Holomorphic Chern-Simons-Witten Theory:
from 2D to 4D Conformal Field Theories

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Abstract. It is well known that rational 2D conformal field theories are connected with Chern-Simons theories defined on 3D real manifolds. We consider holomorphic analogues of Chern-Simons theories defined on 3D complex manifolds (six real dimensions) and describe 4D conformal field theories connected with them. All these models are integrable. We describe analogues of the Virasoro and affine Lie algebras, the local action of which on fields of holomorphic analogues of Chern-Simons theories becomes nonlocal after pushing down to the action on fields of integrable 4D conformal field theories. Quantization of integrable 4D conformal field theories and relations to string theories are briefly discussed.
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1 Introduction

Rational 2D conformal field theories (CFT) form an important subclass of CFT’s in two dimensions. They are completely solvable and may be obtained from 3D Chern-Simons theories by an appropriate choice of a gauge group and coupling constants [1]-[3]. The aim of the present paper is to describe 4D CFT’s, which are as solvable as rational 2D CFT’s, and successful nonperturbative quantization of which may be developed. These integrable 4D CFT’s are connected with holomorphic analogues of Chern-Simons theories [4]-[7] in the same way as rational 2D CFT’s are related to ordinary Chern-Simons theories. This correspondence between 4D and 6D theories is different from the AdS/CFT correspondence being extensively discussed in the literature. We also consider free 4D CFT’s describing fields of arbitrary spin $s = 0, 1/2, 1, ...$ (and negative helicity) on 4D manifolds with the self-dual Weyl tensor.

In many respects the progress in 2D CFT’s was related to the existence of infinite-dimensional symmetry algebras among which the Virasoro and affine Lie algebras play the most important role. In contrast with significant progress in understanding 2D CFT’s the knowledge of 4D CFT’s is much less explicit, and not so many exact results are obtained (see e.g. [8]-[13] and references therein). Usually, one connects this with the fact that, unlike the 2D case, the conformal group in four dimensions (which serves for the classification of primary fields in the limit of flat 4D space [14]) is finite-dimensional, and constraints arising from conformal invariance are not sufficient for a detailed description of 4D CFT’s. This is true for the general case, but for a special subclass of 4D CFT’s - integrable 4D CFT’s - this is a wrong impression based on the consideration of only local (manifest) symmetries.

Integrable 4D CFT’s are related to self-dual theories, which was discussed in the paper [7]. By now it is well known that self-duality in four dimensions is connected with holomorphy in six dimensions [15]-[22]. Correspondingly, 4D self-duality/6D holomorphy and symmetry algebras of 4D self-dual theories are the replacement of 2D self-duality/2D holomorphy and 2D conformal invariance. The discussion of this topic in the present paper is based on the results of [7] and much of the notation and terminology is taken directly from that work. In Appendices we recall definitions of those notions which we shall use.

2 Chern-Simons theories

2.1 Definitions and notation

Let $X$ be an oriented real 3D smooth manifold, $G$ a semisimple compact Lie group, $\mathfrak{g}$ its Lie algebra, $P$ a principal $G$-bundle over $X$, $A$ a connection 1-form on $P$ and $F(A) = dA + A \wedge A$ its curvature. Locally $A$ is a $\mathfrak{g}$-valued 1-form on $X$, and $F(A)$ is the $\mathfrak{g}$-valued 2-form on $X$.

Suppose a representation of $G$ in the complex vector space $\mathbb{C}^n$ is given. We associate the complex vector bundle $\mathcal{E} = P \times_G \mathbb{C}^n$ with the principal bundle $P$, and we shall use both the principal bundle $P$ and the vector bundle $\mathcal{E}$ in the description of Chern-Simons theories.

Chern-Simons (CS) theories (see e.g. [1]-[3],[23]-[25] and references therein) describe locally constant $G$-bundles, that is, bundles with locally constant transition matrices. Put another way, a locally constant $G$-bundle is the bundle with a flat connection $A$ and the field equations of the CS theory are

$$F(A) = dA + A \wedge A = 0. \quad (2.1)$$

Locally eqs.(2.1) are solved trivially, and on any sufficiently small open set $U \subset X$ we have $A = \phi^{-1} d\phi$, where $\phi(x) \in G$. So, locally $A$ is a pure gauge.
2.2 Moduli space of flat connections

We denote by $\mathcal{A}$ the space of (irreducible) connections on $P$ and by $\mathcal{G}$ the infinite-dimensional group of gauge transformations $A \mapsto A^g = g^{-1}Ag + g^{-1}dg$ (automorphisms of $P$ which induce the identity transformation of $X$). Then the moduli space $\mathcal{M}_X$ of flat connections on $P$ is the space of solutions to eqs.(2.1) modulo gauge transformations,

$$\mathcal{M}_X = \{A \in \mathcal{A} : F(A) = 0\} / \mathcal{G}. \quad (2.2)$$

In other words, the moduli space $\mathcal{M}_X$ of flat connections on $P$ (and on the vector bundle $\mathcal{E}$) is the space of gauge nonequivalent solutions to eqs.(2.1).

There is another description of the moduli space $\mathcal{M}_X$ as the space of homomorphisms from the fundamental group $\pi_1(X)$ of the manifold $X$ to the group $G$ modulo conjugation by $G$,

$$\mathcal{M}_X = \text{Hom}(\pi_1(X), G) / G. \quad (2.3)$$

Just this description of the moduli space of flat connections on $\mathcal{E}$ was used most often in papers on CS theories. For technical complications in the case when $X$ is not connected, see e.g. [25].

Below we want to consider the sheaf description of locally constant bundles $\mathcal{E} \to X$ from which both (2.2) and (2.3) descriptions of the moduli space $\mathcal{M}_X$ of flat connections on $\mathcal{E}$ follow. This description is known in mathematics (see e.g. [26] and references therein), and it can be compared with the description of holomorphic bundles over complex manifolds. This will be useful for introducing a holomorphic analogue of the CS theory. For simplicity one usually considers locally constant bundles $\mathcal{E}$ which are equivalent to the trivial one as smooth vector bundles (if one takes connected and simply connected structure groups $G$). The generalization to topologically nontrivial bundles is straightforward (see e.g. [2], [24]-[26]). In §2.3, we shall introduce all necessary sheaves, and in §2.4 we shall describe moduli of flat connections in terms of cohomology sets. For definitions of sheaves, cohomology sets etc, see Appendices.

2.3 Sheaves $\mathcal{G}$, $\mathcal{G}$ and $\mathcal{A}$

Let us consider the sheaf $\mathcal{G}$ (of germs) of smooth maps from $X$ into the group $G$ and its subsheaf $\mathcal{G} := X \times G$, continuous sections of which are locally constant maps from $X$ into $G$. Sections of the sheaf $\mathcal{G}$ over an open set $U \subset X$ are smooth $G$-valued functions $\phi \in \mathcal{G}(U)$ on $U$. Consider also the sheaf $\mathcal{A}^q$ of smooth $q$-forms on $X$ with values in the Lie algebra $\mathfrak{g}$ ($q = 0,1,...$). The space of sections of the sheaf $\mathcal{A}^q$ over an open set $U \subset X$ is the space $\mathcal{A}^q(U)$ of smooth $\mathfrak{g}$-valued $q$-forms on $U$.

The sheaf $\mathcal{G}$ acts on the sheaves $\mathcal{A}^q$, $q = 0,1,...$, with the help of the adjoint representation $\text{Ad}$. In particular, for any open set $U \subset X$ we have

$$A \mapsto \text{Ad}_A \phi A = \phi^{-1}A\phi + \phi^{-1}d\phi, \quad (2.4a)$$

$$F \mapsto \text{Ad}_A F = \phi^{-1}F \phi, \quad (2.4b)$$

where $\phi \in \mathcal{G}(U)$, $A \in \mathcal{A}^1(U)$, $F \in \mathcal{A}^2(U)$.

Denote by $i : \mathcal{G} \to \mathcal{G}$ an embedding of $\mathcal{G}$ into $\mathcal{G}$. We define a map $\delta^0 : \mathcal{G} \to \mathcal{A}^1$ given for any open set $U$ of the space $X$ by the formula

$$\delta^0 \phi = -(d\phi)\phi^{-1}, \quad (2.5)$$

where $\phi \in \mathcal{G}(U)$, $\delta^0 \phi \in \mathcal{A}^1(U)$. Let us also introduce an operator $\delta^1 : \mathcal{A}^1 \to \mathcal{A}^2$, defined for an open set $U \subset X$ by the formula

$$\delta^1 A = da + A \wedge A, \quad (2.6)$$
where $A \in \mathcal{A}^1(U)$, $\delta^1A \in \mathcal{A}^2(U)$. In other words, the maps of sheaves $\delta^0 : \mathcal{G} \rightarrow \mathcal{A}^1$ and $\delta^1 : \mathcal{A}^1 \rightarrow \mathcal{A}^2$ are defined by means of localization. Finally, we denote by $\mathcal{A}$ the subsheaf in $\mathcal{A}^1$, consisting of locally defined $g$-valued 1-forms $A$ such that $\delta^1A = 0$, i.e. sections $A \in \mathcal{A}(U)$ of the sheaf $\mathcal{A} = \text{Ker} \delta^1$ over $U \subset X$ satisfy eqs. (2.1).

### 2.4 Locally constant bundles and cohomology sets

It is not difficult to verify that the sequence of sheaves

$$1 \rightarrow \mathcal{G} \xrightarrow{i} \mathcal{G} \xrightarrow{\delta^0} \mathcal{A} \xrightarrow{\delta^1} 0 \quad (2.7)$$

is exact. From (2.7) we obtain the exact sequence of cohomology sets [26]

$$e \rightarrow H^0(X, \mathcal{G}) \xrightarrow{i} H^0(X, \mathcal{G}) \xrightarrow{\delta^0} H^0(X, \mathcal{A}) \xrightarrow{\delta^1} H^1(X, \mathcal{G}) \xrightarrow{\rho} H^1(X, \mathcal{G}), \quad (2.8)$$

where $e$ is a marked element (identity) of the considered sets, and the map $\rho$ coincides with the canonical embedding induced by the embedding of sheaves $i : \mathcal{G} \rightarrow \mathcal{G}$.

By definition, (all) locally constant $G$-bundles $\mathcal{E}$ over $X$ are parametrized by the set $H^1(X, \mathcal{G})$. The kernel $\text{Ker} \rho = \rho^{-1}(e)$ of the map $\rho$ coincides with a subset of those elements from $H^1(X, \mathcal{G})$ which are mapped into the class $e \in H^1(X, \mathcal{G})$ of smoothly trivial bundles. By virtue of the exactness of the sequence (2.8), the space $\text{Ker} \rho$ is bijective to the quotient space $H^0(X, \mathcal{A})/H^0(X, \mathcal{G})$. But $H^0(X, \mathcal{A})$ is the space of global solutions to eqs. (2.1), $H^0(X, \mathcal{G})$ coincides with the group of gauge transformations $G$, and therefore $H^0(X, \mathcal{A})/H^0(X, \mathcal{G})$ coincides with the moduli space $\mathcal{M}_X$ of flat connections on $\mathcal{E}$. So we have

$$\mathcal{M}_X = H^0(X, \mathcal{A})/H^0(X, \mathcal{G}) \simeq \text{Ker} \rho \subset H^1(X, \mathcal{G}), \quad (2.9)$$

i.e. there is a one-to-one correspondence between the moduli space $\mathcal{M}_X$ of flat connections on $\mathcal{E}$ and the moduli space $\text{Ker} \rho = \rho^{-1}(e) \subset H^1(X, \mathcal{G})$ of those locally constant bundles $\mathcal{E} \rightarrow X$ which are trivial as smooth bundles.

**Remark.** One may also consider deformations of a locally constant bundle $\mathcal{E} \rightarrow X$ which is not topologically trivial. If this bundle is associated with a principal bundle $P$ and parametrized as a smooth bundle by a modulus $p \in H^1(X, \mathcal{G})$, then instead of the sheaf $\mathcal{A}^1$ one should take the sheaf of 1-forms with values in the bundle $\text{Ad}P = P \times_G g$ and instead of the sheaf $\mathcal{G}$ one should take the sheaf $\mathcal{G}'$ of sections of the bundle $\text{Int} P$ (the bundle of groups $P \times_G G$ where $G$ acts on itself by conjugation). Then we shall have exact sequences analogous to (2.7), (2.8) but with these new sheaves, and the point $e \in H^1(X, \mathcal{G}')$ will correspond to the point $p \in H^1(X, \mathcal{G})$.

If the structure group $G$ of the bundle $P$ is connected and simply connected, then $H^1(X, \mathcal{G}) = e$ and therefore $\text{Ker} \rho = H^1(X, \mathcal{G})$. That is, for topologically trivial $G$-bundles we have

$$H^0(X, \mathcal{A})/H^0(X, \mathcal{G}) \simeq H^1(X, \mathcal{G}). \quad (2.10)$$

Bijection (2.10) is a non-Abelian variant of the isomorphism between Čech and de Rham cohomologies.

### 2.5 Rational 2D conformal field theories

It is well known that 3D Chern-Simons theories on $X$ are connected with 2D conformal field theories if one supposes that a 3-manifold $X$ has the form $\Sigma \times \mathbb{R}$ (or $\Sigma \times S^1$), where $\Sigma$ is a 2-manifold with or without a boundary [1]-[3]. Notice that the moduli space $\mathcal{M}_{\Sigma \times \mathbb{R}}$ of flat connections on a $G$-bundle
over $\Sigma \times \mathbb{R}$ is bijective to the moduli space $\mathfrak{M}_\Sigma$ of flat connections on the induced $G$-bundle over $\Sigma$ (for proof see e.g. [25]). Moreover, if $\Sigma$ has a boundary, the quantum Hilbert space $H_\Sigma$ is an infinite-dimensional representation space of the chiral algebra of a CFT on $\Sigma$. In particular, if $\Sigma = D_0$ is a disk (e.g. $|z| \leq 1$) on the complex plane $\mathbb{C}$, then the CS theory on $D_0 \times \mathbb{R}$ reproduces the chiral (holomorphic) version of the WZNW model [1]-[3]. The phase space of this model is the space of based loops $LG/G$.

The specification of a 3D CS theory is performed by a choice of gauge group $G$, coupling constants and a manifold $X$. The transition to 2D CFT is carried out by a choice of a 2-manifold $\Sigma$ (disk, annulus, torus,...) in the splitting $X = \Sigma \times \mathbb{R}$ (or $X = \Sigma \times S^1$) and by a choice of sources on $\Sigma$ [1]-[3],[23]-[25]. For instance, we recover the above-mentioned chiral WZNW theory without sources if we choose a connected and simply connected gauge group $G$ and $X = D_0 \times \mathbb{R}$. If we take connected but non-simply connected groups $G$, we obtain a special subclass of rational 2D CFT's [2, 24]. One can also consider disconnected groups of type $N \ltimes G$, where $G$ is a connected group with a discrete automorphism group $N$, which leads to so-called orbifold models [2, 24].

At last, an important subclass of 2D CFT's is formed by $G/H$ coset models resulting from CS theories with a gauge group $G \times H$ when coupling constants of gauge fields of the groups $G$ and $H$ have opposite signs [2]. For a more detailed discussion, see [1]-[3],[23]-[25].

### 3 Holomorphic Chern-Simons-Witten theories

#### 3.1 Definitions and notation

Let us consider a smooth six-dimensional manifold $Z$ with an integrable almost complex structure $\mathcal{J}$. Then $Z$ is a complex 3-manifold, and one can introduce a cover $\mathcal{U} = \{U_\alpha\}$ of $Z$ and complex coordinates $z_\alpha : U_\alpha \to \mathbb{C}^3$. Let $G = G^C$ be a complex semisimple Lie group, $g = g^C$ its Lie algebra, $P'$ a principal $G$-bundle over $Z$ and $E' = P' \times_G \mathbb{C}^n$ a smooth complex vector bundle of rank $n$ associated with $P'$.

Let $\hat{B}$ be the $(0,1)$-component of a connection 1-form on the bundle $E'$. Suppose that $\hat{B}$ satisfies the equations

$$\hat{\partial} \hat{B} + \hat{B} \wedge \hat{B} = 0,$$

where $\hat{\partial}$ is the $(0,1)$-part of the exterior derivative $d = \partial + \bar{\partial}$. Equations (3.1) mean that the $(0,2)$-part of the curvature $F$ of the bundle $E'$ is equal to zero: $F^{0,2} := \hat{\partial}^2 \hat{B} = (\hat{\partial} + \hat{B})^2 = 0$ and therefore the bundle $E'$ is holomorphic. We shall call eqs.(3.1) the field equations of holomorphic Chern-Simons-Witten (CSW) theory. This theory describes holomorphic bundles over a complex 3-manifold $Z$. We shall also use the abbreviation ‘hCSW theory’ for it.

Equations (3.1) were suggested by Witten [4] for a special case of bundles over Calabi-Yau (CY) 3-folds $Z$ as equations of a holomorphic analogue of the ordinary Chern-Simons theory. Witten obtained eqs.(3.1) from open $N = 2$ topological strings with a central charge $\hat{c} = 3$ (6D target space), and the CY restriction $c_1(Z) = 0$ arised from $N = 2$ superconformal invariance of a sigma model used in constructing the topological string theory. The connection of eqs.(3.1) with topological strings was also considered in [5]. For the hCSW theory on a CY 3-fold $Z$, eqs. (3.1) follow from the action [4, 5]

$$S_0 = \frac{1}{2} \int_Z \hat{\Omega} \wedge \text{Tr}(\hat{B} \wedge \hat{\partial} \hat{B} + \frac{2}{3} \hat{B} \wedge \hat{B} \wedge \hat{B}),$$

where $\hat{\Omega}$ is a nowhere vanishing holomorphic $(3,0)$-form on $Z$. The theory (3.1), (3.2) describes nonequivalent holomorphic structures on the bundle $E' \to Z$. 

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
Equations (3.1) on CY 3-folds were considered by Donaldson and Thomas [6] in the frames of the programme on extending the results of Casson, Floer and Donaldson to manifolds of dimension D > 4. Donaldson and Thomas [6] pointed out that one may consider a more general situation with eqs.(3.1) on complex manifolds Z which are not Calabi-Yau. We shall consider this more general case of bundles over arbitrary complex 3-manifolds.

### 3.2 Sheaves \( \hat{\mathcal{H}}, \hat{S} \) and \( \hat{B} \)

To describe the moduli space of flat \((0,1)\)-connections we have

\[
\mathcal{H} \to \mathcal{Z} \to \mathcal{Z}^\times 
\]

where \( \hat{\mathcal{H}} \) is a subsheaf of the sheaf \( \hat{\mathcal{S}} \), and there exists a canonical embedding \( i : \hat{\mathcal{H}} \to \hat{\mathcal{S}} \). Consider also the sheaf \( \hat{\mathcal{B}}^{0,1,q} \) of smooth \((q,0)\)-forms on \( \mathcal{Z} \) with values in the Lie algebra \( \mathfrak{g} \). The sheaf \( \hat{\mathcal{S}} \) acts on the sheaves \( \hat{\mathcal{B}}^{0,q} \) (\( q = 1, 2, \ldots \)) with the help of the adjoint representation. In particular, for any open set \( \mathcal{U} \subset \mathcal{Z} \) we have

\[
\hat{B} \to \text{Ad}_{\hat{\psi}} \hat{B} = \hat{\psi}^{-1} \hat{B} \hat{\psi} + \hat{\psi}^{-1} \hat{\partial} \hat{\psi}, \tag{3.3a}
\]

\[
\hat{F} \to \text{Ad}_{\hat{\psi}} \hat{F} = \hat{\psi}^{-1} \hat{F} \hat{\psi}, \tag{3.3b}
\]

where \( \hat{\psi} \in \hat{\mathcal{S}}(\mathcal{U}), \hat{B} \in \hat{\mathcal{B}}^{0,1}(\mathcal{U}), \hat{F} \in \hat{\mathcal{B}}^{0,2}(\mathcal{U}) \).

Let us define a map \( \hat{\delta}^0 : \hat{S} \to \hat{B}^{0,1} \) given for any open set \( \mathcal{U} \) of the space \( \mathcal{Z} \) by the formula

\[
\hat{\delta}^0 \hat{\psi} = - (\hat{\partial} \hat{\psi}) \hat{\psi}^{-1}, \tag{3.4}
\]

where \( \hat{\psi} \in \hat{\mathcal{S}}(\mathcal{U}), \hat{\delta}^0 \hat{\psi} \in \hat{\mathcal{B}}^{0,1}(\mathcal{U}), \hat{d} = \hat{\partial} + \hat{\partial} \). Let us also introduce an operator \( \hat{\delta}^1 : \hat{B}^{0,1} \to \hat{B}^{0,2} \), defined for any open set \( \mathcal{U} \subset \mathcal{Z} \) by the formula

\[
\hat{\delta}^1 \hat{B} = \hat{\partial} \hat{B} + \hat{B} \hat{\Lambda} \hat{B}, \tag{3.5}
\]

where \( \hat{B} \in \hat{\mathcal{B}}^{0,1}(\mathcal{U}), \hat{\delta}^1 \hat{B} \in \hat{\mathcal{B}}^{0,2}(\mathcal{U}) \). In other words, the maps of sheaves \( \hat{\delta}^0 : \hat{S} \to \hat{B}^{0,1} \) and \( \hat{\delta}^1 : \hat{B}^{0,1} \to \hat{B}^{0,2} \) are defined by means of localizations.

Denote by \( \hat{\mathcal{B}} \) the subsheaf in \( \hat{\mathcal{B}}^{0,1} \) consisting (of germs) of \((0,1)\)-forms \( \hat{B} \) with values in \( \mathfrak{g} \) such that \( \hat{\delta}^1 \hat{B} = 0 \), i.e. sections \( \hat{B} \) over any open set \( \mathcal{U} \) of the sheaf \( \hat{\mathcal{B}} = \text{Ker} \hat{\delta}^1 \) satisfy the equations

\[
\mathcal{E}^{0,2}(\hat{\mathcal{B}}) = \hat{\partial} \hat{\mathcal{B}} + \hat{\mathcal{B}} \hat{\Lambda} \hat{\mathcal{B}} = 0, \tag{3.6}
\]

where \( \hat{\mathcal{B}} \in \hat{\mathcal{B}}^{0,1}(\mathcal{U}) \). So the sheaf \( \hat{\mathcal{B}} \) can be identified with the sheaf of flat \((0,1)\)-connections \( \hat{\partial} \hat{\mathcal{B}} = \hat{\partial} + \hat{\hat{B}} \) on the complex vector bundle \( \mathcal{E}' \) over \( \mathcal{Z} \) (and on the principal bundle \( \mathcal{P}' \)).

### 3.3 Moduli space of flat \((0,1)\)-connections

Let us consider the sheaves \( \hat{\mathcal{S}}, \hat{\mathcal{B}}^{0,1} \) and \( \hat{\mathcal{B}}^{0,2} \). The triple \( \{ \hat{\mathcal{S}}, \hat{\mathcal{B}}^{0,1}, \hat{\mathcal{B}}^{0,2} \} \) with the maps \( \hat{\delta}^0 \) and \( \hat{\delta}^1 \) is a resolution of the sheaf \( \hat{\mathcal{H}} \), i.e. the sequence of sheaves

\[
1 \to \hat{\mathcal{H}} \xrightarrow{i} \hat{\mathcal{S}} \xrightarrow{\hat{\delta}^0} \hat{\mathcal{B}}^{0,1} \xrightarrow{\hat{\delta}^1} \hat{\mathcal{B}}^{0,2}, \tag{3.7}
\]

where \( i \) is an embedding, is exact [26].
By virtue of the exactness of the sequence (3.7), we have
\[ \text{Im} \delta^0 = \text{Ker} \delta^1 = \hat{B}. \]  
(3.8)
Since \( \delta^0 \) is the projection connected with the action (3.3a) of the sheaf \( \hat{S} \) on \( \hat{B}^{0,1} \), the sheaf \( \hat{S} \) acts transitively with the help of \( \text{Ad} \) on \( \hat{B} \) and \( \hat{B} \simeq \hat{S}/\hat{\mathcal{H}} \). Thus, we obtain the exact sequence of sheaves
\[ 1 \longrightarrow \hat{\mathcal{H}} \longrightarrow \hat{S} \xrightarrow{\delta^0} \hat{B} \xrightarrow{\delta^1} 0. \]  
(3.9)
From (3.9) we obtain the exact sequence of cohomology sets
\[ e \longrightarrow H^0(Z, \hat{\mathcal{H}}) \xrightarrow{i^*} H^0(Z, \hat{S}) \xrightarrow{\delta^0} H^0(Z, \hat{B}) \xrightarrow{\delta^1} H^1(Z, \hat{\mathcal{H}}) \xrightarrow{\hat{\varphi}} H^1(Z, \hat{S}), \]  
(3.10)
where \( e \) is a marked element (identity) of the considered sets, and the map \( \hat{\varphi} \) coincides with the canonical embedding, induced by the embedding of sheaves \( i : \hat{\mathcal{H}} \rightarrow \hat{S} \).

There is a one-to-one correspondence between the set \( H^1(Z, \hat{\mathcal{H}}) \) and the set of equivalence classes (moduli space) of holomorphic \( \mathbf{G} \)-bundles over \( Z \). The kernel \( \ker \hat{\varphi} = \hat{\varphi}^{-1}(e) \) of the map \( \hat{\varphi} \) coincides with a subset of those elements from \( H^1(Z, \hat{\mathcal{H}}) \), which are mapped into the class \( e \in H^1(Z, \hat{S}) \) of topologically (and smoothly) trivial bundles. This means that representatives of the subset \( \ker \hat{\varphi} \) are those transition matrices \( \mathbf{F} \in Z^1(\mathfrak{U}, \hat{\mathcal{H}}) \) for which there exists a splitting
\[ \mathbf{F}_{\alpha\beta} = \psi^{-1}_{\alpha} \psi_{\beta} \]  
(3.11)
with smooth matrix-valued functions \( \{ \psi_{\alpha} \} \subset C^0(\mathfrak{U}, \hat{S}) \). Here \( C^0(\mathfrak{U}, \hat{S}) \) is a group of 0-cochains with the coefficients in \( \hat{S} \), and \( Z^1(\mathfrak{U}, \hat{\mathcal{H}}) \) is a set of 1-cocycles with the coefficients in \( \hat{\mathcal{H}} \) (for definitions, see Appendices).

The moduli space \( \mathcal{M}_Z \) of flat (0,1)-connections on \( P' \) (and on \( E' \)) is the space of gauge nonequivalent global solutions to eqs.(3.1). By definition we have \( \mathcal{M}_Z = H^0(Z, \hat{\mathcal{B}})/H^0(Z, \hat{S}) \), where \( H^0(Z, \hat{\mathcal{B}}) \) is the space of global solutions to eqs.(3.1) and \( H^0(Z, \hat{S}) \) is the group of gauge transformations of the hCSW theory. From the exactness of the sequence (3.10) we obtain
\[ \mathcal{M}_Z = H^0(Z, \hat{\mathcal{B}})/H^0(Z, \hat{S}) \simeq \ker \hat{\varphi} \subset H^1(Z, \hat{\mathcal{H}}), \]  
(3.12)
i.e. the moduli space \( \mathcal{M}_Z \) of global solutions of eqs.(3.6) on \( Z \) is bijective to the moduli space of holomorphic bundles over \( Z \) which are trivial as smooth bundles. Transition matrices of such bundles have the form (3.11).

**Remarks**

1. Using the sheaves \( \hat{S} \) and \( \hat{B} \), considered in §3.2, one can introduce a Dolbeault 1-cohomology set \( H^{0,1}_{\partial \hat{B}}(Z) \) as a set of orbits of the group \( H^0(Z, \hat{S}) \) in the set \( H^0(Z, \hat{B}) \), i.e.
\[ H^{0,1}_{\partial \hat{B}}(Z) := H^0(Z, \hat{B})/H^0(Z, \hat{S}). \]  
(3.13)
It follows from (3.12) that \( H^{0,1}_{\partial \hat{B}}(Z) \simeq \ker \hat{\varphi} \), and this is a non-Abelian variant of the isomorphism between Čech and Dolbeault cohomologies. Notice that deformations of a topologically nontrivial holomorphic vector bundle \( E' \rightarrow Z \) can be described by introducing sheaves of sections of the bundles \( \text{Ad}P' \), \( \text{Int}P' \) etc in the same way as it has been done in Remark from §2.4.

2. Up to now we have considered a complex gauge group \( \mathbf{G} \). The point is that in reduction of hCSW theories to 4D CFT theories the imposing of reality conditions in some cases can be more tricky than the choice of a compact subgroup \( \mathbf{G} \subset \mathbf{G}^{\mathbb{C}} \) (see e.g. [17, 18]). Later on we shall return to this problem.

3. From the discussions in §§3.2,3.3 one can see that the description of holomorphic bundles is analogous to the description of locally constant bundles. The difference is mainly in the sheaves describing these theories.
3.4 Theories on Calabi-Yau 3-folds

Let us consider a CY 3-fold $Z$. Then, besides a complex structure $J$ on $Z$, there exist a Kähler 2-form $\omega$ and a Ricci-flat metric, and the canonical line bundle of $Z$ is trivial. There is a one-to-one correspondence between complex structures on the bundle $E' \to Z$ and the operators $\bar{\partial}_B = \bar{\partial} + B$ satisfying the integrability conditions $\bar{\partial}_B^2 = 0$. These operators do not depend on the Kähler form $\omega$ and on the $(3,0)$-form $\hat{\Omega}$ from (3.2) (a section of the canonical bundle of $Z$). Hence the description of §§3.2,3.3 of the moduli space of holomorphic bundles $E' \to Z$ is not changed.

Some refinement arises if we impose an additional (mild) topological condition of stability on the bundle $E'$ (for definitions, see [27, 28] and references therein). Then the curvature 2-form $F$ of a connection on $E'$ has to satisfy the additional equations

$$F \wedge \omega \wedge \omega = 0, \quad (3.14)$$

which are equivalent to [27, 28]

$$\Lambda F = 0. \quad (3.15a)$$

Here $\Lambda$ is an algebraic ‘trace’ operation inverse to multiplication by $\omega$. In local coordinates $\{z^a\}$ eqs.(3.15a) have the form

$$\omega^{ab} F_{ab} = 0, \quad (3.15b)$$

where $\{\omega^{ab}\}$ are components of the bivector which is inverse to the Kähler (1,1)-form $\omega = \omega_{\bar{a}b} dz^a \wedge d\bar{z}^b$.

Connections on the holomorphic bundle $E'$ satisfying eqs.(3.15) are called the Hermitian-Einstein or Hermitian-Yang-Mills connections [27, 28]. Solutions of eqs.(3.15) depend on the Kähler 2-form $\omega$. The moduli space $M'_{CY}$ of Hermitian-Yang-Mills connections is a (dense) subset in the set (3.12), $M'_{CY} \subset M_{CY}$. Equations (3.6), (3.15) are the field equations of the hCSW theory on a CY 3-fold $Z$.

The study of holomorphic CSW theories on CY 3-folds $Z$ is of interest because of their connection with $N=2$ topological strings [4, 5]. These theories are also connected with 4D CFT’s, which we shall discuss below in §4. The study of geometry of the moduli space of stable holomorphic bundles $E'$ over CY 3-folds is also important for understanding extended mirror symmetry and compactification of the heterotic strings (see e.g. [29] and references therein). Extended mirror symmetry exchanges pairs $(Z, E')$ and $(\tilde{Z}, \tilde{E}')$, where $Z, \tilde{Z}$ are CY 3-folds and $E' \to Z, \tilde{E}' \to \tilde{Z}$ are stable holomorphic bundles. One of these bundles may coincide with the tangent bundle. We recover standard mirror symmetry if both of these bundles are tangent bundles. The discussion of this interesting topic is beyond the scope of our paper.

4 Integrable 4D conformal field theories

4.1 Holomorphic 4D CSW theories on special complex 3-manifolds

A choice of different gauge groups and coupling constants leads to different CS/hCSW theories. As has been discussed in §2.5, CS theories on a real 3-manifold $X$ are equivalent to 2D CFT’s if one supposes that $X$ is a trivial fibre bundle over a 2-manifold $\Sigma$ with the fibre $\mathbb{R}$ or $S^1$. Analogously, holomorphic CSW theories on a complex 3-manifold $Z$ are connected with 4D CFT’s if one supposes that $Z$ is a bundle over a real 4-manifold $M$ with two-dimensional fibres.

We consider two main cases. In the first case a complex 3-manifold $Z$ is the total space of a fibre bundle over a real oriented 4-manifold $M$ with $S^2$ as a typical fibre (Riemann surface of genus zero). Then, considering $Z$ as a bundle associated with the bundle of orthonormal frames on $M$, it may
be proved that the Weyl tensor of the manifold $M$ with a metric $g$ is self-dual and such manifolds $M$ are called self-dual \cite{15,18,20}-\cite{22}. In the second case we shall consider a Calabi-Yau 3-fold $Z$ which is a direct product of a 4-manifold $M$ and a torus $T^2$ (Riemann surface of genus one) and therefore $Z = M \times T^2$ is a trivial fibre bundle over $M$. In this case $M$ must be a hyperKähler 4-manifold, for instance, $K3$ or $T^4$ with the changed canonical orientations.

Remember that the 2D WZNW model is a ‘generating’ model for rational 2D CFT’s. An analogous role in four dimensions is played by the self-dual Yang-Mills (SDYM) model. The SDYM theory is connected with the holomorphic CSW theory in the same way as the (chiral) WZNW theory is connected with the ordinary CS theory. Indeed, in the case when $Z$ is a bundle over a self-dual 4-manifold $M$ (then $Z \equiv Z(M)$ is the twistor space of $M$) the hCSW model on $Z$ can be reduced to the SDYM model on $M$. This follows from the existence of a one-to-one correspondence between self-dual vector bundles $E$ (bundles with self-dual connections $A$) over $M$ and such holomorphic vector bundles $E'$ over the twistor space $Z(M)$ that are holomorphically trivial on any projective line $\mathbb{CP}^1 \hookrightarrow Z(M)$ in $Z(M)$ \cite{16}-\cite{19}. From the group-theoretic and cohomological point of view this correspondence has been analyzed in \cite{7}.

We shall show that the SDYM model can also be obtained from the hCSW model on a CY 3-fold $Z = M \times T^2$. In this case gauge fields ‘live’ on a hyperKähler 4-manifold $M$. Notice that by using the so-called Yang gauge \cite{30} the field equations and action \cite{27,31} for the SDYM model so resemble the field equations and action for the 2D WZNW model that in the paper \cite{32} the SDYM theory was called the 4D WZW theory.

In the case of the ordinary CS theory the moduli space of flat connections on a bundle over $X = \Sigma \times \mathbb{R}$ is bijective to the moduli space of flat connections on the induced bundle over $\Sigma$, since the connection component along $\mathbb{R}$ can always be removed by a gauge transformation. In the case of the holomorphic CSW theory on a complex 3-manifold $Z$, which is a fibration over a 4-manifold $M$, for holomorphic connections it is impossible to remove their components along fibres, since fibres are two-dimensional as real manifolds. This is why the moduli space $\mathcal{M}_M$ of a 4D conformal field theory on $M$ connected with the hCSW theory on $Z$ is a subspace in the moduli space $\mathcal{M}_Z$ of the hCSW theory (see §3.3). Below in §4.2 we shall describe an embedding of the moduli space $\mathcal{M}_M$ of self-dual gauge fields on $M$ into the moduli space $\mathcal{M}_Z$ of flat $(0,1)$-connections on the bundle $E' \to Z$.

### 4.2 Moduli space of self-dual Yang-Mills connections

Consider a complex 3-manifold $Z$ which is the fibre bundle with the canonical projection $\pi : Z \to M$ and fibres $\mathbb{CP}^1_x$ over the points $x$ from a self-dual 4-manifold $M$. The typical fibre $\mathbb{CP}^1$ has the $SU(2)$-invariant complex structure $j$ (see \cite{7}), and the vertical distribution $V = \text{Ker} \, \pi_*$ inherits this complex structure. A restriction of $V$ to each fibre $\mathbb{CP}^1_x$, $x \in M$, is the tangent bundle to that fibre. The Levi-Civita connection on the Riemannian manifold $M$ generates the splitting of the tangent bundle $T(Z)$ into a direct sum

$$T(Z) = V \oplus H$$

(4.1)

of the vertical $V$ and horizontal $H$ distributions.

Using the complex structures $j$ and $\mathcal{J}$ on $\mathbb{CP}^1$ and $Z$ respectively, one can split the complexified tangent bundle of $Z$ into a direct sum

$$T^C(Z) = (V^{1,0} \oplus H^{1,0}) \oplus (V^{0,1} \oplus H^{0,1})$$

(4.2)

of subbundles of type $(1,0)$ and $(0,1)$. So we have the integrable distribution $V^{0,1}$ of antiholomorphic vector fields.
Having the canonical distribution $V^{0,1}$ on the space $Z$, we introduce the sheaf $S$ of partially holomorphic maps $\psi : Z \to G$, which are annihilated by vector fields from $V^{0,1}$. In other words, sections of the sheaf $S$ over open subsets $U = U \times \Omega \subset Z$ ($U \subset M, \Omega \subset \mathbb{CP}^1$) are $G$-valued functions $\psi$ on $U$, which are holomorphic on $\Omega \subset \mathbb{CP}^1 \xrightarrow{\iota} Z$, $x \in U$. It is obvious that the sheaf $H = H \oplus H^{0,1}$ is a sheaf of holomorphic maps from $Z$ into $G$, i.e. smooth maps which are annihilated by vector fields from $V^{0,1} \oplus H^{0,1}$, is a subsheaf of $S$, and $S$ is a subsheaf of the sheaf $\hat{S}$ from §3.2. Notice that in this section we use the notation $H$ without hat for the sheaf $H$.

Consider now the sheaves $B^{0,q}$ introduced in §3.2. Let $B^{0,1}$ be the subsheaf of $(0,1)$-forms from $\hat{B}^{0,1}$ vanishing on the distribution $V^{0,1}$. The sheaf $S$ acts on the sheaves $B^{0,1}$ and $\hat{B}^{0,q}$ by means of the adjoint representation. In particular, for $B^{0,1}$ and $\hat{B}^{0,2}$ we have the same formulae (3.3) with the replacement $\psi$ by $\psi \in S(U)$,

\[
B \mapsto \text{Ad} \psi B = \psi^{-1}B \psi + \psi^{-1}\partial \psi, \quad (4.3a)
\]

\[
F \mapsto \text{Ad} \psi F = \psi^{-1}F \psi, \quad (4.3b)
\]

where $B \in B^{0,1}(U)$, $F \in \hat{B}^{0,2}(U)$.

The map $\delta^0$, introduced in §3.2, induces a map $\delta^0 : S \to B^{0,1}$, defined for any open set $U$ of the space $Z$ by the formula

\[
\delta^0 \psi = -(\partial \psi)\psi^{-1}, \quad (4.4)
\]

where $\psi \in S(U)$, $\delta^0 \psi \in B^{0,1}(U)$. Analogously, the operator $\bar{\delta}^1$ induces a map $\bar{\delta}^1 : B^{0,1} \to \hat{B}^{0,2}$, given for any open set $U \subset Z$ by the formula

\[
\bar{\delta}^1 B = \bar{\delta} B + B \wedge B, \quad (4.5)
\]

where $B \in B^{0,1}(U)$, $\bar{\delta}^1 B \in \hat{B}^{0,2}(U)$.

At last, let us denote by $B$ the subsheaf of $B^{0,1}$ consisting (of germs) of $g$-valued $(0,1)$-forms $\tilde{B}$ such that $\tilde{\delta}^1 B = 0$, i.e. sections $B$ of the sheaf $B = \text{Ker} \tilde{\delta}^1$ satisfy the equations

\[
\tilde{\delta} B + B \wedge B = 0. \quad (4.6)
\]

Restricting $\delta^0$ to $S \subset \hat{S}$ and $\bar{\delta}^1$ to $B^{0,1} \subset \hat{B}^{0,1}$, we obtain the exact sequence of sheaves

\[
1 \to H \xrightarrow{i} S \xrightarrow{\delta^0} \hat{B}^{0,1} \xrightarrow{\bar{\delta}^1} \hat{B}^{0,2}, \quad (4.7)
\]

where $1$ is the identity of the sheaf $H$. From (4.7) we obtain the exact sequence of sheaves

\[
1 \to H \xrightarrow{i} S \xrightarrow{\delta^0} B \xrightarrow{\bar{\delta}^1} 0, \quad (4.8)
\]

since $\text{Im} \delta^0 = \text{Ker} \bar{\delta}^1$ (the exactness of the sequence (4.7)), and $S$ acts on $B$ transitively ($B \simeq S/H$).

From (4.8) we obtain the exact sequence of cohomology sets

\[
e \to H^0(Z, H) \xrightarrow{i_*} H^0(Z, S) \xrightarrow{\delta^0_*} H^0(Z, B) \xrightarrow{\bar{\delta}^1_*} H^1(Z, H) \xrightarrow{\varphi} H^1(Z, S), \quad (4.9)
\]

where the map $\varphi$ is an embedding, induced by the embedding of sheaves $i : H \to S$. The kernel $\text{Ker} \varphi = \varphi^{-1}(e)$ of the map $\varphi$ coincides with a subset of those elements from $H^1(Z, H)$, which are mapped into the class $e \in H^1(Z, S)$ of smoothly trivial bundles over $Z$, which are holomorphically trivial on any projective line $\mathbb{CP}^1_x \hookrightarrow Z$, $x \in M$. This means that representatives of the subset $\text{Ker} \varphi$ of the 1-cohomology set $H^1(Z, H)$ are those transition matrices $F \in Z^1(U, H)$ for which there exists a decomposition

\[
F_{\alpha\beta} = \psi^{-1}_\alpha \psi^\beta, \quad (4.10)
\]
with smooth $G$-valued functions $\{\psi_\alpha\} \in C^1(U, S)$ that are *holomorphic* in local complex coordinate $\lambda$ on $\mathbb{C}P^1$. Here $U = \{U_\alpha\}$ is the cover of the space $Z$ introduced already in §3.1.

In the case under consideration it follows from the twistor correspondence [16]-[18] that the moduli space $\mathcal{M}_M^C$ of complex self-dual gauge fields on a self-dual 4-manifold $M$ is parametrized by the set $H^0(Z, \mathcal{B})/H^0(Z, \mathcal{S})$, where $H^0(Z, \mathcal{B})$ is the space of global solutions to eqs.(4.6), and the group $H^0(Z, \mathcal{S})$ is isomorphic to the group of gauge transformations on the self-dual bundle $\mathcal{E}$ over $M$ [7]. From the exactness of the sequence (4.9) it follows that the set $\ker \varphi \subset H^1(X, \mathcal{H})$ is bijective to the moduli space $\mathcal{M}_M^C$ of (complex) solutions to the SDYM equations on $M$,

$$\mathcal{M}_M^C = H^0(Z, \mathcal{B})/H^0(Z, \mathcal{S}) \simeq \ker \varphi \subset H^1(X, \mathcal{H}).$$

(4.11)

If we want to obtain real self-dual gauge fields with values in the compact Lie algebra $g \subset g$ and their moduli space $\mathcal{M}_M \subset \mathcal{M}_M^C$, we should impose some specific reality conditions on all objects ($G$-valued functions $\psi_\alpha$, $(0,1)$-forms $B$ etc). For discussion of the reality conditions see e.g. [17]-[19].

### 4.3 Self-dual gauge fields on hyperKähler manifolds

In §3.4 we discussed holomorphic CSW theories on a CY 3-fold $Z$. It was recalled that after imposing the stability condition, connections on the holomorphic bundle $E'$ over $Z$ have to satisfy the Donaldson-Uhlenbeck-Yau equations (3.6),(3.15). To go over to four dimensions, let us consider the simplest case $Z = M \times T^2$ (see §4.1), where $M$ is a 4-manifold. Then from the CY restriction it follows that $M$ is a hyperKähler manifold. Recall that hyperKähler 4-manifolds are a particular case of self-dual 4-manifolds, where in addition to self-duality of the Weyl tensor the Ricci tensor is equals to zero [18, 33].

So, we have the trivial bundle $Z = M \times T^2 \to M$ with the fibre $T^2$. Suppose, as in §4.2, that components of a connection on $E'$ along the fibre $T^2$ of the bundle $Z \to M$ are equal to zero. Then the Donaldson-Uhlenbeck-Yau equations on $Z$ are reduced to the SDYM equations on the hyperKähler manifold $M$. Thus, holomorphic CSW theories on CY 3-folds are also connected with SDYM theories but now defined on hyperKähler 4-manifolds. Clearly, the SDYM is a very distinguished theory.

Besides the two analyzed cases (fibres $S^2$ and $T^2$ in the bundle $\pi : Z \to M$) it would be interesting to consider also the following more general cases: (1) components of a connection on the bundle $E' \to Z$ are not equal to zero along fibres of the bundle $Z \to M$; (2) the space $Z$ is a (nontrivial) bundle over a 4-manifold $M$ with a Riemann surface $\Sigma$ of genus $\geq 2$ as a typical fibre.

### 4.4 Moduli space of local solutions to the SDYM equations

It has been noted many times that the chiral WZNW theory on a disk $D_0 \simeq \mathbb{C}P^1 - \{\infty\}$ is one of the most important theories among 2D CFT’s. It is recovered from the CS theory on $X = D_0 \times \mathbb{R}$ [1]-[3]. In four dimensions, to this model there corresponds the SDYM model on an open set $U \subset M$, which is in a sense ‘generic’ since it describes all *local* solutions of the SDYM equations. We shall briefly show how to recover this model.

Fix an open set $U \subset M$ such that $Z|_U \simeq U \times \mathbb{C}P^1$ and choose coordinates $x^\mu$ on $U$. Consider the restriction of the twistor bundle $\pi : Z \to M$ to $U$ and put $\mathcal{P} := Z|_U$. The space $\mathcal{P}$ is an open subset of $Z$, and, as a real manifold, $\mathcal{P}$ is diffeomorphic to the direct product $U \times \mathbb{C}P^1$. A metric $g$ on $U$ is not flat, and a conformal structure $[g]$ on $U$ is coded into a complex structure $\mathcal{J}$ on $\mathcal{P}$ [15, 18]. We again have a natural one-to-one correspondence between solutions of the SDYM equations on $U$ and holomorphic bundles $E'$ over $\mathcal{P}$, holomorphically trivial on (real) projective lines $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}$, $\forall x \in U$.
Having described in § 4.2 the connection between the hCSW theory on $Z$ and the SDYM theory on $M$, we consider smooth solutions $A$ of the SDYM equations on $U$ (local solutions). The moduli space $\mathcal{M}_U$ of real local solutions to the SDYM equations on $U$ is an infinite-dimensional space and contains germs of all global solutions on $M$. The moduli space $\mathcal{M}_U$ and the moduli space $\mathcal{M}_C^U$ of complex solutions have been described in [7]. In particular, it is shown that $\mathcal{M}_C^U$ is a double coset space

$$\mathcal{M}_C^U \simeq C^0(\mathfrak{u}, \mathcal{H}) \backslash C^1(\mathfrak{u}, \mathcal{H}) / C^1_\Delta(\mathfrak{u}, \mathcal{H}),$$

(4.12)

where the groups of cochains $C^0(\mathfrak{u}, \mathcal{H}), C^1(\mathfrak{u}, \mathcal{H})$ of the cover $\mathfrak{u}$ of the space $\mathcal{P}$ with values in $\mathcal{H}$ are considered as local groups, and the group $C^1_\Delta(\mathfrak{u}, \mathcal{H})$ is a (diagonal) subgroup in $C^1(\mathfrak{u}, \mathcal{H})$ [7]. The action $\rho_0$ of the group $C^0(\mathfrak{u}, \mathcal{H})$ on the space $C^1(\mathfrak{u}, \mathcal{H})/C^1_\Delta(\mathfrak{u}, \mathcal{H})$ is described in Appendix B.

Analogously, $\mathcal{M}_U$ is a double coset space

$$\mathcal{M}_U \simeq C^0(\mathfrak{u}, \mathcal{H}) \backslash C^1(\mathfrak{u}, \mathcal{H}) / C^1_\tau(\mathfrak{u}, \mathcal{H}),$$

(4.13)

where the real subgroups $C^0(\mathfrak{u}, \mathcal{H}), C^1(\mathfrak{u}, \mathcal{H})$ and $C^1_\tau(\mathfrak{u}, \mathcal{H})$ of the local groups $C^0(\mathfrak{u}, \mathcal{H}), C^1(\mathfrak{u}, \mathcal{H})$ and $C^1_\Delta(\mathfrak{u}, \mathcal{H})$ are described in [7]. Concerning (4.12) and (4.13), recall that $K \backslash G/H$ denotes the double coset structure resulting from division of a group $G$ by the right action of $H$ and left action of $K$.

To sum up, self-duality in 4D is connected with holomorphy in 6D and is a generalization of chirality in 2D. The SDYM model on an open ball $U \subset \mathbb{R}^4$ is an analogue of the chiral WZNW model on a disk $D_0 \subset \mathbb{R}^2 \simeq \mathbb{C}$. The double coset structure (4.13) of the moduli space of solutions to the SDYM equations on $U$ is an analogue of the coset structure $LG/G$ of the moduli space of solutions to the chiral WZNW model on $D_0$. So, quantization of the SDYM model on an open ball $U \subset \mathbb{R}^4$ is an urgent task for constructing of integrable 4D conformal quantum field theories.

### 4.5 Relatives of the SDYM model

Having considered the basic nonlinear integrable 4D conformal field theory - the SDYM theory - one may move on to a consideration of its “relatives”. Just as for ordinary CS theories, it is interesting to study holomorphic CSW theories on the twistor space $Z(M)$ of a self-dual 4-manifold $M$ when we have:

1) connected and simply connected gauge group $G$,
2) connected but not simply connected gauge group $G$,
3) disconnected groups of type $N \rtimes G$, where $G$ is a connected group with a discrete automorphism group $N$.

The case 1) is standard. The case 2) was also considered in the literature, where it was shown, for instance, that instantons for non-simply connected gauge groups can have fractional topological charge (see e.g. [34]). It seems that the case 3) of disconnected groups was not discussed in the literature on the SDYM model. In this case we can obtain analogues of orbifold models studied in 2D CFT’s [35].

It also seems interesting to study coset SDYM models which can be introduced analogously to 2D/3D coset models (see e.g. [2]) by using the Donaldson-Nair-Schiff action [27, 31] and appropriate boundary conditions. It is strange that such models have not been considered in the literature.
5 Solvable theories in two, four and six dimensions

5.1 Free conformal field theories in four dimensions

Among 2D quantum field theories the important role is played by free conformal field theories (see e.g. [36] and references therein). Among free 4D CFT’s on a 4-manifold \( M \) there is an important subclass of completely solvable models connected with the holomorphic CSW theory on the twistor space \( Z(M) \) of \( M \). We shall first describe these theories in 4D and then in 6D setting.

Let us consider a self-dual 4-manifold \( M \) with a metric \( g \). Let we are given a homeomorphism of an open subset \( U \subset M \) on an open subset of the space \( \mathbb{R}^4 \). Then the coordinates \( x^\mu \) on \( \mathbb{R}^4 \), \( \mu, \nu, ... = 1, ..., 4 \), define local coordinates on \( U \subset M \), which we also denote by \( x^\mu \). Speaking further about objects in the local coordinates, we mean \( x^\mu \), e.g. for the metric \( g \) in the local coordinates we have \( g = g_{\mu \nu} dx^\mu dx^\nu \).

On tangent spaces \( T_x M \) we use the metric with the components \( \delta_{\alpha \beta} \), where \( \alpha, \beta, ... = 1, ..., 4 \) are tangent indices, and introduce a (local) orthonormal frame field \( e_\alpha = e_\alpha^\mu \partial_\mu = \epsilon_\alpha^\mu \partial / \partial x^\mu \) for the tangent bundle \( T^\alpha \) and a (local) orthonormal coframe field \( \epsilon^\alpha = \epsilon^\alpha_\mu dx^\mu \) for the cotangent bundle \( T^* M \), \( g = \delta_{\alpha \beta} e^\alpha e^\beta \), \( \epsilon_\alpha^\beta = \delta^\beta_\alpha \).

We denote the Levi-Civita connection on \( M \) by \( \nabla \). Let \( \nabla_\alpha := \nabla_{e_\alpha} \) be its component along \( e_\alpha \). Using the Pauli matrices \( \sigma^a = (\sigma^a_{AA'}) \), where \( a, b, ... = 1, 2, 3 \), \( A = 1, 2 \), \( A' = 1, 2 \), we introduce the matrices \( \sigma^a = \{ i \sigma^a, 1 \} \) with the components \( \sigma^a_{AA'} \) and put \( \nabla_{AA'} := \sigma^a_{AA'} \nabla_\alpha \). Having raised the index with the help of \( \epsilon^{AB} = -\epsilon^{BA} \), \( \epsilon^{12} = 1 \), we obtain the operator \( \nabla^{AA'}_\alpha := \epsilon^{BA} \nabla_{BA'} \).

Let \( \varphi_A...A_n \) be a smooth complex n-index symmetric spinor field on \( M (n > 0) \) with conformal weight \(-1\). These fields have spin \( s = n/2 \) and negative helicity [21, 37]. They are chiral primary fields in the terminology of 4D CFT’s. Equations of motion of such free massless fields interacting with the gravity field (metric \( g \) on \( M \)) have the following form [21, 37]:

\[
\nabla^{A_1}_{A_2} \varphi_{A_1...A_n} = 0. \quad (5.1a)
\]

One can also consider a scalar field \( \varphi_0 \) of spin \( s = 0 \), the field equation for which is

\[
\Box \varphi_0 + \frac{1}{6} R \varphi_0 = 0. \quad (5.1b)
\]

Here \( \Box := \delta^{\alpha \beta} \nabla_\alpha \nabla_\beta \) is the Laplacian on \( M \), and \( R \) is the scalar curvature.

Equations (5.1) are field equations of free 4D CFT’s describing chiral spinor fields of spin \( s = n/2 \) \((n \geq 0)\) interacting with a self-dual background gravitational field \( g \) (the Weyl tensor for \( g \) is self-dual). One can also introduce the interaction of these spinor fields with self-dual Yang-Mills fields. To do this, let us consider a principal \( G \)-bundle \( P \rightarrow M \) with a self-dual connection \( D \) on \( P \). Let \( D_\alpha := D_{e_\alpha} \) be its component along \( e_\alpha \). Denote by \( \nabla_\alpha := \nabla_{e_\alpha} \otimes 1 + 1 \otimes D_\alpha \) the components of tensor-product connection containing both the Levi-Civita and Yang-Mills connections and introduce the operator \( \nabla^{AA'}_\alpha := \epsilon^{BA} \sigma^a_{BA'} \nabla_\alpha \).

Denote by \( \text{AdP} \) a vector bundle with the base \( M \) and a typical fibre \( g \) generated by the adjoint representation of the group \( G \), \( \text{AdP} = P \times_{\text{Ad} G} g \). Let \( \tilde{\varphi}_{A_1...A_n} \) be a smooth n-index symmetric spinor field on \( M (n \geq 0) \) with values in \( \text{AdP} \). Locally this is a \( g \)-valued field on open subsets of \( M \). Now one can introduce equations of motion of free massless chiral fields \( \tilde{\varphi}_{A_1...A_n} \) interacting with a background self-dual gauge and gravitational fields as [38]

\[
\nabla^{A_1}_{A_2} \tilde{\varphi}_{A_1...A_n} = 0, \quad (5.2a)
\]

\[
\Box \tilde{\varphi} + \frac{1}{6} R \tilde{\varphi} = 0, \quad (5.2b)
\]

where \( \Box := \delta^{\alpha \beta} \nabla_\alpha \nabla_\beta \). Equations (5.2) are field equations of 4D CFT’s describing free \( g \)-valued chiral spinor fields of spin \( s = n/2 \) \((n \geq 0)\) in a self-dual background.
5.2 Free 4D conformal field theories in holomorphic setting

Free 4D CFT’s described in §5.1 can be reformulated as theories of free holomorphic fields on the twistor space $Z$ of the self-dual manifold $M$ [21, 37, 38]. Namely, let us define the tautological line bundle $L \to Z$ by taking the restriction $L|_{\mathbb{CP}^1}$ of $L$ to each fibre $\mathbb{CP}^1 \to Z$ to be the standard holomorphic tautological line bundle $L_{x}$ over $\mathbb{CP}^1$ with the first Chern class $c_1(L_x) = -1$. Denote by $L^k$ the $k$-th tensor power of the bundle $L$ and by $O(-k)$ the sheaf of holomorphic sections of the bundle $L^k \to Z$. Then the cohomology group $H^1(Z, O(-n-2))$ is isomorphic to the space $\mathcal{M}_n$ of smooth solutions to eqs.(5.1) on $M$ [21, 37],

$$\mathcal{M}_n \simeq H^1(Z, O(-n-2)).$$

(5.3)

Here $n \geq 0$ is the spin of the chiral spinor fields $\phi_{A_1...A_n}$ from (5.1).

Now let us consider fields $\tilde{\phi}_{A_1...A_n}$ on $M$ satisfying eqs.(5.2). Using the projection $\pi : Z \to M$ of the twistor space on the self-dual 4-manifold $M$ and the twistor correspondence, we can pull back the bundle $P \to M$ to the bundle $P' := \pi^*P$ over $Z$. Let us consider the associated bundles $AdP' := P' \times_{AdG} g$ and $AdP' \otimes L^k$. A self-dual connection $D$ on the bundle $P$ induces the holomorphic connection on the bundle $AdP'$ satisfying eqs.(4.6) of the holomorphic Chern-Simons-Witten theory and therefore the bundle $AdP' \otimes L^k$ is holomorphic. Denote by $O^g(-k)$ the sheaf of holomorphic sections of the bundle $AdP' \otimes L^k$. There is an isomorphism

$$\mathcal{M}_n^g \simeq H^1(Z, O^g(-n-2))$$

(5.4)

between the first cohomology group $H^1(Z, O^g(-n-2))$ of the space $Z$ with the coefficients in the sheaf $O^g(-n-2)$ and the solution space $\mathcal{M}_n^g$ of eqs.(5.2) [38].

Using the isomorphisms (5.3) and (5.4), one can write down formulae for general solutions of eqs.(5.1) and (5.2). Thus, the above-described free 4D CFT’s are not only integrable but also explicitly solvable. However, we shall call them free integrable 4D CFT’s.

5.3 Interconnections of 4D and 6D theories

In §§4,5.1,5.2 we have considered interconnections of holomorphic Chern-Simons-Witten theories in three complex dimensions (six real dimensions) with integrable conformal field theories in four real dimensions. This consideration can be summarized in the following “commutative diagram”:

\[
\begin{array}{ccc}
\text{Holomorphic CSW theories on complex 3-manifolds} & \approx & \text{Holomorphic CSW theories and their relatives on Calabi-Yau 3-folds} \\
\text{Holomorphic CSW theories and their relatives on twistor spaces of self-dual 4-manifolds} & \longrightarrow & \text{Holomorphic CSW theories and their relatives on Calabi-Yau 3-folds} \\
\downarrow & & \downarrow \\
\text{Integrable 4D CFT’s on 4-manifolds with self-dual Weyl tensor} & \longrightarrow & \text{Integrable 4D CFT’s on 4-manifolds with self-dual Riemann tensor} \\
\end{array}
\]

(5.5)

The arrows mean that one theory can be derived from another one. For instance, integrable 4D CFT’s on 4-manifolds with self-dual Riemann tensor can be derived from hCSW theories on twistor spaces and from hCSW theories on CY 3-folds.

Almost all theories described above arise in string theory. Namely,
• integrable 4D CFT’s on 4-manifolds with the self-dual Riemann tensor are connected with $\mathcal{N} = 2$ strings (4D target space) [39, 40];

• holomorphic CSW theories on Calabi-Yau 3-folds are connected with $\mathcal{N} = 2$ topological strings (6D target space) [4, 5];

• holomorphic CSW theories on twistor spaces of Ricci-flat self-dual 4-manifolds are connected with $\mathcal{N} = 4$ topological strings [41, 42];

• holomorphic CSW theories on complex 3-manifolds can be connected with generalized $\mathcal{N} = 2$ topological string theories [5, 42].

It would be interesting to study the last case. Notice also that heterotic $\mathcal{N} = (2, 1)$ strings [40, 43] are connected with integrable 4D CFT’s on 4-manifolds with the self-dual Weyl tensor, and we shall describe this elsewhere.

5.4 Parallels between 2D and 4D conformal theories

In § 2.5 we have discussed 2D CFT’s arising from ordinary CS theories in 3 real dimensions, and in § 4 we have discussed 4D CFT’s arising from holomorphic CSW theories in 3 complex dimensions. Parallels between 2D and 4D conformal field theories are summarized in the following table:

<table>
<thead>
<tr>
<th>2D</th>
<th>4D</th>
</tr>
</thead>
<tbody>
<tr>
<td>chiral WZNW models</td>
<td>SDYM models</td>
</tr>
<tr>
<td>WZNW models with non-simply connected gauge groups</td>
<td>SDYM models with non-simply connected gauge groups describe instantons with fractional topological charge</td>
</tr>
<tr>
<td>orbifold models</td>
<td>orbifold models (can be introduced)</td>
</tr>
<tr>
<td>coset models</td>
<td>coset models (can be introduced)</td>
</tr>
<tr>
<td>free chiral CFT’s</td>
<td>free chiral CFT’s describing (chiral) fields of negative helicity and arbitrary spin $s \geq 0$</td>
</tr>
</tbody>
</table>

In this paper we discussed how holomorphic CSW theories on complex 3-manifolds are connected with integrable CFT’s on real 4-manifolds and showed that these theories are similar to (chiral) rational 2D CFT’s or (chiral) free 2D CFT’s. It is well known that significant progress in understanding CFT’s in two dimensions was related to the existence of the Virasoro-Kač-Moody symmetries imposing severe restrictions on correlation functions and uniquely determining them in genus zero. Naturally, in studying integrable 4D CFT’s the following questions arise:

1. What are analogues of affine Lie algebras of 2D CFT’s ?
2. What is an analogue of the Virasoro algebra ?

We shall answer these questions in § 6 using the Čech approach to the sheaf cohomologies. To see that symmetry algebras of integrable 4D CFT’s really generalize symmetry algebras of 2D CFT’s, in § 6.1 we shall describe the Virasoro and affine Lie algebras in terms of the Čech cohomology.
6 Cohomological symmetry algebras

6.1 The Čech description of the Virasoro and affine Lie algebras

Let us consider a two-dimensional sphere $S^2$. The sphere $S^2$ can be covered by two coordinate patches $\Omega_1, \Omega_2$, with $\Omega_1$ the neighbourhood of $\lambda = 0$, and $\Omega_2$, the neighbourhood of $\lambda = \infty$, where $\lambda$ is a complex coordinate on $S^2 = \mathbb{C} \cup \infty$. The sphere $S^2$, considered as a complex projective line $\mathbb{CP}^1 = \Omega_1 \cup \Omega_2$, is the complex manifold obtained by patching together $\Omega_1$ and $\Omega_2$ with the coordinates $\lambda$ on $\Omega_1$ and $\zeta$ on $\Omega_2$ related by $\zeta = \lambda^{-1}$ on $\Omega_1 \cap \Omega_2$. For example, if $\Omega_1 = \{ \lambda \in \mathbb{C} : |\lambda| < \infty \} = \mathbb{C}$ and $\Omega_2 = \{ \lambda \in \mathbb{C} \cup \infty : |\lambda| > 0 \} = \mathbb{C} \cup \infty$, then $\Omega_1 \cap \Omega_2$ is the multiplicative group $\mathbb{C}^*$ of complex numbers $\lambda \neq 0, \infty$. In the following we shall consider this two-set open cover $\mathcal{O} = \{ \Omega_1, \Omega_2 \}$ of the Riemann sphere $\mathbb{CP}^1$.

Remember that the affine Lie algebra $\mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ (without a central term) is the algebra of $\mathfrak{g}$-valued meromorphic functions on $\mathbb{CP}^1 \simeq \mathbb{C}^* \cup \{0\} \cup \{\infty\}$ with the poles at $\lambda = 0, \lambda = \infty$ and holomorphic on $\Omega_1 = \Omega_1 \cap \Omega_2 \simeq \mathbb{C}^*$. Hence, it is a subalgebra in the algebra

$$C^1(\mathcal{O}, \mathcal{O}_{\mathcal{CP}^1}) \simeq \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$$

(6.1) of 1-cochains of the cover $\mathcal{O} = \{ \Omega_1, \Omega_2 \}$ of $\mathbb{CP}^1$ with values in the sheaf of holomorphic maps from $\mathbb{CP}^1$ into the Lie algebra $\mathfrak{g}$. For definitions, see Appendices. Notice that (central) extensions of the algebra (6.1) will appear after passing to quantum theory.

Elements of the Virasoro algebra $Vir^0$ (with zero central charge) are meromorphic vector fields on $\mathbb{CP}^1$ having poles at the points $\lambda = 0, \lambda = \infty$ and holomorphic on the overlap $\Omega_{12} = \Omega_1 \cap \Omega_2 \simeq \mathbb{C}^* = \mathbb{CP}^1 \setminus \{0\} \setminus \{\infty\}$. This algebra has the following Čech description. Let us consider the sheaf $\mathcal{V}_{\mathbb{CP}^1}$ of holomorphic vector fields on $\mathbb{CP}^1$. Then, for the space of Čech 1-cochains with values in $\mathcal{V}_{\mathbb{CP}^1}$ we have

$$C^1(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1}) \simeq Vir^0 \oplus Vir^0.$$  

(6.2)

Notice that for $\{v_{12}, v_{21}\} \in C^1(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1})$ the antisymmetry condition cannot be imposed on cohomology indices of the holomorphic vector fields $v_{12}, v_{21}$ since it is not preserved under commutation. So in the general case we have $v_{21} \neq v_{12}$.

The space $Z^1(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1}) = \{ v \in C^1(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1}) : v_{12} = -v_{21} \}$ of 1-cocycles of the cover $\mathcal{O}$ of $\mathbb{CP}^1$ with values in the sheaf $\mathcal{V}_{\mathbb{CP}^1}$ coincides with the algebra $Vir^0$ as a vector space since

$$Z^1(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1}) \simeq (Vir^0 \oplus Vir^0)/\text{diag}(Vir^0 \oplus Vir^0).$$  

(6.3)

Further, by virtue of the equality

$$H^1(\mathbb{CP}^1, \mathcal{V}_{\mathbb{CP}^1}) = 0,$$  

(6.4)

which means the rigidity of the complex structure of $\mathbb{CP}^1$, any element $v_{12} = -v_{21}$ from $Z^1 \simeq Vir^0$ can be represented in the form

$$v_{12} = v_1 - v_2.$$  

(6.5)

Here, $v_1$ can be extended to a holomorphic vector field on $\Omega_1$, and $v_2$ can be extended to a holomorphic vector field on $\Omega_2$.

It follows from (6.3)–(6.5) that the algebra $Vir^0$ is connected with the algebra

$$C^0(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1})$$

(6.6)

of 0-cochains of the cover $\mathcal{O}$ with values in the sheaf $\mathcal{V}_{\mathbb{CP}^1}$ by the (twisted) homomorphism

$$f^0 : C^0(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1}) \longrightarrow C^1(\mathcal{O}, \mathcal{V}_{\mathbb{CP}^1}) \leftrightarrow$$

(6.7a)
The symmetry algebra is the algebra \( U_{CP}^\text{Vir} \), and on the intersection
\( U_{0} = U_{1} \cap U_{2} \), \( U_{1} = U \times \Omega_{1} \), \( U_{2} = U \times \Omega_{2} \),
with the coordinates \( \{ x^{\mu}, \lambda, \bar{\lambda} \} \) on \( U_{1} \) and \( \{ x^{\mu}, \zeta, \bar{\zeta} \} \) on \( U_{2} \). The two-set open cover \( \Omega = \{ \Omega_{1}, \Omega_{2} \} \)
of the Riemann sphere \( \mathbb{C}P^{1} \) has been described in §6.1. We shall consider the intersection \( U_{12} \) of
\( U_{1} \) and \( U_{2} \),
\[ U_{12} := U_{1} \cap U_{2} = U \times (\Omega_{1} \cap \Omega_{2}), \]
with the coordinates \( x^{\mu} \) on \( U \), \( \lambda, \bar{\lambda} \) on \( \Omega_{12} := \Omega_{1} \cap \Omega_{2} \). Thus, the twistor space \( P \) is a trivial bundle
\( \pi : P \to U \) over \( U \) with the fibre \( \mathbb{C}P^{1} \), where \( \pi : \{ x^{\mu}, \lambda, \bar{\lambda} \} \to \{ x^{\mu} \} \) is the canonical projection. For more details see [7].

The space \( P \) is a complex manifold. One can introduce complex coordinates \( \{ z_{1}^{1} \} \) on \( U_{1} \), \( \{ z_{2}^{2} \} \) on \( U_{2} \), and on the intersection \( U_{12} \) of charts \( U_{1} \) and \( U_{2} \) these coordinates are connected by the holomorphic transition function \( f_{12}, z_{1}^{1} = f_{12}^{*}(z_{2}^{2}) \) (see [7] for explicit expressions).

In §§6.3, 6.4 we shall briefly describe symmetry algebras of the SDYM equations on \( U \) answering thus the questions of §5.4 on 4D analogues of the Virasoro and affine Lie algebras (without central terms).

6.3 Analogues of affine Lie algebras in integrable 4D CFT’s

An affine-type symmetry algebra of integrable 4D CFT’s is connected with the algebra \( G_{h} \) of functions that are holomorphic on \( U_{12} = U_{1} \cap U_{2} \subset P \) and take values in the Lie algebra \( g \) of a complex Lie group \( G \). The algebra \( G_{h} \) with pointwise commutators generalizes affine Lie algebras.

The symmetry algebra is the algebra
\[ C^{1}(U, \mathcal{O}_{P}^{g}) \simeq G_{h} \oplus G_{h} \]
of 1-cochains of the cover \( U = \{ U_{1}, U_{2} \} \) of the space \( P \) with values in the sheaf \( \mathcal{O}_{P}^{g} \) of holomorphic maps from \( P \) into the Lie algebra \( g \). We mainly consider the case \( g = sl(n, \mathbb{C}) \).

The algebra (6.10) acts on the transition matrix \( F_{12}^{0} = 1 \) of the trivial holomorphic bundle \( E_{0}^{0} = P \times \mathbb{C}^{n} \) corresponding to the vacuum solution \( A^{0} = 0 \) of the SDYM equations on \( U \) in the Penrose-Ward construction. This action generates infinitesimal deformations of the bundle \( E_{0}^{0} \) described by \( \delta F^{0} \) and variations \( \delta A^{0} \) of the trivial solution \( A^{0} \). If we want to consider infinitesimal variations of a nontrivial self-dual gauge potential \( A \) and a transition matrix \( F_{12}^{0} \) in a holomorphically nontrivial bundle \( E^{0} \to P \), we have to use a more general sheaf than the sheaf \( \mathcal{O}_{P}^{g} \) from (6.10) [45]. Namely,
Consider a principal fibre bundle $P' \to P$ with a transition matrix $F_{12}$ on $U_{12} = U_1 \cap U_2$ for the cover $\Omega = \{U_1, U_2\}$ of $P$. Then, introduce the associated bundle $\text{Ad}P' = P' \times \text{Ad}G \mathfrak{g}$ of Lie algebras and the sheaf of holomorphic sections of the bundle $\text{Ad}P'$. Since this sheaf is isomorphic to the sheaf $O^\mathfrak{g}_P$ considered above, we also denote it by $O^\mathfrak{g}_P$.

Now we introduce the algebra of 1-cochains of the cover $\Omega$ with values in the sheaf of holomorphic sections of the bundle $\text{Ad}P'$. This algebra is isomorphic to the algebra $(6.10)$, and we also denote it by $C^1(\Omega, O^\mathfrak{g}_P)$. For a description of infinitesimal symmetries of the SDYM equations we need also the sheaf $\mathcal{S}^\mathfrak{g}_P$ of smooth maps from $P$ into $\mathfrak{g}$ which are holomorphic along fibres $\mathbb{C}P^1_x$ of the bundle $P \to U$. Then we introduce the algebra $C^0(\Omega, \mathcal{S}^\mathfrak{g}_P)$ of 0-cochains of the cover $\Omega$ with values in $\mathcal{S}^\mathfrak{g}_P$. Elements of the algebra $C^0(\Omega, \mathcal{S}^\mathfrak{g}_P)$ are the collection $\{\phi_1, \phi_2\} \in C^0(\Omega, \mathcal{S}^\mathfrak{g}_P)$ of sections of the sheaf $\mathcal{S}^\mathfrak{g}_P$, where $\phi_1$ is a section over $U_1$, and $\phi_2$ is a section over $U_2$. Now we briefly describe the action of $C^1(\Omega, O^\mathfrak{g}_P)$ on $F_{12}$ and $A = A_\mu dx^\mu$ following the papers [7, 45].

The action of $\theta = \{\theta_{12}, \theta_{21}\} \in C^1(\Omega, O^\mathfrak{g}_P)$ on the transition matrix $F_{12}$ is

$$F_{12} \mapsto \delta_\theta F_{12} = \theta_{12} F_{12} - F_{12} \theta_{21}. \tag{6.11}$$

If we choose the compact gauge Lie algebra $\mathfrak{g} = su(n) \subset \mathfrak{g} = sl(n, \mathbb{C})$ and want to preserve the reality of gauge fields, then in $C^1(\Omega, O^\mathfrak{g}_P)$ we should choose a subalgebra (real form) $C^1_\tau(\Omega, O^\mathfrak{g}_P)$ imposing the conditions [7, 45]

$$\theta_{12}^1(x, -\overline{\lambda}^{-1}) = -\theta_{21}(x, \lambda),$$

where the coordinates $x, \lambda, \overline{\lambda}$ have been introduced in (6.8), and $\dagger$ denotes Hermitian conjugation.

Recall that to a self-dual gauge potential $A = A_\mu dx^\mu$ there corresponds the $(0,1)$-part $\partial_B$ of a connection on the bundle $E' \to P$ satisfying eqs.(4.6) and the transition matrix $F_{12} = \psi_1^{-1} \psi_2$, where $\mathfrak{g}$-valued functions $\psi_{1,2}$ are defined from the equations $(\partial + B) \psi_1 = 0$ and $(\partial + B) \psi_2 = 0$ on $U_1$ and $U_2$, respectively [7]. We introduce $\psi_1(\delta_\theta F_{12}) \psi_2^{-1}$ and from the formula $\delta_\theta F_{12} = (\delta_\theta \psi_1^{-1}) \psi_2 + \psi_1^{-1} \delta_\theta \psi_2$ it follows that

$$\psi_1(\delta_\theta F_{12}) \psi_2^{-1} \in Z^1(\Omega, \mathcal{S}^\mathfrak{g}_P).$$

Since $H^1(\mathbb{C}P^1, O^\mathfrak{g}_{\mathbb{C}P^1}) = H^1(P, \mathcal{S}^\mathfrak{g}_P) = 0$, there always exists a splitting

$$\psi_1(\delta_\theta F_{12}) \psi_2^{-1} = \phi_1(\theta) - \phi_2(\theta), \tag{6.12}$$

where $\{\phi_1(\theta), \phi_2(\theta)\} \in C^0(\Omega, \mathcal{S}^\mathfrak{g}_P)$. Smooth $\mathfrak{g}$-valued function $\phi_1(\theta)$ on $U_1$, holomorphic in $\lambda$, and smooth $\mathfrak{g}$-valued function $\phi_2(\theta)$ on $U_2$, holomorphic in $1/\lambda$, give a solution of the infinitesimal variant of the Riemann-Hilbert problem. Moreover, formula (6.12) defines a (twisted) isomorphism between the algebra $C^1(\Omega, O^\mathfrak{g}_P)$ and a subalgebra in $C^0(\Omega, \mathcal{S}^\mathfrak{g}_P)$ obtained through solutions of infinitesimal Riemann-Hilbert problems.

Having $\phi(\theta) = \{\phi_1(\theta), \phi_2(\theta)\} \in C^0(\Omega, \mathcal{S}^\mathfrak{g}_P)$, one can define an action of the algebra $C^1(\Omega, O^\mathfrak{g}_P)$ on smooth objects $\psi = \{\psi_1, \psi_2\}$ and $B = \{B^{(1)}, B^{(2)}\}$ on $P$. Namely, this algebra acts on $\psi_1, \psi_2$ and $B^{(1)}, B^{(2)}$ as follows

$$\delta_\theta \psi_1 = -\phi_1(\theta) \psi_1, \quad \delta_\theta \psi_2 = -\phi_2(\theta) \psi_2, \quad \delta_\theta B^{(1)} = \partial_B \phi_1(\theta) + [B^{(1)}, \phi_1(\theta)], \quad \delta_\theta B^{(2)} = \partial_B \phi_2(\theta) + [B^{(2)}, \phi_2(\theta)], \tag{6.13a}$$

where $\phi(\theta)$ corresponds to $\theta$ through eqs.(6.12). Notice that this is not a gauge transformation, because $\phi_1(\theta) \neq \phi_2(\theta)$ on $U_{12} \subset P$.

At last, one can define an action of the algebra $C^1(\Omega, O^\mathfrak{g}_P)$ on the self-dual gauge potential $A = A_\mu dx^\mu$ using the complex coordinates $y^1 = x^1 + ix^2, y^2 = x^3 - ix^4, \overline{y}^1 = x^1 - ix^2, \overline{y}^2 = x^3 + ix^4$. 

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on $\mathbb{R}^4 \simeq \mathbb{C}^2$. Let us rewrite components of the connection on $E \to U$ in these coordinates: $D_{y_1} = \partial_{y_1} + A_{y_1}$, $D_{y_2} = \partial_{y_2} + A_{y_2}$, $D_{\bar{y}_1} = \partial_{\bar{y}_1} + A_{\bar{y}_1}$, $D_{\bar{y}_2} = \partial_{\bar{y}_2} + A_{\bar{y}_2}$. Then we have [7, 45]

$$\delta_0 A_{y_1} = \text{Res}_{\lambda=0} \left[ \lambda^{-2} (D_{y_2} \phi_2 (\theta) + \lambda D_{y_1} \phi_2 (\theta)) \right] := \oint_{S^1} \frac{d\lambda}{2\pi i \lambda^2} (D_{y_2} \phi_2 (\theta) + \lambda D_{y_1} \phi_2 (\theta)), \quad (6.14a)$$

$$\delta_0 A_{y_2} = - \text{Res}_{\lambda=0} \left[ \lambda^{-2} (D_{y_1} \phi_2 (\theta) - \lambda D_{y_2} \phi_2 (\theta)) \right] := - \oint_{S^1} \frac{d\lambda}{2\pi i \lambda^2} (D_{y_1} \phi_2 (\theta) - \lambda D_{y_2} \phi_2 (\theta)), \quad (6.14b)$$

$$\delta_0 A_{\bar{y}_1} = \text{Res}_{\lambda=0} \left[ \lambda^{-1} (D_{y_2} \phi_1 (\theta) - \lambda D_{y_1} \phi_1 (\theta)) \right] := \oint_{S^1} \frac{d\lambda}{2\pi i \lambda} (D_{y_2} \phi_1 (\theta) - \lambda D_{y_1} \phi_1 (\theta)), \quad (6.14c)$$

$$\delta_0 A_{\bar{y}_2} = \text{Res}_{\lambda=0} \left[ \lambda^{-1} (D_{y_1} \phi_1 (\theta) + \lambda D_{y_2} \phi_1 (\theta)) \right] := \oint_{S^1} \frac{d\lambda}{2\pi i \lambda} (D_{y_1} \phi_1 (\theta) + \lambda D_{y_2} \phi_1 (\theta)), \quad (6.14d)$$

where the contour $S^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$ circles once around $\lambda = 0$ and the contour integral determines residue $\text{Res}$ at the point $\lambda = 0$. It follows from (6.14) that the action of the algebra $\mathcal{C}^1(\mathcal{U}, \mathcal{O}^\mathbb{R}_P)$ on $\{ A_\mu \}$ is nonlocal, i.e. $\delta_0 A_\mu$ depend on values of $\{ A_\mu \}$ at all points $x \in U$ since $\phi_{1,2}(\theta)$ hiddenly depend on $\{ A_\mu \}$.

6.4 An analogue of the Virasoro algebra in integrable 4D CFT’s

In §§5.4, 5.5, 8.2 of the paper [7] the local group $\mathcal{F}$ of biholomorphisms of the twistor space $\mathcal{P}$ and its action on the space of local solutions to the SDYM equations were described. To this group there corresponds the algebra $\mathcal{C}^0(\mathcal{U}, \mathcal{V}_P)$ of 0-cochains of the cover $\mathcal{U} = \{ U_1, U_2 \}$ of $\mathcal{P}$ with values in the sheaf $\mathcal{V}_P$ (of germs) of holomorphic vector fields on $\mathcal{P} = U_1 \cup U_2$. In particular, the algebra $\mathcal{H}^0(\mathcal{P}, \mathcal{V}_P)$ of global sections of the sheaf $\mathcal{V}_P$ corresponds to biholomorphisms of $\mathcal{P}$ preserving the transition function $f_{12}$. However, these algebras are not correct generalizations of the Virasoro algebra.

An analogue of the Virasoro algebra is the algebra $\mathcal{V}_P(U_{12})$ of holomorphic vector fields on $U_{12} = U_1 \cap U_2 \subset \mathcal{P}$. It is a subalgebra of the algebra

$$\mathcal{C}^1(\mathcal{U}, \mathcal{V}_P) \simeq \mathcal{V}_P(U_{12}) \oplus \mathcal{V}_P(U_{12}) \quad (6.15)$$

of 1-cochains of the cover $\mathcal{U}$ with values in the sheaf $\mathcal{V}_P$. Elements of the algebra $\mathcal{C}^1(\mathcal{U}, \mathcal{V}_P)$ are collections of vector fields

$$\chi = \{ \chi_{12}, \chi_{21} \} = \{ \chi_{12}^a \frac{\partial}{\partial z_{12}^a}, \chi_{21}^b \frac{\partial}{\partial z_{21}^b} \} \quad (6.16)$$

with ordered “cohomology indices”.

Let us define the following action of the algebra $\mathcal{C}^1(\mathcal{U}, \mathcal{V}_P)$ on the complex coordinates on $\mathcal{P}$:

$$\delta_\chi f_{12}^a = \chi_{12}^a \frac{\partial f_{12}^a}{\partial z_{21}^b} \chi_{21}^b \quad \iff \quad \delta_\chi f_{12}^0 \frac{\partial}{\partial z_{12}^a} = \chi_{12} = \chi_{21}. \quad (6.17)$$

The algebra $\mathcal{C}^0(\mathcal{U}, \mathcal{V}_P)$ acts on the transition function $f_{12}$ of the space $\mathcal{P}$ by the formula

$$h^0 : \mathcal{C}^0(\mathcal{U}, \mathcal{V}_P) \ni \{ \chi_1, \chi_2 \} \mapsto \{ \chi_1 - \chi_2, \chi_2 - \chi_1 \} \in \mathcal{C}^1(\mathcal{U}, \mathcal{V}_P) \quad (6.18)$$

of the algebra $\mathcal{C}^0(\mathcal{U}, \mathcal{V}_P)$ into the algebra $\mathcal{C}^1(\mathcal{U}, \mathcal{V}_P)$.

Let us consider a subalgebra

$$\mathcal{C}^1_{\Delta}(\mathcal{U}, \mathcal{V}_P) = \{ \chi \in \mathcal{C}^1(\mathcal{U}, \mathcal{V}_P) : \chi_{12} = \chi_{21} \} \quad (6.19)$$
of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$. Then, the space

$$Z^1(\mathcal{U}, \mathcal{V}_P) = \{ \chi \in C^1(\mathcal{U}, \mathcal{V}_P) : \chi_{12} = -\chi_{21} \}$$

of 1-cocycles of the cover $\mathcal{U}$ with values in $\mathcal{V}_P$ is isomorphic to the quotient space

$$Z^1(\mathcal{U}, \mathcal{V}_P) \simeq C^1(\mathcal{U}, \mathcal{V}_P)/C^1(\mathcal{U}, \mathcal{V}_P).$$

So we can always split $2\chi \in C^1(\mathcal{U}, \mathcal{V}_P)$ in symmetric $\chi^{\text{sym}} := \{ \chi_{12} + \chi_{21}, \chi_{12} + \chi_{21} \} \in C^1(\mathcal{U}, \mathcal{V}_P)$ and antisymmetric $\chi^{\text{ant}} := \{ \chi_{12} - \chi_{21}, \chi_{21} - \chi_{12} \} \in Z^1(\mathcal{U}, \mathcal{V}_P)$ parts.

Notice that $\delta \chi := \{ \delta \chi f_{12}, \delta \chi f_{21} \} \in Z^1(\mathcal{U}, \mathcal{V}_P)$, and the quotient space

$$H^1(\mathcal{U}, \mathcal{V}_P) := h^0(C^0(\mathcal{U}, \mathcal{V}_P))\backslash Z^1(\mathcal{U}, \mathcal{V}_P)$$

(6.20)
describes nontrivial infinitesimal deformations of the complex structure of $\mathcal{P}$. For the cover $\mathcal{U} = \{ \mathcal{U}_1, \mathcal{U}_2 \}$ charts $\mathcal{U}_1, \mathcal{U}_2$ are Stein manifolds, and we have $H^1(\mathcal{P}, \mathcal{V}_P) = H^1(\mathcal{U}, \mathcal{V}_P)$. In contrast with the 2D case (6.4) we now have $H^1(\mathcal{P}, \mathcal{V}_P) \neq 0$. Hence, the transformations (6.17) of the transition function in general change the conformal structure on $\mathcal{P}$ and therefore change the conformal structure on $U$.

We consider a holomorphic bundle $E' \to \mathcal{P}$ which corresponds to a self-dual bundle $E \to U$. Then $C^1(\mathcal{U}, \mathcal{V}_P)$ is the symmetry algebra of the following system of equations:

$$\bar{\partial}^2 = 0, \quad \bar{\partial} B + B \wedge B = 0,$$

(6.21)

where the equations $\bar{\partial}^2 = 0$ are the integrability conditions of an almost complex structure on $\mathcal{P}$, and the equations $\bar{\partial}^2_B = \bar{\partial} B + B \wedge B = 0$ are the conditions of holomorphy of the bundle $E'$ over $\mathcal{P}$.

The symmetric part $\chi^{\text{sym}}$ of $\chi \in C^1(\mathcal{U}, \mathcal{V}_P)$ does not change the transition function $f_{12}$ (see (6.17)) and the complex structure of $\mathcal{P}$. Hence, one may define the following holomorphic action of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ on transition matrices $\mathcal{F}_{12}$ of the holomorphic bundle $E'$:

$$\delta \chi \mathcal{F}_{12} = \chi^{\text{sym}}(\mathcal{F}_{12}) = \chi_{12}(\mathcal{F}_{12}) + \chi_{21}(\mathcal{F}_{12}).$$

(6.22)

To define an action of this algebra on smooth objects $B = \{ B^{(1)}, B^{(2)} \}, \{ \psi_1, \psi_2 \}$ on $\mathcal{P}$ and on a self-dual gauge potential $A$ on $U$, one should: 1) substitute $\delta \chi \mathcal{F}_{12}$ from (6.22) into (6.12) instead of $\delta \theta \mathcal{F}_{12}$ and obtain $\phi_1(\chi), \phi_2(\chi), 2) substitute \phi_{1,2}(\chi)$ into (6.13) instead of $\phi_{1,2}(\theta), 3) substitute \phi_{1,2}(\chi)$ into (6.14) instead of $\phi_{1,2}(\theta)$ and obtain $\delta \chi A_{\nu_1}$ etc.

6.5 Infinitesimal deformations of complex structures on $\mathcal{P}$

Recall that a conformal structure $[g]$ on $U \subset M$ is called self-dual if the Weyl tensor for any metric $g$ on $U$ in the conformal equivalence class $[g]$ is self-dual [18]. By virtue of the twistor correspondence [15, 18] the moduli space of self-dual conformal structures on an open subset $U$ of a 4-manifold $M$ is bijective to the moduli space of complex structures on the twistor space $\mathcal{P}$ of $U$. Thus, deformations of the complex structure on $\mathcal{P}$ are equivalent to deformations of the conformal structure on $U$.

All algebras of infinitesimal symmetries of the self-dual gravity equations known by now (see e.g. [47] and references therein) are subalgebras in the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$. The action of the algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ transforms $f_{12}$ into an equivalent transition function and therefore preserves the conformal structure on $U$. At the same time, the action of the algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ on transition matrices of holomorphic bundles $E' \to \mathcal{P}$ is not trivial.
To define an action of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ on smooth objects on $\mathcal{P}$ (functions, q-forms etc), we should introduce:

1) the sheaf $\mathcal{T}^{1,0}$ of $(1,0)$ vector fields on $\mathcal{P}$, holomorphic along fibres $\mathbb{C}P_1^{\mathcal{Z}}$ of the bundle $\mathcal{P} \to U$;
2) the sheaf $\mathcal{W}$ of $\partial$-closed $(0,1)$-forms $W$ on $\mathcal{P}$ with values in $\mathcal{T}^{1,0}$, vanishing on the distribution $\mathcal{V}^{0,1}$ (see § 4.2).

On any open set $\mathcal{U} \subset \mathcal{P}$ sections $W \in \mathcal{W}(\mathcal{U})$ of the sheaf $\mathcal{W}$ have to satisfy the equations

$$\partial W = 0$$

(6.23)

Then we have the exact sequence of sheaves

$$0 \rightarrow \mathcal{V}_P \rightarrow \mathcal{T}^{1,0} \rightarrow \mathcal{W} \rightarrow 0$$

(6.24)

and the corresponding exact sequence of cohomology spaces

$$0 \rightarrow H^0(\mathcal{P}, \mathcal{V}_P) \rightarrow H^0(\mathcal{P}, \mathcal{T}^{1,0}) \rightarrow H^0(\mathcal{P}, \mathcal{W}) \rightarrow H^1(\mathcal{P}, \mathcal{V}_P) \rightarrow 0,$$

(6.25)

describing infinitesimal deformations of the complex structure of the twistor space $\mathcal{P}$.

From (6.25) it follows that for any element $\delta_\chi f \in Z^1(\mathcal{U}, \mathcal{V}_P) \subset Z^1(\mathcal{U}, \mathcal{T}^{1,0})$ there exists an element $\varphi = \{\varphi_1, \varphi_2\} \in C^0(\mathcal{U}, \mathcal{T}^{1,0})$ such that

$$\delta_\chi f = \{\chi_{12} - \chi_{21}, \chi_{21} - \chi_{12}\} = \{\varphi_1(\chi) - \varphi_2(\chi), \varphi_2(\chi) - \varphi_1(\chi)\} \in h^0(C^0(\mathcal{U}, \mathcal{T}^{1,0})),$$

(6.26)

where $h^0 : \{\varphi_1, \varphi_2\} \mapsto \{\varphi_1 - \varphi_2, \varphi_2 - \varphi_1\}$ is the twisted homomorphism of the algebra $C^0(\mathcal{U}, \mathcal{T}^{1,0})$ into the algebra $C^1(\mathcal{U}, \mathcal{T}^{1,0})$. Then for infinitesimal transformations of the complex coordinates on $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ we have

$$\delta_\chi z^a_1 := \varphi^a_1(\chi; z_1, \bar{z}_1), \quad \delta_\chi z^a_2 := \varphi^a_2(\chi; z_2, \bar{z}_2).$$

(6.27)

To preserve the reality of the conformal structure on $U$, one should define real subalgebras of the algebras $C^1(\mathcal{U}, \mathcal{V}_P)$ and $C^0(\mathcal{U}, \mathcal{T}^{1,0})$ by analogy with §§ 6.6, 7.7 of [7]. We shall not write down transformations of the conformal structure on $U$ since this will require a lot of additional explanations.

Formula (6.26) defines a (twisted) isomorphism between the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ and a subalgebra in $C^0(\mathcal{U}, \mathcal{T}^{1,0})$. The action (6.27) of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ based on this isomorphism is not holomorphic, i.e. it changes the complex structure of $\mathcal{P}$. Having this action one can define an action of $C^1(\mathcal{U}, \mathcal{V}_P)$ on any object on $\mathcal{P}$. In particular, we have the nonholomorphic action

$$\tilde{\delta}_\chi F_{12} = \tilde{\delta}_\chi z^a_1 \frac{\partial F_{12}}{\partial z^a_1} = \varphi_1(F_{12}) = \varphi_2(F_{12}) + \chi_{12}(F_{12})$$

(6.28)

of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ on transition matrices $F_{12}$ of the holomorphic bundle $E' \to \mathcal{P}$. If $\varphi^a_1$ and $\varphi^a_2$ are holomorphic functions on $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively, then $\{\varphi_1, \varphi_2\} \in C^0(\mathcal{U}, \mathcal{V}_P) \subset C^0(\mathcal{U}, \mathcal{T}^{1,0})$ and formula (6.28) defines the holomorphic action of the algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ on transition matrices $F_{12}$ of the bundle $E'$ over the twistor space $\mathcal{P}$. 

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7 Quantization programme

7.1 Symmetry algebras and their representations

In §§ 6.3-6.5 we have described the action of the algebra $C^1(U, V_P) + C^1(U, O^g_P)$ ($\oplus$ denotes the semidirect sum) on the holomorphic transition function of the twistor space $P$ and on holomorphic transition matrices in the bundles $E' \rightarrow P$. There is a subalgebra in the algebra $C^0(U, T^{1,0}) + C^0(U, S^g_P)$ which is isomorphic to the algebra $C^1(U, V_P) + C^1(U, O^g_P)$ and is defined with the help of solutions to the infinitesimal Riemann-Hilbert problems (6.12), (6.26). This subalgebra acts on smooth objects defined on $P$. The action of the symmetry algebra on smooth objects in four dimensions is defined with the help of contour integrals (residue at the point $\lambda = 0$). We summarize the description of the symmetry algebras of CFT’s in two dimensions and their analogues in four and six dimensions in the following table:

<table>
<thead>
<tr>
<th>2D</th>
<th>6D holomorphic setting</th>
<th>6D smooth setting</th>
<th>4D</th>
</tr>
</thead>
<tbody>
<tr>
<td>the Virasoro algebra $\Leftrightarrow C^1(\mathcal{O}, V_{\mathbb{C}P^1})$</td>
<td>the algebra of 1-cochains $C^1(U, V_P)$</td>
<td>a subalgebra in the algebra $C^0(U, T^{1,0})$</td>
<td>the algebra of (nonlocal) transformations $\delta(\lambda) = \text{Res}(\lambda), \lambda \in C^1(U, V_P)$</td>
</tr>
<tr>
<td>the affine Lie algebra $g \otimes \mathbb{C}[\lambda, \lambda^{-1}] \Leftrightarrow C^1(\mathcal{O}, O^g_{\mathbb{C}P^1})$</td>
<td>the algebra of 1-cochains $C^1(U, O^g_P)$</td>
<td>a subalgebra in the algebra $C^0(U, S^g_P)$</td>
<td>the algebra of (nonlocal) transformations $\delta(\lambda) = \text{Res}(\lambda), \lambda \in C^1(U, O^g_P)$</td>
</tr>
</tbody>
</table>

Notice that the subalgebra $C^0(U, V_P)$ of the algebra $C^1(U, V_P)$ also plays an important role. The action of this algebra does not change the complex structure on $P$ and therefore does not change the conformal structure on $U \subset M$. In many particular cases of a choice of a 4-manifold $M$, the subalgebra $H^0(P, V_P)$ of this algebra is finite-dimensional and coincides with the algebra of the conformal group in the “flat limit”. As a first step, fields of any 4D CFT should be classified using representations of this algebra.

Recall that at a quantum level, rational 2D CFT’s are solvable for energy spectrum and correlation functions. Moreover, in genus zero the quantization of these theories in fact amounts to the description of the symmetry algebras of CFT’s in two dimensions and their analogues in four and six dimensions in the following table:

Analogous results can be expected for integrable 4D CFT’s. Indeed, for the basic representative of these models - the SDYM model on an open ball $U \subset \mathbb{R}^4$ - the moduli space is the double coset space (4.13) and the algebra $C^1(U, O^g_P)$ is the complexification of the Lie algebra of the symmetry group $C^1(U, H)$ from (4.13). Hence, as in the case of the chiral 2D WZNW theory on a disk, symmetries completely determine the phase space of the SDYM model on $U$ and quantization of this model amounts to constructing representations of the algebra $C^1(U, O^g_P)$ and to choosing in them the subspaces which are invariant w.r.t. the action of the subalgebra $C^0(U, O^g_P)$ of $C^1(U, O^g_P)$. Extensions of all these algebras arise only as quantum effects of normal ordering.

7.2 Quantization of the SDYM model

In this section we want to discuss quantization of integrable 4D CFT’s. Remembering the connection between 2D CFT’s and ordinary 3D CS theories, one may come to the reasonable conclusion
that the quantization of integrable 4D CFT’s may be much more successful if we use their connection with holomorphic CSW theories.

When quantizing the holomorphic CSW theory on twistor spaces, one may use the results on the quantization of the ordinary CS theory (see e.g. [1]-[3],[23] and references therein) after a proper generalization. We are mainly interested in (canonical) quantization of the SDYM model on $U$ since this model is the closest analogue of the chiral 2D WZNW model on a disk. As such, we have to put $\hat{B}_3=0$ in eqs.(3.1) on the twistor space $P$ of $U$, which leads to the equations (cf.(4.6))

$$\partial B + B \wedge B = 0$$  \hspace{1cm} (7.1)

equivalent to the SDYM equations on $U$, as has been discussed in this paper. The comparison with the ordinary CS theory in the Hamiltonian approach shows that the coordinate $\bar{\lambda}$ conjugate to $\lambda \in \mathbb{C}P^1$ may be considered as (complex) time of the holomorphic CSW theory.

Some problems related to quantization of the SDYM model were discussed in [31, 32, 48]. The quantization was carried out in four dimensions in terms of $L$-valued fields $A_\mu$ or in terms of a $G$-valued scalar field by using the Yang gauge. But the obtained results are fragmentary; the picture is not complete and far from what we have in 2D CFT’s.

Here we shall discuss the quantization of the SDYM model using its connection with the holomorphic CSW theory. In quantization of constrained systems one can use two standard approaches: 1) one first solves the constraints and then performs the quantization of the moduli space; 2) one first quantizes the free theory and then imposes (quantum) constraints. Both approaches will be discussed. We shall write down the list of questions and open problems whose solutions are necessary to give the holomorphic CSW and the SDYM theories a status of quantum field theories.

1. One should rewrite a symplectic structure $\hat{\omega}$ on the space of gauge potentials or their relatives in terms of fields on the twistor space $P$. This 2-form $\hat{\omega}$ induces a symplectic structure $\hat{\omega}$ on the moduli space $M_U$ of solutions to eqs.(7.1), and the cohomology class $[\hat{\omega}] \in H^2(M_U, \mathbb{R})$ has to be integral.

2. Over the moduli space $M_U$, one should define a complex line bundle $L$ with the Chern class $c_1(L) = [\hat{\omega}]$. Then $L$ admits a connection with the curvature 2-form equal to $\hat{\omega}$.

3. A choice of a conformal structure $[g]$ on $U$ induces the complex structure $J$ on the twistor space $P$ and endows the moduli space $M_U$ with a complex structure which we shall denote by the same letter $J$. Then the bundle $L$ over $(M_U, J)$ has a holomorphic structure, and a quantum Hilbert space of the SDYM theory can be introduced as the space $H_J$ of (global) holomorphic sections of $L$.

4. Is it possible to introduce the bundle $L \rightarrow M_U$ as the holomorphic determinant line bundle $\text{Det} \partial_B$ of the operator $\partial_B = \bar{\partial} + B$ on $P$?

5. The action functional of the holomorphic CSW theory on a Calabi-Yau 3-fold has a simple form (3.2) analogous to the action of the standard CS theory. How should one modify this action if we go over to the case of an arbitrary complex 3-manifold?

6. One should lift the action of the symmetry groups and algebras described in this paper up to an action on the space $H_J$ of holomorphic sections of the bundle $L$ over $M_U$. What is an extension (central or not) of these groups and algebras? Finding an extension $\hat{C}^1(U, O_P^g)$ of the algebra $C^1(U, O_P^g)$ is equivalent to finding a curvature of the bundle $L$ since this curvature represents a local anomaly. It is also necessary to find an extension $\hat{C}^1(U, V_P)$ of the algebra $C^1(U, V_P)$.

7. What can be said about representations of the algebras $C^1(U, V_P)$ and $C^1(U, O_P^g)$? Which of these representations are connected with the Hilbert space $H_J$?

8. In the quantum hCSW and SDYM theories there exist Sugawara-type formulae, i.e. generators of the algebra $\hat{C}^1(U, V_P)$ can be expressed in terms of generators of the algebra $\hat{C}^1(U, O_P^g)$. 

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This follows from the fact that any transformation of transition matrices of a holomorphic bundle $E' \to P$ under the holomorphic action of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ can be compensated by an action of the algebra $C^1(\mathcal{U}, \mathcal{O}_P^\mathbb{R})$. What are the explicit formulae connecting the generators of these algebras?

9. One should write down Ward identities resulting from the symmetry algebra $C^1(\mathcal{U}, \mathcal{V}_P) + C^1(\mathcal{U}, \mathcal{O}_P^\mathbb{R})$. To what extent do these identities define correlation functions?

Clearly, to carry out this quantization programme, it will be necessary to overcome a number of technical difficulties.

Remarks

1. It is known that on the moduli space $\mathcal{M}_U$ of solutions to the SDYM equations on $U$ one can introduce the hyperKähler structure [49]. On the other hand, in [7] it has been shown that $\mathcal{M}_U$ is the double coset space (4.13). There is no contradiction between these structures. In fact, Joyce [50] has shown that on many double coset spaces one can introduce 3 complex structures.

2. In the finite-dimensional case, any double coset space has the form $K\backslash G/H$, where groups $G, H$ and $K$ are finite-dimensional. Quantization of a homogeneous space $G/H$ is well known (orbit method [51], geometric quantization [52]) and is reduced to the construction of representations of the group $G$ on polarized sections of a complex line bundle over $G/H$. In the chiral 2D WZNW model this approach is used for quantization of the homogeneous space $\mathcal{L}G/\mathcal{G}$ of the loop group $\mathcal{L}G$. By quantizing the biquotient space $K\backslash G/H$, we first have to associate a representation space of the group $G$ with the homogeneous space $G/H$ (this is standard) and then choose in it subspaces which are invariant under the action of the group $K$. Quantization of the biquotient moduli space $\mathcal{M}_U$ will be an infinite-dimensional variant of this construction.

7.3 The analytic geometry of integrable 4D CFT’s

Let us introduce an almost complex structure on a real 6-manifold $Z$, that is equivalent to assigning the operator $\bar{\partial}$ on $Z$. Integrability conditions of this almost complex structure are equivalent to the equations

$$\bar{\partial}^2 = 0$$

and different complex structures on $Z$ correspond to different operators $\bar{\partial}$.

Having the complex 3-manifold $(Z, \bar{\partial})$, let us consider a complex vector bundle $E'$ over $Z$. Assignment of an almost complex structure on $E'$ is equivalent to the introduction of the $(0,1)$-component $\bar{\partial}_B = \bar{\partial} + B$ of connection on $E'$. Integrability of this almost complex structure is equivalent to the validity of equations

$$\bar{\partial}_B^2 = 0 \iff \bar{\partial}B + B \wedge B = 0.$$ 

In particular, we shall be interested in the case when $Z$ is the twistor space of a self-dual 4-manifold $M$ and $(0,1)$-connections $\bar{\partial}_B = \bar{\partial} + B$ are defined on a holomorphic bundle $E' \to Z$, which is connected with a self-dual bundle $E \to M$. For such connections we have the equations

$$\bar{\partial}_B^2 = 0 \iff \bar{\partial}B + B \wedge B = 0,$$

which are equivalent, as was discussed, to the SDYM equations on $M$.

The general picture arising as a result of quantization of the SDYM model on a self-dual 4-manifold $M$ and of the holomorphic CSW theory on the twistor space $Z$ of $M$ resembles the one that arises in the quantization of 2D CFT’s and ordinary CS theories and is as follows. Let $[g]$ be a self-dual conformal structure on a 4-manifold $M$ and let $\mathcal{J}$ be a complex structure on the twistor space $Z$ of $M$. As has already been noted, there exists a bijection [15, 18] between the moduli
space of self-dual conformal structures on $M$ and the moduli space $\mathcal{X}$ of complex structures on $Z$. Note that energy-momentum tensors of integrable 4D CFT’s on $M$ with a metric $g$ are connected with infinitesimal variations of the metric $g$ on $M$ and, as we have said above, variations of the metric on $M$ are proportional to variations of the complex structure $J$ on the twistor space $Z$ of $M$. Notice that the energy-momentum tensor for the SDYM model can be obtained by using the Donaldson-Nair-Schiff action [27, 31]. It seems that this has not been done in the literature.

A choice of a complex structure $J$ on the twistor space $Z$ endows the moduli space of solutions to the SDYM equations on $M$ with a complex structure which we shall denote by the same letter $J$. We denote this moduli space with the complex structure $J$ by $\mathcal{M}_J$. Then one can introduce a bundle

$$\tilde{\mathcal{M}} \rightarrow \mathcal{X}$$

(7.3)

with fibres $\mathcal{M}_J$ at the points $J \in \mathcal{X}$. The total space $\tilde{\mathcal{M}}$ of this bundle is the moduli space of solutions to eqs.(7.2). In other words, this is the moduli space of pairs $(Z, E')$, where $Z$ is a complex 3-manifold and $E'$ is a holomorphic bundle over $Z$. Study of this moduli space is important for understanding extended mirror symmetries.

**Remark.** If we consider holomorphic bundles $E'$ over Calabi-Yau 3-folds $Z$, the moduli space $\mathcal{X}$ of complex structures on $Z$ should be replaced by a “larger” moduli space $\hat{X}$ of pairs (a complex structure $J$ on $Z$, a Kähler structure $\omega$ on $Z$). The moduli space $\mathcal{M}_{J,\omega}$ of stable holomorphic bundles depends not only on the complex structure $J$ on $Z$, but also on the Kähler structure $\omega$ on $Z$. Then, instead of the bundle (7.3) one should consider a bundle

$$\hat{\mathcal{M}} \rightarrow \hat{X}$$

(7.4)

with fibres $\mathcal{M}_{J,\omega}$ at the points $(J, \omega) \in \hat{X}$. One should use the total space $\hat{\mathcal{M}}$ of the bundle (7.4) in the consideration of the extended mirror symmetry between CY 3-folds with holomorphic bundles.

Further, let $\mathcal{M}_J$ be the moduli space of solutions to the SDYM equations on $M$ and let $H_J$ be the quantum Hilbert space of holomorphic sections of the line bundle $\mathcal{L}$ over $\mathcal{M}_J$. The space $H_J$ depends on $J \in \mathcal{X}$ and one can introduce a holomorphic vector bundle

$$\hat{H} : \hat{\mathcal{M}} \rightarrow \mathcal{X}$$

(7.5)

with fibres $H_J$ at the points $J \in \mathcal{X}$. Then one may raise the question about the existence of a (projectively) flat connection on the bundle (7.5). If such a connection exists, then as the quantum Hilbert space one may take the space of covariantly constant sections of the vector bundle $\hat{H}$. Thus, the Friedan-Shenker approach [53] to 2D CFT’s can be generalized to integrable 4D CFT’s. In this geometric approach the energy-momentum tensor of an integrable 4D CFT will induce a (flat) connection on the bundle (7.5), and equations defining covariantly constant sections of the bundle (7.5) will be equivalent to Ward identities. Moreover, it can be expected that for CFT’s on an open set $U \subset M$, an (n+1)-point amplitude with energy-momentum insertion will be related to an n-point amplitude and its derivative w.r.t. moduli parameters from the space $\mathcal{X}$ because of the Ward identities associated to the algebra $\hat{C}(\mathcal{U}, \mathcal{V}_p)$ (cf. [54]). Analogously, the Ward identities for the algebra $\hat{C}(\mathcal{U}, \mathcal{O}_p)$ will involve derivatives w.r.t. moduli parameters from $\mathcal{M}_J$. The described symmetries can essentially simplify the study of integrable 4D conformal quantum field theories.
8 Conclusion

In this paper we studied the moduli spaces of solutions to the field equations of the ordinary Chern-Simons theory, holomorphic Chern-Simons-Witten theory and the SDYM theory. Integrable 4D CFT’s, their connection with holomorphic CSW theories, parallels with 2D CFT’s and symmetries have been described. We have also discussed the programme of quantization of the SDYM model on a self-dual 4-manifold $M$ based on the equivalence of this model to a subsector of the holomorphic CSW model on the twistor space $Z$ of $M$.

One of the purposes of this paper was the advance of integrable 4D CFT’s as perspective candidates for the role of 4D quantum field theories with vanishing beta functions. It was argued that integrable conformal field theories in four dimensions can be developed to such an extent of generality as free and rational conformal field theories in two dimensions. The main argument is the invariance of integrable 4D CFT’s under the action of the infinite-dimensional algebras, which will put severe restrictions on the Green functions through the Ward identities and will strongly constraint underlying field theories. Much work remains to be done.

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Appendix A. Sheaves of groups

Let us consider a topological space $X$ and recall the definitions of a presheaf and a sheaf of groups over $X$ (see e.g. [55, 56]).

One has a presheaf $\{G(U), r_U^V\}$ of groups over a topological space $X$ if with any nonempty open set $U$ of the space $X$ one associates a group $G(U)$ and with any two open sets $U$ and $V$ with $V \subset U$ one associates a homomorphism $r_U^V : G(U) \to G(V)$ satisfying the following conditions: (i) the homomorphism $r_U^U : G(U) \to G(U)$ is the identity map $\text{id}_U$; (ii) if $W \subset V \subset U$, then $r_W^V = r_W^U \circ r_U^V$.

A sheaf of groups over a topological space $X$ is a topological space $G$ with a local homeomorphism $\pi : G \to X$. This means that any point $s \in G$ has an open neighbourhood $V$ in $G$ such that $\pi(V)$ is open in $X$ and $\pi : V \to \pi(V)$ is a homeomorphism. A set $G_x = \pi^{-1}(x)$ is called a stalk of the sheaf $G$ over $x \in X$, and the map $\pi$ is called the projection. For any point $x \in X$ the stalk $G_x$ is a group, and the group operations are continuous.

A section of a sheaf $G$ over an open set $U$ of the space $X$ is a continuous map $s : U \to G$ such that $\pi \circ s = \text{id}_U$. A set $G(U) := \Gamma(U, G)$ of all sections of the sheaf $G$ of groups over $U$ is a group. Corresponding to any open set $U$ of the space $X$ the group $G(U)$ of sections of the sheaf $G$ over $U$ and to any two open sets $U, V$ with $V \subset U$ the restriction homomorphism $r_U^V : G(U) \to G(V)$, we obtain the presheaf $\{G(U), r_U^V\}$ over $X$. This presheaf is called the canonical presheaf.

On the other hand, one can associate a sheaf with any presheaf $\{G(U), r_U^V\}$. Let

$$G_x = \lim_{x \in U} G(U)$$

be a direct limit of sets $G(U)$. There exists a natural map $r_U^x : G(U) \to G_x, x \in U$, sending elements from $G(U)$ into their equivalence classes in the direct limit. If $s \in G(U)$, then $s_x := r_U^x(s)$
is called a germ of the section \( s \) at the point \( x \), and \( s \) is called a representative of the germ \( s_x \). Put another way, two sections \( s, s' \in \mathcal{U}(U) \) are called equivalent at the point \( x \in U \) if there exists an open neighbourhood \( V \subset U \) such that \( s|_V = s'|_V \); the equivalence class of such sections is called the germ \( s_x \) of section \( s \) at the point \( x \). Put

\[
\mathcal{S} = \bigcup_{x \in X} \mathcal{S}_x
\]

and let \( \pi : \mathcal{S} \to X \) be a projection mapping points from \( \mathcal{S}_x \) into \( x \). The set \( \mathcal{S} \) is equipped with a topology, the basis of open sets of which consists of sets \( \{ s_x, x \in U \} \) for all possible \( s \in \mathcal{U}(U), U \subset X \). In this topology \( \pi \) is a local homeomorphism, and we obtain the sheaf \( \mathcal{S} \).

Let \( X \) be a smooth manifold. Consider a complex (non-Abelian) Lie group \( \mathbf{G} = G^C \) and define a presheaf \( \{ \hat{\mathcal{S}}(U), r^U_V \} \) of groups by putting

\[
\hat{\mathcal{S}}(U) := \{ C^\infty \text{-maps } f : U \to \mathbf{G} \},
\]

and using the canonical restriction homomorphisms \( r^U_V \) when for \( f \in \hat{\mathcal{S}}(U) \) its image \( r^U_V(f) \) equals \( f|_V \in \hat{\mathcal{S}}(V), V \subset U \). To each elements \( \alpha_x \) and \( \beta_x \) from \( \hat{\mathcal{S}}_x := r^U_x(\hat{\mathcal{S}}(U)) \) one can put into correspondence their pointwise multiplication \( \alpha_x \beta_x \). To this presheaf \( \{ \hat{\mathcal{S}}(U), r^U_V \} \) there corresponds the sheaf \( \hat{\mathcal{S}} \) of germs of smooth maps of the space \( X \) into the group \( \mathbf{G} \).

Suppose now that \( X \) is a complex manifold. Then one can define a presheaf \( \{ \hat{\mathcal{H}}(U), r^U_V \} \) of groups assuming that

\[
\hat{\mathcal{H}}(U) \equiv \mathcal{O}^\mathbf{G}(U) := \{ \text{holomorphic maps } h : U \to \mathbf{G} \},
\]

and associate with it the sheaf \( \mathcal{H} \equiv \mathcal{O}^\mathbf{G} \) of germs of holomorphic maps of the space \( X \) into the complex Lie group \( \mathbf{G} \).

Appendix B. Cohomology sets and vector bundles

We consider a complex manifold \( X \) and a sheaf \( \mathcal{S} \) coinciding with either the sheaf \( \hat{\mathcal{S}} \) or the sheaf \( \hat{\mathcal{H}} \) introduced in Appendix A. One can also consider a sheaf \( \mathcal{S} \) of Abelian groups with addition as a group operation.

Čech cohomology sets \( H^0(X, \mathcal{S}) \) and \( H^1(X, \mathcal{S}) \) of the space \( X \) with values in the sheaf \( \mathcal{S} \) of groups are defined as follows [55, 56].

Let there be given an open cover \( \mathcal{U} = \{ U_\alpha \}, \alpha \in I, \) of the manifold \( X \). The family \( \{ U_0, ..., U_q \} \) of elements of the cover such that \( U_0 \cap ... \cap U_q \neq \emptyset \) is called a q-simplex. The support of this simplex is \( U_0 \cap ... \cap U_q \). Define a 0-cochain with coefficients in \( \mathcal{S} \) as a map \( f : \langle \alpha \rangle \mapsto f_\alpha \), where \( \alpha \in I \) and \( f_\alpha \) is a section of the sheaf \( \mathcal{S} \) over \( U_\alpha \),

\[
f_\alpha \in \mathcal{S}(U_\alpha) := \Gamma(U_\alpha, \mathcal{S}). \tag{B.1}
\]

A set of 0-cochains is denoted by \( C^0(\mathcal{U}, \mathcal{S}) \) and is a group under the pointwise multiplication (or addition in the Abelian case).

Consider now the ordered set of two indices \( \langle \alpha, \beta \rangle \) such that \( \alpha, \beta \in I \) and \( U_\alpha \cap U_\beta \neq \emptyset \). Define a 1-cochain with the coefficients in \( \mathcal{S} \) as a map \( f : \langle \alpha, \beta \rangle \mapsto f_{\alpha\beta} \), where \( f_{\alpha\beta} \) is a section of the sheaf \( \mathcal{S} \) over \( U_\alpha \cap U_\beta \),

\[
f_{\alpha\beta} \in \mathcal{S}(U_\alpha \cap U_\beta) := \Gamma(U_\alpha \cap U_\beta, \mathcal{S}). \tag{B.2}
\]

A set of 1-cochains is denoted by \( C^1(\mathcal{U}, \mathcal{S}) \) and is a group under the pointwise multiplication (or addition in the Abelian case).
Subsets of cocycles $Z^q(\mathcal{U}, \mathcal{G}) \subset C^q(\mathcal{U}, \mathcal{G})$ for $q = 0, 1$ are defined by the formulae

$$Z^0(\mathcal{U}, \mathcal{G}) = \{ f \in C^0(\mathcal{U}, \mathcal{G}) : f_\alpha f_\beta^{-1} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset \},$$

$$Z^1(\mathcal{U}, \mathcal{G}) = \{ f \in C^1(\mathcal{U}, \mathcal{G}) : f_{\beta\alpha} = f_{\alpha\beta}^{-1} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset; \ f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset \}. \quad (B.4)$$

Notice that if one considers a sheaf $\mathcal{G}$ of Abelian groups, the cocycle conditions (B.4) should be replaced by the conditions $f_{\alpha\beta} + f_{\beta\alpha} = 0$, $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$ and analogously in all other definitions. It follows from (B.3) that $Z^0(\mathcal{U}, \mathcal{G})$ coincides with the group $H^0(\mathcal{X}, \mathcal{G}) := \mathcal{G}(\mathcal{X}) \equiv \Gamma(\mathcal{X}, \mathcal{G})$ of global sections of the sheaf $\mathcal{G}$. The set $Z^1(\mathcal{U}, \mathcal{G})$ is not in general a subgroup of the group $C^1(\mathcal{U}, \mathcal{G})$. It contains the marked element $1$, represented by the 1-cocycle $f_{\alpha\beta} = 1$ for any $\alpha, \beta$ such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$.

For $h \in C^0(\mathcal{U}, \mathcal{G})$, $f \in Z^1(\mathcal{U}, \mathcal{G})$ let us define an action $\rho_0$ of the group $C^0(\mathcal{U}, \mathcal{G})$ on the set $Z^1(\mathcal{U}, \mathcal{G})$ by the formula

$$\rho_0(h, f)_{\alpha\beta} = h_\alpha f_{\alpha\beta} h_\beta^{-1}. \quad (B.5)$$

So we have a map $\rho_0 : C^0 \times Z^1 \ni (h, f) \mapsto \rho_0(h, f) \in Z^1$. A set of orbits of the group $C^0$ in $Z^1$ is called a $1$-cohomology set and denoted by $H^1(\mathcal{U}, \mathcal{G})$. In other words, two cocycles $f, \tilde{f} \in Z^1$ are called equivalent, $f \sim \tilde{f}$, if

$$\tilde{f} = \rho_0(h, f) \quad (B.6)$$

for some $h \in C^0$, and the 1-cohomology set $H^1 = \rho_0(C^0) \setminus Z^1$ one calls a set of equivalence classes of 1-cocycles. Finally, we should take the direct limit of these sets $H^1(\mathcal{U}, \mathcal{G})$ over successive refinement of the cover $\mathcal{U}$ of $\mathcal{X}$ to obtain $H^1(\mathcal{X}, \mathcal{G})$, the 1-cohomology set of $\mathcal{X}$ with the coefficients in $\mathcal{G}$. In fact, one can always choose a cover $\mathcal{U} = \{ \mathcal{U}_\alpha \}$ such that it will be $H^1(\mathcal{U}, \mathcal{G}) = H^1(\mathcal{X}, \mathcal{G})$ and therefore it will not be necessary to take the direct limit of sets. This is realized, for instance, when the coordinate charts $\mathcal{U}_\alpha$ are Stein manifolds (see e.g. [55]).

Now consider the case when $\mathcal{G}$ is the sheaf of germs of (smooth or holomorphic) functions with values in the complex Lie group $\mathbf{G}$. Suppose we are given a representation of $\mathbf{G}$ in $\mathbb{C}^n$. It is well known that any 1-cocycle $\{ f_{\alpha\beta} \}$ from $Z^1(\mathcal{U}, \mathcal{G})$ defines a unique complex vector bundle $E'$ over $\mathcal{X}$, obtained from the direct products $\mathcal{U}_\alpha \times \mathbb{C}^n$ by glueing with the help of $f_{\alpha\beta} \in \mathbf{G}$. Moreover, two 1-cocycles define isomorphic complex vector bundles over $\mathcal{X}$ if and only if the same element from $H^1(\mathcal{X}, \mathcal{G})$ corresponds to them. Thus, we have a one-to-one correspondence between the set $H^1(\mathcal{X}, \mathcal{G})$ and the set of equivalence classes of complex vector bundles of the rank $n$ over $\mathcal{X}$. Smooth bundles are parametrized by the set $H^1(\mathcal{X}, \mathcal{S})$ and holomorphic bundles are parametrized by the set $H^1(\mathcal{X}, \mathcal{H})$, where the sheaves $\mathcal{S}$ and $\mathcal{H}$ were described in Appendix A. For more details see e.g. [55, 56].

29
References


