applied as a underestimated confidence level.
5. Other detections of the scaling index

5.1. Sample of rich clusters with mass estimation

Similar to eq.(2), the number density of \( r_{cl} \) clusters with mass in the range of \( M \) to \( M + dM \) is

\[
N(M, r_{cl})dM = -\frac{\partial}{\partial M} n(> M, r_{cl})dM.
\] (14)

One can then define a mean mass of \( r_{cl} \) clusters with richness \( n(> M, r_{cl}) < n_0 \) by

\[
\overline{M}(r_{cl}) = \int_{n(> M, r_{cl}) < n_0} M N(M, r_{cl})dM.
\] (15)

Using eqs.(5), (6) and (7), we have

\[
\log \overline{M}(r_{cl}) = \alpha \log r_{cl} + \text{const}.
\] (16)

This means, the index \( \alpha \) determined by fitting eq.(16) with a sample of mass measurements of clusters with richness \( n(> M, r_{cl}) < n_0 \) should be the same as that given by abundance solution of eqs.(6) and (7). This prediction can be tested if we have fair samples of masses of clusters with richness \( n(> M, r_{cl}) < n_0 \) and over a given radius range of \( r_{cl} \).

Despite we still lack cluster mass samples covering a large radius range of \( r_{cl} \) and with the desired completeness, the current data are already able to preliminarily test the prediction eq.(16). For instance, it is generally believed that the weak gravitational lensing clusters being studied are among the richest clusters. Weak lensing mass estimate gives only a lower bound to the total cluster mass because of the unknown mean density in the so-called control annulus (Fahlman et al 1994). Nevertheless, the radius dependence of the cluster mass given by weak gravitational lensing, \( M_{wl}(r_{cl}) \), is insensitive to the control field which contributes only a constant component to the mass distribution. Moreover, under the isothermal assumption, we have \( M(r_{cl})/r_{cl} \propto \sigma_1^2 \), where \( \sigma_1 \) is the sight-of-line velocity dispersion of the cluster galaxies. Since the relation \( M(r_{cl})/r_{cl} \propto \sigma_1^2 \) is independent of richness, a velocity dispersion normalized mass, \( M_{wl}(r_{cl})/\sigma_1^2 \), appears to be less dependent on richness. So, the cluster mass sample consisting of \( M_{wl}(r_{cl}) \) and velocity dispersion measurements would be good for fitting with eq.(16). There are 9 clusters which have both velocity dispersion measurements and weak gravitational lensing mass \( M_{wl} \) estimates in the radius range of 0.15 to 2 \( \text{h}^{-1}\text{Mpc} \) (Wu & Fang 1997a). Although \( M_{wl}(r_{cl}) \) from the weak lensing actually corresponds to a projected mass within \( r_{cl} \), the values of \( \alpha \) given by either the projection or the 3-D masses will be roughly the same at large radius \( r_{cl} > 1\text{h}^{-1}\text{Mpc} \). A best fitting of this weak lensing data yields \( \alpha = 0.71 \pm 0.20 \).

Another data set of rich cluster masses can be selected from X-ray measurements. The largest sample of X-ray clusters with mass estimates published to date is given by White, Jones & Forman (1998), which contains 226 X-ray cluster masses for 207 clusters derived from a deprojection of Einstein Observatory X-ray imaging data. Meanwhile, by extensively searching literature there
are additional 152 X-ray determined cluster/group masses available. These data were obtained by either the similar approach to WJF (e.g. White & Fabian 1995; Ettori, Fabian & White 1998; etc.) or the analysis of the ROSAT PSPC X-ray observations (e.g. Pildis. et al 1995; David, Jones & Forman 1995; Cirimele et al. 1997; etc.). All the available 144 measurements of cluster masses from X-ray observations with \( r_{cl} > 0.5 h^{-1} \text{Mpc} \) are plotted in Fig. 6.

The X-ray and optical measurements of some gravitational lensing clusters have shown that the gravitational lensing clusters on average correspond to X-ray temperature \( T \geq 7.5 \text{keV} \) and velocity dispersion of \( \sigma_1 \geq 1200 \text{km s}^{-1} \). Therefore, it is reasonable to select richest clusters by the conditions of \( T \geq 7 \text{keV} \) and \( \sigma_1 \geq 1000 \text{km s}^{-1} \). There are 11 measurements satisfying these conditions. These data are plotted as star-added circles in Fig. 6. It yields \( \alpha = 0.66 \pm 0.40 \) in the range of \( 1.5 < r_{cl} < 4 h^{-1} \text{Mpc} \).

It is interesting to see from Fig. 6 that the X-ray mass distribution has a clear upper envelope, and all the clusters selected by \( T \geq 7 \text{keV} \) and \( \sigma_1 \geq 1000 \text{km s}^{-1} \) distribute along the envelope. It implies that the clusters at the envelope are among the richest clusters at the given \( r_{cl} \). Recall that observations may easily lose less massive clusters, but tend to pick up the bright and massive ones. This selection effect is severe for producing a reliable sample of less rich clusters, but benefit to the completeness of sample of rich clusters. Therefore, one can employ the envelope clusters to fit with eq.(16). To this end, we have binned the data set of cluster masses into 8 logarithmic intervals according to radius from \( r_{cl} = 0.5 - 4.0 h^{-1} \text{Mpc} \). Within each bin, the mean value of the second and the third largest clusters is used as the envelope value. Two of the 8 bins having less than 2 clusters are not considered as reliable measurements of the envelope and are thus not used. All envelope values are shown in Fig. 6, in which the vertical error bars are the mass difference between the second and third masses of the most massive clusters. The least-square fitting gives \( \alpha = 0.52 \pm 0.25 \) in the radius range of 0.5 to 3 \( h^{-1} \text{Mpc} \). To reduce the possible effect of the bin size selection, we re-calculated \( \alpha \) using different bin sizes. It turns out that differences among these results are not larger than 1\( \sigma \).

All the detections of eq.(16) with independent ensembles of rich cluster mass estimates have yielded the same number of \( \alpha \) within error limits. For clusters with “richness” \( n(> M, r_{cl}) \leq 1(50 h^{-1} \text{ Mpc})^{-3} \) in the range of \( 1 < r_{cl} < 4 h^{-1} \text{ Mpc} \) the mean value of \( \alpha \) is \( \simeq 0.63 \pm 0.10 \). These values of \( \alpha \) basically agree with the \( \alpha \) detected by the abundance of IRAS clusters (§4.3). Therefore, current data are in good consistence with the scaling invariance of rich cluster abundance.

5.2. Deviation from a constant \( \alpha \)

As has been discussed in §2, the scaling index \( \alpha \) remains as a constant on smaller scales and later time, but will show “going up” on larger scales and early time on which the quasilinear evolution still plays important role. Therefore, one can expect that the “going up” behavior will
be more significant at higher redshifts. An effective measure of the “going up” is the abundance ratio $n[> (r_{cl}/r_0)^\alpha M, r_{cl}]/n(> M, r_0)$ at higher redshifts. If the scaling invariance of abundance perfectly holds with a constant index $\alpha$, we have

$$\frac{n[> (r_{cl}/r_0)^\alpha M, r_{cl}]}{n(> M, r_0)} = 1$$

(17)

Therefore, the radius $r$ beyond which this ratio no longer remains equal to unity is a measure of the importance of the quasilinear evolution.

Fig. 7 plots the ratio $n[> (r_{cl}/r_0)^\alpha M, r_{cl}]/n(> M, r_0)$ of simulated samples with parameters $\alpha = 0.63$, $r_0 = 1.5h^{-1}$ Mpc and $M = 5.5 \times 10^{14}h^{-1}M_\odot$. Fig. 7 shows that all the abundance ratios become larger than 1 on large scales. Namely the real values of $\alpha$ for the simulated samples are larger than the assumed 0.63 on larger scales. This is the “going up”, indicating the deviation from constant $\alpha$.

Fig. 7 shows that this “going up” behavior is significantly different for different models. The SCDM curves are much more quickly and strongly “going up” than LCDM and OCDM. This is because clusters in the SCDM formed later than in the LCDM and OCDM. In the latter case, the ratio $n[> (r_{cl}/r_0)^\alpha M, r_{cl}]/n(> M, r_0)$ remains equal to about unity in the range of $r_{cl} = 1.5 - 6h^{-1}$ Mpc and $z \leq 0.8$, while for the former the corresponding radius range is much smaller. We have re-calculated Fig. 7 by changing the value of $\alpha$. The results show that in the range $\alpha \sim 0.43 - 0.77$, the OCDM and LCDM always have a larger radius range of the invariance than SCDM. Therefore, the radius range in which the abundance scaling invariance holds is effective for discriminating models.

The fitting done in §5.1 has shown that a constant $\alpha$ of 0.52 - 0.70 in the radius $r_{cl}$ from 1 to 4 $h^{-1}$ Mpc is consistent with all data. Moreover, considering most gravitational lensing and X-ray clusters used in §5.1 are of moderate redshift, the numbers of $\alpha \sim 0.52 - 0.70$ and $1 < r_{cl} < 4h^{-1}$ Mpc are actually real for moderate redshift. Thus, the SCDM will be in difficulty with the measurements of scaling invariance. It should be pointed out that the radius range $1 < r_{cl} < 4h^{-1}$ Mpc in §5.1 is for the physical scale of relevant clusters. A physical radius range 1 - 4 $h^{-1}$ Mpc corresponds to a comoving range of 1.5 - 6 $h^{-1}$ Mpc at redshift $\sim 0.5$. Therefore, this test is more robust at higher redshift.

It may be difficult to directly test the scaling at high redshifts, because mass determination is often limited by the luminosity(or surface brightness) detection threshold of the survey. In this case we may turn to the correlation function. The scaling gives testable prediction on the correlation lengths of clusters on different scales. This will be investigated in Xu, Fang & Deng (1998).
6. Conclusions

We have studied the scaling invariance of abundance of rich clusters. Both the N-body simulation and the available observational data have confirmed the existence of the scaling invariance. This scaling is characterized by index $\alpha$ which can be determined by the abundances of clusters on different scale $r_{cl}$, or by fitting mass-radius relation of clusters. The scaling gains a further support from the following result: the $\alpha$ given by X-ray and gravitational lensing mass estimates is the same as that from the IRAS cluster abundance. Despite the significance level of current results is not high, it is worth of revealing the general behavior of the scaling of cosmic clustering in different evolutionary stages. It can already be employed for discrimination among models.

The LCDM and OCDM abundances can always have a scaling in a larger scale range ($\sim 1 - 6h^{-1}$ Mpc) with $\alpha \sim 0.5 - 0.7$ for $z \leq 0.8$. While the SCDM can only provide a smaller scaling range ($\sim 1 - 3h^{-1}$ Mpc) with $\alpha < \sim 0.8$ and $z < 0.8$. If $\alpha$ is allowed to be $\geq 0.80$, the SCDM can provide a scaling in the range of $1 - 4h^{-1}$ Mpc for $z \leq 0.8$. However, the SCDM is difficult to fit $\alpha \simeq 0.50 - 0.70$ and $r_{cl} \sim 1 - 4h^{-1}$ Mpc simultaneously. This result may cause some troubles for the SCDM.

In general, the mass density profiles of clusters in the low density models are steeper than that of the corresponding clusters in higher $\Omega_M$ models (Jing et al. 1995). In other words, the index $\alpha$ for the $\Omega_M = 1$ model is always larger than that in a low density model ($\Omega_M < 1$). Therefore, the current result of the mass-radius scaling seems to favor models with a lower mass density.

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A. The DWT decomposition and reconstruction

A.1. The discrete wavelet transform

Let us briefly introduce the DWT analysis of large scale structures, for the details referring to (Fang & Pando 1997; Pando & Fang 1996, 1998, and references therein). We consider here a 1-D mass density distribution $\rho(x)$ or contrast $\delta(x) = [\rho(x) - \bar{\rho}]/\bar{\rho}$, which are mathematically random fields over a spatial range $0 \leq x \leq L$. It is not difficult to extend all results developed in this section into 2-D and 3-D because the DWT bases for higher dimension can be constructed by a direct product of 1-D bases.

Like the Fourier expansion of the field $\delta(x)$, the DWT expansion of the field $\delta(x)$ is given by

$$\delta(x) = \sum_{j=0}^{\infty} \sum_{l=0}^{2j-1} \tilde{w}_{j,l} \psi_{j,l}(x) \quad (A1)$$
where \( \psi_{j,l}(x) \), \( j = 0, 1, ..., l = 0...2^j - 1 \) are the bases of the DWT. Because these bases are orthogonal and complete, the wavelet function coefficient (WFC), \( \tilde{w}_{j,l} \), is computed by

\[
\tilde{w}_{j,l} = \int \delta(x) \psi_{j,l}(x) \, dx.
\] (A2)

The wavelet transform bases \( \psi_{j,l}(x) \) are generated from the basic wavelet \( \psi(x/L) \) by a dilation, \( 2^j \), and a translation \( l \), i.e.

\[
\psi_{j,l}(x) = \left( \frac{2^j}{L} \right)^{1/2} \psi(2^j x/L - l).
\] (A3)

The basic wavelet \( \psi \) is designed to be continuous, admissible and localized. Unlike the Fourier bases \( \exp(i2\pi nx/L) \), which are non-local in physical space, the wavelet bases \( \psi_{j,l}(x) \) are localized in both physical space and Fourier (scale) space. In physical space, \( \psi_{j,l}(x) \) is centered at position \( lL/2^j \), and in Fourier space, it is centered at wavenumber \( 2\pi \times 2^j/L \). Therefore, the DWT decomposes the density fluctuating field \( \delta(x) \) into domains \( j,l \) in phase space, and for each basis the corresponding area in the phase space is as small as that allowed by the uncertainty principle. WFC \( \tilde{w}_{j,l} \) and its intensity \( |\tilde{w}_{j,l}|^2 \) describe, respectively, the fluctuation of density and its power on scale \( L/2^j \) at position \( lL/2^j \) (Pando & Fang 1998).

**A.2. Reconstruction of density field**

Using the completeness of the DWT basis, one can reconstruct the original density field from the coefficient \( \tilde{w}_{j,l} \). To achieve this, DWT analysis employs another set of functions consisting of the so-called scaling functions, \( \phi_{j,l} \), which are generated from the basic scaling \( \phi(x/L) \) by a dilation, \( 2^j \), and a translation \( l \), i.e.

\[
\phi_{j,l}(x) = \left( \frac{2^j}{L} \right)^{1/2} \phi(2^j x/L - l).
\] (A4)

The basic scaling \( \phi \) is essentially a window function with width \( x/L = 1 \). Thus, the scaling functions \( \phi_{j,l}(x) \) are also windows, but with width \( (1/2^j)L \), and centered at \( lL/2^j \). The scaling function \( \phi_{j,l}(x) \) are orthogonal with respect to the index \( l \), but not for \( j \). This is a common property of windows, which can be orthogonal in physical space, but not in Fourier space.

For Daubechies wavelets, the basic wavelet and the basic scaling are related by recursive equations as (Daubechies 1992)

\[
\phi(x/L) = \sum_l a_l \phi(2x/L - l)
\]

\[
\psi(x/L) = \sum_l b_l \phi(2x/L + l)
\] (A5)

where coefficients \( a_l \) and \( b_l \) are different for different wavelet. In this paper, we use the Daubechies 4 wavelet (D4), for which \( a_0 = (1 + \sqrt{3})/4, \ a_1 = (3 + \sqrt{3})/4, \ a_2 = (3 - \sqrt{3})/4, \ a_3 = (1 - \sqrt{3})/4. \)
From Eq. (A5), one can show that the scaling functions \( \phi_{j,l}(x) \) are always orthogonal to the wavelet bases \( \psi_{j',l'}(x) \) if \( j \leq j' \), i.e.

\[
\int \phi_{j,l}(x) \psi_{j',l'}(x) dx = 0, \quad \text{for} \quad j \leq j'.
\]  

(A6)

Therefore, \( \phi_{j,l}(x) \) can be expressed by \( \psi_{j',l'}(x) \) as

\[
\phi_{j,l}(x) = \sum_{j'=0}^{\infty} \sum_{l'=0}^{2^{j'-1} - 1} c_{j,j',l'} \psi_{j',l'}(x) = \sum_{j'=0}^{j-1} \sum_{l'=0}^{2^{j'-1} - 1} c_{j,j',l'} \psi_{j',l'}(x).
\]  

(A7)

The coefficients \( c_{j,j',l'} = \int \phi_{j,l}(x) \psi_{j',l'}(x) dx \) can be determined by \( a_l \) and \( b_l \).

Using \( \phi_{j,l}(x) \), we construct a density field on scale \( j \) as

\[
\rho_j(x) = \frac{2}{2^{j-1}} \sum_{l=0}^{2^{j-1} - 1} w_{j,l} \phi_{j,l}(x),
\]  

(A8)

where \( w_{j,l} \) is called the scaling function coefficient (SFC) given by

\[
w_{j,l} = \int_0^L \rho(x) \phi_{j,l}(x) dx.
\]  

(A9)

Since the scaling function \( \phi_{j,l}(x) \) is window-like, the coefficient \( w_{j,l} \) is actually a “count-in-cell” in a window on scale \( j \) at position \( l \).

Using Eqs. (A1), (A7), (A8) and (A9), one can find

\[
\rho_j(x) = \bar{\rho} \sum_{j'=0}^{j-1} \sum_{l'=0}^{2^{j'-1} - 1} \tilde{w}_{j',l'} \psi_{j',l'}(x) + \bar{\rho}.
\]  

(A10)

Namely, \( \rho_j(x) \) contains all terms of density fluctuations \( \tilde{w}_{j',l'} \psi_{j',l'}(x) \) of \( j' < j \), but not terms of \( j' \geq j \). From Eqs. (A1) and (A10), we have

\[
\rho(x) = \rho_j(x) + \bar{\rho} \sum_{j'=j}^{\infty} \sum_{l'=0}^{2^{j'-1} - 1} \tilde{w}_{j',l'} \psi_{j',l'}(x).
\]  

(A11)

One can also define the smoothed density contrasts on scale \( j \) to be

\[
\delta_j(x) = \frac{\rho_j(x) - \bar{\rho}}{\bar{\rho}} = \sum_{j'=0}^{j-1} \sum_{l'=0}^{2^{j'-1} - 1} \tilde{w}_{j',l'} \psi_{j',l'}(x).
\]  

(A12)

Eqs. (A11) and (A12) show that \( \rho_j(x) \) and \( \delta_j(x) \) are smoothed density fields on scale \( j \). One can construct the density field \( \rho_j(x) \) or \( \delta_j(x) \) on finer and finer scales by WFCs \( \tilde{w}_{j,l} \) till to the precision of the original field. Since the sets of bases \( \psi_{j,l} \) and \( \phi_{j,l} \) are complete, the original field can be reconstructed without lost information.
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Fig. 1.— Mass functions of clusters identified with radii $r_{cl} = 1.5, 3.0, 6.0$ and $12 \, h^{-1}\text{Mpc}$ for SCDM, LCDM and OCDM models at $z = 0$. $n(> M, r_{cl})$ is the number density of clusters with mass larger than $M$ within radius $r_{cl}$. $M$ is in unit of $h^{-1}\odot$. The observed data are for clusters with radius $r_{cl} = 1.5h^{-1}\text{Mpc}$ (Bahcall & Cen 1992).

Fig. 2.— Mass-radius scaling of clusters given by the solution of abundance equation (6). The “richness” is taken to be $n = 1(90h^{-1}\text{Mpc})^{-3}$. The thin, dashed, and thick lines are for SCDM, LCDM and OCDM, respectively.

Fig. 3.— The mean mass-radius relation $M(r_{cl})$ of clusters with different identification scales $r_{cl}$. The “richness” for all $r_{cl}$ clusters is $n = 1(90h^{-1}\text{Mpc})^{-3}$. The dashed and solid lines are for SCDM and OCDM, respectively.

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Fig. 4.— Mass function of $r_{cl} = 1.5h^{-1}\text{Mpc}$ IRAS clusters (circles with error bars) in the shell $[5000,7500]\text{km s}^{-1}$. The stars are for the mass function of Bahcall & Cen (1992). The dashed line shows the mass function by FOFs method of Jing & Fang (1994). The thick line shows the mass function by DWT method averaged over 5-realizations of our $310h^{-1}\text{Mpc}$ box simulations. The thin line shows the mass function by DWT method from 1 realization of $155h^{-1}\text{Mpc}$ box simulation.

Fig. 5.— Mass-radius scaling of clusters given by the solution of abundance equation (2). The thick, thin, and dashed lines are for LCDM, OCDM and SCDM, respectively. The data with error bars come from the clusters identified from IRAS galaxies. The richness parameter is taken to be $1(50h^{-1}\text{Mpc})^{-3}$.

Fig. 6.— All the available 144 measurements (circles) of cluster masses from X-ray observations with $r_{cl} > 0.5h^{-1}\text{Mpc}$. The 11 richest clusters ($T > 7 \text{keV}$ and $\sigma > 1000\text{km s}^{-1}$) with $r_{cl} > 1.5h^{-1}\text{Mpc}$ are marked additionally with stars. Squares with error bars are the data of upper envelope clusters. The horizontal error bars showing the widths of radius bins, and the vertical error bars being the difference between the masses of the second and the third most massive clusters within each bin. The solid line is an equal-weight least square fit to the envelope data.

Fig. 7.— $n[> M_{th}(r_{cl}), r_{cl}]/n_{1.5}$ vs. $r_{cl}$ in the range of $r_{cl} = 1.5-12h^{-1}\text{Mpc}$ at redshifts $z \sim 0.2, 0.5$ and 0.8, for SCDM, LCDM and OCDM. Here $M_{th}(r_{cl}) = (r_{cl}/r_{1.5})^\alpha M_{1.5}$ and $n_{1.5} = n(> M_{1.5}, 1.5)$. The parameters are taken to be $M_{1.5} = 5.5 \times 10^{14}h^{-1}\text{M}_\odot$, $r_{1.5} = 1.5h^{-1}\text{Mpc}$, and $\alpha = 0.63$. The dot-dashed horizontal lines are for perfect scale invariance with $\alpha = 0.63$. All curves for simulated samples show the “going up” of $\alpha$ on large scales.
Mass Profile

- SCDM
- OCDM

$\log_{10}(r (h^{-1} \text{Mpc}))$

$\log_{10}(M(r)/M(1.5))$

$r_{c} = 6$

$6\,\,3\,\,1.5\,\,0.75\,\,6\,\,3\,\,1.5\,\,0.75$
\[ \log_{10} \left[ \frac{M(r_{cl})}{M(1.5)} \right] \]

- \[ 2500 < v < 5000 \]
- \[ 5000 < v < 7500 \]
- \[ 7500 < v < 10000 \]