Self Inhibiting Heat-Flux
I. Whistlers Quasilinear Theory

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ABSTRACT
Heat-transfer through weakly magnetized diffuse astrophysical plasmas excites whistlers. This leads to electron whistler resonant scattering, a reduction of the electron mean-free path, and heat-flux inhibition. However, only whistlers propagating at a finite angle to the magnetic field (off-axis) can scatter the heat-flux carrying electrons. Thus, the level of heat-flux-inhibition along the magnetic field lines depends on the presence of off-axis whistlers.

We obtain a solution of the Boltzmann equation with the whistler wave equation and show that if \( \epsilon^{th} \beta_e \gg 10^{-4} \), where \( \epsilon^{th} \) is the thermal Knudsen number, and \( \beta_e \) is the ratio of the electron pressure to the magnetic energy density, scattering of heat-flux carrying electrons by off-axis whistlers, which are shown to propagate at about 65°, is efficient enough to lead to heat-flux inhibition along field lines. The inhibition so obtained is proportional to \( (\epsilon^{th} \beta_e)^{-1} \).

Key words: Heat-conduction, Whistlers

1 INTRODUCTION

The presence of a cold gas embedded in a hot gas is a common phenomenon. This situation occurs in many astrophysical systems such as the interstellar medium (ISM), the intracluster gas in galaxy clusters (ICM), AGNs and Lyman \( \alpha \) systems. These multi temperature-density configurations were and still are the subject of intensive theoretical study. Field (1965) carried out a linear stability analysis of optically thin, collision dominated magnetized plasma. He found the critical wavelength (the Field length) that distinguishes perturbation wavelengths that form cold clumps from those that don’t. Perturbation wavelengths shorter than the Field length are heated by heat-conduction and fail to form cold clumps whereas the longer perturbation wavelengths are cooled enough by radiation losses to condense into cold clumps. The Field length retains its qualitative role in the non-linear regime (Cowie & Songaila 1977; Cowie & McKee 1977; McKee & Cowie 1977).

At large Knudsen numbers there is heat-flux saturation. This is considered by Cowie & McKee (1977), Balbus & McKee (1982), Slavin & Cox (1992), and Dalton & Balbus (1993). Chun & Rosner (1993) consider the effects of a non-local Maxwellian electron distribution on Field’s analysis, and Bandiera & Chan (1994a,b) generalize their work and emphasize the role of the non-local thermoelectric effect in thermal evaporation processes. Apart from the details considered in these works, the Field length retains its qualitative and quantitative role. All the collision dominated plasma theories predict that scales smaller than the Field length evaporate (heated). For typical ISM or ICM parameters, the thermal evaporation theory predicts a hotter ISM and ICM than what is actually observed. This is also the case for Lyman-\( \alpha \) systems and the standard AGN model.

For the ISM, McKee & Ostriker (1977) proposed that the observed cold gas is constantly replenished by supernovae explosions. However, this does not necessarily apply to other astrophysical environments in which cold and hot gas are observed to co-exist. In broad line emission regions surrounding quasistellar objects (Begeleman & McKee 1990; McKee & Begeleman 1990) cold and hot gas coexist, but the presence of a small scale replenishment source is not clear. The ICM at the cores of some clusters is composed of interacting cold and hot gas (Canizares, Market & Donahue 1988; Fukazawa et al. 1994). Detailed studies show that thermal evaporation as predicted by classical theory (Spitzer 1962)
should be very efficient in the ICM (Cowie & Binney 1977; Binney & Cowie 1981; Fabian & Nulsen 1977; Loewenstein & Fabian 1990; Fabian 1994). These works interpret the ICM observations as a reduction of the magnitude of the thermal conductivity below that predicted by classical theory. The evidence for heat flux inhibition at the ISM, ICM, AGNs, and Lyman-α systems is indirect. Direct confirmation of heat flux inhibition exists in the solar corona and wind for three decades (Montgomery, Bame & Hundhausen 1968). Recent measurements performed with Ulysses (Scime et al. 1994) confirm that between 1.2 and 5.4 AU, heat flux is inhibited compared to its Spitzer value.

This work concentrates on the ICM observations, with a strong emphasis on the "cooling flow model" (CFM; cf. Fabian et al. 1984, 1991 Fabian 1994 for a review). The CFM reproduces the observed properties of the ICM in the central part of galaxy clusters very well, but only with the assumption of strong heat flux inhibition, i.e. heat flux is ignored. The justification of this hypothesis has been heuristic, and has presumed that heat flux inhibition is obtained by a tangled magnetic field ("in a manner which is not yet clear"). This paper considers heat flux inhibition in the frame work of a rigorous discussion, and attempts to quantify the level of heat flux inhibition. In doing so we partially rely on works that consider heat flux inhibition in the solar wind (Gary & Feldman 1977; Gary et al. 1994 and Scime et al. 1994), due to whistlers.

The ICM observations require heat conductivity suppression (heat flux inhibition) compared to the Spitzer value by at least two orders of magnitude, and sometimes up to four orders of magnitude (Pistinner & Shaviv 1996; Pistinner, Levinson & Eichler 1996).

Reducing the electron mean free path would lead to heat flux inhibition. Mechanisms that would accomplish this can be divided into two main theoretical classes:

(i) Increase of the path length of the magnetic field lines, hence the length over which parallel heat transport must operate.

(ii) Plasma waves that resonantly scatter the heat conducting electrons.

The first mechanism is realized when a tangled magnetic field pervades the plasma (for details cf. Cowie & Binney 1981, Rosner & Tucker 1989; Tao 1995; Pistinner & Shaviv 1996). The second comes about when self excited electron plasma "waves exist" (Gary & Feldman 1977; Jafelice 1992; Levinson & Eichler 1992; Pistinner et al. 1996). These plasma waves are expected to scatter the heat carrying electrons, thereby reducing their mean free path.

The classification made above is quantitative (Goldstein, Klimas & Sandri 1975), and it follows from the different physical processes involved. More specifically, let \( l_B \) and \( r_L \) be the coherence length of the magnetic field and the electron gyroradius respectively, then:

(i) Mechanism (i) (the tangling mechanism) is realized when the condition \( r_L \ll l_B \) holds, and does not scatter electrons (except for adiabatic mirroring).

(ii) Mechanism (ii), in which electromagnetic plasma waves scatter electrons “non-adiabatically” works if \( r_L \approx l_B \).

Large scale tangling, (i), is now seriously constrained and in any case “questionable” (Tao 1995; Pistinner & Shaviv 1996). It seems that only a very particular family of magnetic field correlation functions can inhibit heat conduction. These correlation functions have Lorentzian rather than Gaussian properties, and the physical basis for their origin is not yet clear. In contrast electromagnetic electron plasma waves (ii), obtained from a collisionless plasma model (Levinson & Eichler 1992), provide large inhibition factors when applied to the CFM (Pistinner, et al. 1996). The model of Levinson & Eichler (1992) follows Gary & Feldman (1977) and invokes whistler-electron interaction to suppress the heat transfer. In light of our findings (Pistinner et al. 1996) and the observational information and interpretation accumulated during the past three decades, heat conduction theory in diffuse astrophysical plasmas should be reexamined. Scattering by plasma turbulence should be considered along with Coulomb collisions.

Plasma turbulence results from collective phenomena, and may lead to electron-waves resonant pitch-angle scattering. Non-magnetized plasmas are studied in detail, and under some conditions ion acoustic turbulence leads to heat flux inhibition (Galeev & Natanzon 1984). Jaffliche (1992) considers this possibility in the context of the CFM. Although he ignores strict requirements for ion-acoustic waves excitation, he finds insufficient heat inhibition. Levinson & Eichler (1992) follow Gary & Feldman (1977) and consider heat flux inhibition by whistlers. Gary & Feldman (1977) find that the inhibition obtained by whistler-electron scattering in the (significantly magnetized) solar wind is not large. Levinson & Eichler (1992) show that collision dominated plasmas are subject to Weibel type instabilities that generate the whistler electromagnetic mode effectively, and find strong heat flux inhibition in weakly magnetized plasmas. The differences between the conclusions of Gary & Feldman and Levinson & Eichler (1992) result from different magnitudes of the assumed magnetic contribution to total pressures: Whereas Gary & Feldman assume comparable magnetic and gas pressures as is appropriate for the solar wind, Levinson & Eichler (1992) consider a weakly magnetized plasma.

Levinson & Eichler (1992) conclude that if the magnetic pressure is small compared to the gas pressure, heat flux inhibition along field lines is effective. In any case, both treatments neglect geometric details of the plasma processes involved. In particular, the assumption of a BGK scattering operator used by Levinson & Eichler 1992 (followed by Pistinner et al. 1996) to calculate the particle distribution function, which neglects the pitch angle dependence of the scattering, was not explicitly justified. The purpose of this paper is to quantitatively assess the validity of the BGK model in view. We show that all pitch angles scatter at roughly comparable rates, so that the neglect of the pitch angles dependence (made by Gary & Feldman 1977; Levinson & Eichler 1992, Pistinner et al. 1996) is quantitatively valid.

The structure of this paper is: In §3 we consider the governing equations for collision dominated and whistler dominated plasma. In §4 we obtain the formal expression for the electron distribution function first in the absence of whistlers and then in the presence of quasilinear situated whistlers. In §5 we obtain the critical condition for whistlers excitation by a collision dominated plasma. We then discuss the consistency of the small scale plasma turbulence theory, and show
that off-axis whistlers are required for significant heat flux inhibition and that they are excited by the required amount. The expression for the heat flux vector is considered in § 3. We summarize our conclusions in § 4.

2 GOVERNING EQUATIONS

This section outlines the governing equations that lead to the heat flux. The discussion starts with collision dominated plasma theory. It concentrates on the assumptions made in deriving the inhibited heat flux, in particular, the assumption of steady state, which raises the question of kinetic stability of the plasma.

Kinetic plasma instabilities can lead to a plasma pervaded by stochastic electromagnetic fields. These fields may scatter particles faster than binary collisions. To account for a kinetic instability saturation, a set of governing equations that describe this phenomena is required.

An electron-ion plasma, irrespective of whether the plasma is collision or collective phenomena dominated, can be described by two Boltzmann or Fokker-Planck equations (BFPE). For sake of brevity we shall assume that the solution to the ion BFPE is given, and consider only the solution to the electron BFPE. The electron BFPE (e.g. Blandford & Eichler 1987) reads:

\[ \mu \frac{\partial f}{\partial t} + \frac{eE_{\parallel}}{m_e} \left( \frac{\partial f}{\partial v_n} + \frac{1}{2} \frac{\partial^2 f}{\partial v_n^2} \right) - \frac{e(1-\mu^2)}{2m_e} \frac{\partial}{\partial v_n} \frac{\partial f}{\partial v_n} = 0 \]  

(1)

where \( f, \mu, v, B, E_{\parallel}^D \), \( \nu, l_\parallel \) are the electrons distribution function, the cosine of the pitch angle, the electron speed, the magnetic field magnitude, the self consistent DC electric field and the electron scattering rate (collision frequency) and the affine length along the magnetic field lines respectively.

Several assumptions have been made in writing eq. (1):

(i) steady state
(ii) Lorentzian plasma, i.e. energy exchange between particles is ignored
(iii) curvature drifts are ignored
(iv) phase velocity of the whister waves is ignored.

These assumptions are justified in the context of clusters of galaxies, where the large scale hydrodynamical timescale and the collisional timescale are both long compared to the plasma timescales, the scale length of the magnetic field is enormous compared to particle gyroradii, and the whistler phase velocity is small compared to the electron thermal velocity. For completeness we elaborate briefly on each of these assumptions in § 3.

Classical astrophysical plasmas have traditionally been assumed to be thermal i.e. collision dominated. Thus, the Spitzer & Härm (1953) results are traditionally applied to model them. With eq. (1) one readily reproduces the Spitzer & Härm (1953) results (within a factor of five; cf. Table III and eq. 36 in Spitzer & Härm as energy exchange is ignored in eq. 1) along a single magnetic field line. Toward that goal one only need assume that Coulomb forces (collision dominated plasma) yield:

\[ \nu = \nu_{\text{coll}}, \]  

(2)

where

\[ \bar{\nu} \equiv \frac{\nu}{\nu_{\text{th}}} = \bar{\nu}_{\text{coll}} = \bar{\nu}^{-3}, \]  

(3)

and

\[ \nu_{\text{th}} = \nu_{\text{coll}}(\nu_{\text{th}}) = \frac{m_e}{2k_B T} \ln(\Lambda_{\text{coul}}) \]  

(4)

with \( T, k_B, m_e, \omega_p, N_D \) and \( \ln(\Lambda_{\text{coul}}) \) the gas temperature, the Boltzmann constant, the electron mass, the plasma frequency, the Debye number, and the Coulomb logarithm respectively.

To obtain significant electron heat-flux inhibition by whistlers, whistler-electron scattering must be as fast as ion-electron scattering. Thus, the time scale for establishing steady state on microscopic scales is at least as fast as that when whistlers are absent, and we may assume that a modified steady state is formed. Under such a situation eq. (1) describes the evolution of the electron distribution function but, with the following important modification,

\[ \bar{\nu} \equiv \bar{\nu}^{-3} + \nu_w, \]  

(5)

where \( \nu_w = \nu_w/\nu_{\text{th}}, \) and \( \nu_w \) is the electron whistler scattering rate. A few more implicit assumptions are made in writing eq. (1). The equation is written in a frame in which the whistlers’ electric field vanishes. We assume that the magnitude of the DC field is not big enough to lead to inelastic scattering events. The frame in which the scattering is elastic moves with the whistler phase velocity, which has been neglected in comparison to the velocity of the electrons exciting the whistlers.

To obtain \( \nu_w \) one has to choose an approximated non-linear plasma turbulence theory. In this paper we use quasi-linear theory. There are several formulation of quasilinear theory, and we chose to use the semi-classical formulation given by Melrose (1980). From his work, after some algebra, we obtain

\[ \nu_w = \Theta(\mu)\nu_{w,s} + \left[ 1 - \Theta(\mu) \right] \left( \nu_{w,s} + \nu_{w,a} \right) = \nu_{w,s} + \left[ 1 - \Theta(\mu) \right] \nu_{w,a}. \]  

(6)

where

\[ \Theta(\mu) = \begin{cases} 0 & \mu < 0 \\ 1 & \mu > 0 \end{cases}. \]  

(7)

\( \nu_{w,a} \) replaces eq. 3 are given by Melrose (1980).

\section{Conclusion}

The details of the procedure that lead to the modifications to eq. 1 in which eq. 3 replaces eq. 2 are given by Melrose (1980).
\[ \nu_{w,s} = \frac{1}{m_e c^2} \int_{-1}^{1} W(k, \chi) \left( 1 - \chi^2 \right) d\chi \]
\[ \nu_{w,a} = \frac{2}{m_e c^2} \int_{-1}^{1} W(k, \chi) \left( 1 - \chi^2 \right) d\chi \]

where \( W(k, \chi) \), \( \Omega_e \), \( k \), \( e \), \( c \), \( \chi \), \( p \), are the whistler spectrum, the electron gyro-frequency, the whistler wave number, the electron charge, the speed of light, the cosine of the angle between the whistler propagation direction to the field line direction, and the electron momentum respectively. Although the above expressions can be written more compactly, (Melrose 1980), we write them in this way to reflect that \( \nu_{w,a} \) has contribution from asymmetric wave emission while \( \nu_{w,s} \) has contribution from symmetric wave emission.

Equation 8 demonstrates that \( \nu \) is a function of \( W(k, \chi) \). Thus, one requires an equation that accounts for the evolution of the energy density in the waves (or the waves spectrum). Under the assumption of steady state the equation governing the evolution of the whistlers spectrum is:

\[ \chi v_\theta \frac{\partial W}{\partial \mu} = 2W(\gamma^w - \Gamma^w), \]

where \( \gamma^w \) is the quasilinear growth rate (which is essentially the linear growth rate expanded under the assumption \( \zeta \equiv \gamma^w/\nu^w \ll 1 \)), \( \Gamma^w \) is the non-linear damping rate \( \gamma^w \) and \( v_\theta \) is the whistlers group velocity. In the framework of quasilinear theory the explicit expression for \( \gamma^w \) (Melrose 1980) reads:

\[ \gamma^w = \frac{m_e \Omega_e}{k |\mu|} \left( \frac{1}{2} - \frac{1}{2} \frac{1}{m_e c^2} \left( 1 - \frac{2}{3} \frac{1}{m_e c^2} \right) \right) \]

where \( p_R = \frac{m_e \Omega_e}{k |\mu|} \).

We have a set of coupled functional equations that govern the behavior of the plasma. These equations are eq. 9 for \( f \), (with eq. 8 and eq. 8) and eq. 9 for \( W \) (with eq. 10). We emphasize that eq. 9 and eq. 8 will retain their form in any plasma turbulence theory, and that the turbulence theory determines eq. 9, eq. 8, and eq. 10. With this set of coupled functional equations the distribution function can be determined and the heat flux can be calculated self-consistently.

3 SOLUTION OF THE ELECTRONS BFPE

We consider an approximate solution to eq. 9 under the following assumptions:

(i) We assume that a Knudsen expansion is valid, even in the absence of whistler waves.

(ii) We assume that the net current vanishes.

The first assumption implies that the distribution function can be expanded into a series with \( f = f^0 + f^1 \) and that \( f^0 \gg f^1 \gg f^2 \) such that only the first order correction can be retained.

3.1 The Knudsen Expansion

The solution obtained by a Knudsen expansion can be validated a posteriori, and this solution procedure is not sensitive to whistlers' absence or presence in the plasma. A little algebra shows that the solution to zeroth order is given by an isotropic distribution function which we take to be a Maxwell-Boltzmann distribution. With this assumption the first order correction (in dimensionless units) reads:

\[ \frac{\partial f}{\partial \mu} = - \frac{\tilde{v}}{v} \lambda^{th} \frac{\partial f^{MB}}{\partial \mu} = \frac{eE_{DC}}{m_e v_{th} \nu_{th} \tilde{v}} \frac{\partial f^{MB}}{\partial \tilde{v}}, \]

where

\[ f^{MB} = m_e \pi \frac{1}{2} (m_e v_{th})^3 e^{-2 \tilde{v}^2}, \]

\[ e^{coll} = \lambda^{th} \frac{\partial ln(D_C)}{\partial \mu} \tilde{v}^{-4} \equiv \lambda^{th} \tilde{v}^{-4}, \]

are the Maxwell-Boltzmann distribution, the Knudsen number, and \( \lambda^{th} = v_{th}/\nu_{th} \) is a thermal electron collision mean free path. Note that only \( f^{MB} \) is still dimensional but all other quantities are dimensionless. To eliminate \( E_{DC}^{DC} \) from eq. 12 we need to relate it to the spatial variability of \( f^{MB} \). Toward that goal one uses the current free condition.

3.2 The Current Free Distribution Function

In astrophysical conditions, thermoelectric fields would cause the plasma to settle down to a zero current state. The current free condition reads:

\[ \frac{\partial f}{\partial \mu} = \frac{1}{2 \pi} \int d\tilde{v}^{\mu} \int d\mu f = 0. \]

Using eq. 13 in eq. 14 and substituting the result into eq. 12 we find after some algebra that the current free distribution function is:

\[ \frac{\partial f}{\partial \mu} = - \frac{\tilde{v}}{v} \lambda^{th} \left( \tilde{v}^2 - \frac{Y^2}{Y^1} \right) f^{MB}, \]

where

\[ Y^2 = \int_{-\infty}^{\infty} d\tilde{v} \tilde{v}^3 e^{-\tilde{v}^2} Z(\tilde{v}) \]

\[ Y^1 = \int_{-\infty}^{\infty} d\tilde{v} \tilde{v}^2 e^{-\tilde{v}^2} Z(\tilde{v}) \]

\[ Z(\tilde{v}) = \frac{1}{\tilde{v}^{1+2/3}}, \]

and the ratio of \( Y^2/Y^1 \) should be evaluated once the value of \( \tilde{v} \) has been determined. For a collision dominated plasma the value of \( \tilde{v} \) is given by eq. 10. Thus, the value of \( Y^2/Y^1 \) can be evaluated. However, if the plasma is whistler dominated, \( f \) is a functional of \( \tilde{v} \), and not a simple function of it.
Substituting $\tilde{\nu} = \tilde{v}^{-3}$ into eq. 16, one finds $Y^2/Y^1 = 4$ and using this in eq. 15, one obtains

$$f = f^{MB} \left[ 1 + \mu e^{\text{coll}} \left( 4 - \tilde{v}^2 \right) \right],$$

$$f = f^{MB} \left[ 1 + \mu e^{\text{th}} \tilde{v} \left( 4 - \tilde{v}^2 \right) \right].$$

This solution is valid provided that $e^{th} = e^{\text{coll}}(\epsilon^{th}) \leq 2 \times 10^{-2}$ (due to the velocity dependence of the collisions mean free path; Gray & Killkenny 1980). Note that if $e^{th} = 2 \times 10^{-2}$ then for $\tilde{v} = 2 \Rightarrow e^{\text{coll}} = 0.32 \ll 1$, for $\tilde{v} = 4$ the underlying assumptions of the expansion do not hold.

4 SOLUTION OF THE QUASILINEAR EQUATIONS

The aim of this section is to a) calculate when eq. 17 gives rise to whistlers, b) to find the angular dependence of the whistler spectrum, and c) with this whistler spectrum to estimate the modified steady state distribution function (to which the collision dominated plasma distribution function evolves), from which follows the heat flux inhibition factor. Consistency checks of the a priori assumptions made are carried out as we progress. To obtain these goals we consider approximate solutions to the equations given in § 3 under the assumption that whistlers pervade the plasma. However, before we do so we need to establish when this situation would occur and what the qualitative nature of it is.

4.1 The Whistler Quasilinear Growth Rate

The formal expression for the distribution function has been found via the Knudsen expansion and is given by eq. 13. To calculate when whistlers are present in the plasma we need to use eq. 15 in eq. 10. It proves mathematically convenient to formulate all expressions in terms of $k|| = k\chi$ and $\chi$ as the independent variables rather than $k$ and $\chi$.

After some algebra we obtain the following schematic form for the quasilinear growth rate:

$$\gamma^w = \sqrt{\pi} \Omega_0 \beta_e^{-1} k|| P(\tilde{k}||, \chi)[e^{th} \beta_e Q(\tilde{k}||, \chi) P^{-1}(\tilde{k}||, \chi) - 1]$$

where

$$P(\tilde{k}||, \chi) \equiv \frac{2(1 + \chi^2)}{\chi^2} \int_0^1 d\mu (1 - \mu^2) \tilde{v}^2 \exp \left( \frac{1}{\mu^2 \chi^2} \right) = \frac{8}{\sqrt{3} \pi} \tilde{v}^2 \exp \left( \frac{1}{\mu^2 \chi^2} \right)$$

$$Q(\tilde{k}||, \chi) \equiv \int_0^1 \frac{d\mu (1 - \mu^2) \tilde{v}^4 e^{-\tilde{v}^4 (\tilde{v}^2 - \chi^2)^2}}{1 + \mu^2 e^{\text{coll}} (1 + \chi^2) e^{\text{th} \tilde{v}^4 (\tilde{v}^2 - \chi^2)^2}} \tilde{v}^4 \exp \left( \frac{1}{\mu^2 \chi^2} \right)$$

$$= \frac{1}{1 + 2 \sqrt{1 + k| |^2 \chi^2 / \omega^2}} \left[ 1 + k| |^2 \chi^2 / \omega^2 \right] + \frac{1}{1 + 2 \sqrt{1 + k| |^2 \chi^2 / \omega^2}} \left[ 1 + k| |^2 \chi^2 / \omega^2 \right]$$

$$v = \frac{1}{k| |^2 \chi^2 / \omega^2} \beta_e \equiv \frac{4 \mu \rho v}{k| |^2 \chi^2 / \omega^2} = \tilde{v} = \frac{1}{k| |^2 \chi^2 / \omega^2} \beta_e \equiv \frac{4 \mu \rho v}{k| |^2 \chi^2 / \omega^2}.$$

Choosing $\beta_e \ll 1$, we have $\tilde{v} = 1 / \beta_e \ll 1$, and $P_e$ is the electrons pressure, and $P$ the total gas pressure. We have chosen the magnetic field pointing from hot to cold. For this choice whistler propagation is parallel to field lines, in the coldward direction. Thus, for wave growth we must have $\chi > 0$ (or $\tilde{k}|| > 0$). This point is taken into account in the expression written above. The terms $Q(\tilde{k}||, \chi)$ and $P(\tilde{k}||, \chi)$ in eq. 13 represent a growth term due to the $\frac{\partial f}{\partial \tilde{v}}$ term, and the damping term due to the $\frac{\partial f}{\partial \chi}$ term in eq. 10, respectively.

4.2 Stability of the Collision Dominated Solution

Here we show that under a wide range of conditions, a distribution function, were it governed only by collisions, would be unstable to whistler waves which, of course, modify the distribution function.

Using eq. 18 we first establish when a collision dominated plasma becomes unstable. Using $Y^2/Y^1 = 4$ and $v\tilde{v} = 0$ in eq. 14 we have:

$$\gamma^w = \sqrt{\pi} \Omega_0 \beta_e^{-1} k|| \left[ \int_0^1 \frac{d\mu (1 - \mu^2) \tilde{v}^4 e^{-\tilde{v}^4 \chi^2}}{\tilde{v}^2 (\tilde{v}^2 - 4) \exp \left( \frac{1}{\mu^2 \chi^2} \right)} \right].$$

where the subscript $\text{coll}$ denotes whistler growth in a distribution function stabilized by binary collisions only, not by the whistlers themselves. Some rough estimates may be obtained from eq. 24. The first is that, provided $e^{th} \beta_e > 1$, we have $\gamma^w > 0$, in a narrow cone along the axis. The width angle of this cone is roughly estimated to be $\beta_e^{-1}(e^{th})^{-1} < 1$. To obtain the quantitative conditions we note that the integral in eq. 24 can be written in terms of tabulated functions,

$$\gamma^w = \sqrt{\pi} \Omega_0 \beta_e^{-1} k|| \left[ \int_0^1 \frac{d\mu (1 - \mu^2) \tilde{v}^4 e^{-\tilde{v}^4 \chi^2}}{\tilde{v}^2 (\tilde{v}^2 - 4) \exp \left( \frac{1}{\mu^2 \chi^2} \right)} \right]$$

$$= \frac{1}{\sqrt{4 \pi \omega}} \int_0^\tilde{v} dt e^{-t^2}.$$

Marginal stability of the plasma is found numerically in $\beta_e e^{th}, k||, \chi$ space at

$$e^{th} \beta_e = 0.0001 \pm 10^{-5} \Rightarrow \gamma^w(k||, \chi) = 0.22 \pm 10^{-3}, \chi = 1 \pm 10^{-4} = 0 \pm 10^{-11}.$$

The validity of the last solution is subject to the condition

$$\Omega_0 \ll \omega^w \ll \Omega_e$$

$$0.00054 \ll k||^2 \chi \beta_e^{-1} \ll 1$$

$$0.00054 \ll \frac{k||^2 \chi}{\eta} \beta_e^{-1} \ll 1,$$

where

$$\omega^w = \Omega_e k||^2 \beta_e^{-1} \chi.$$

which is the range in which the whistler mode exists. A numerical study of the properties of eq. 24 shows that there

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is a distinct range of \( \tilde{k}_|| \) for which \( \gamma^w_{\text{coll}} > 0 \). This value is bounded between 0.22 < \( \tilde{k}_|| < 0.7 \) for 0.0001 < \( \epsilon^h \beta_c < 1000 \). For \( \tilde{k}_|| = 0.22 \), one finds
\[
180 \gg \beta > 0.01.
\]
(26)
For \( \tilde{k}_|| = 0.7 \) instead of 0.22 we find that:
\[
1800 \gg \beta > 1.
\]
(27)
The underlying physical interpretation of this range of \( \beta \) is as follows: If \( \beta \) is too large, the waves that resonate with thermal electrons can cyclotron resonate with ions as well, which complicates things beyond the scope of this paper. For \( \beta \) too small, the phase velocity of the whistlers exceeds the thermal electron velocity by a sufficient margin that the resonant particles are far out on the tail of the Maxwellian distribution function, and the excitation is too weak to be interesting. In any case, condition eq. (27) is met in clusters of galaxies, and condition eq. (26) at the ISM. The angular dependence of \( \gamma \) on eq. (21) on \( \chi \) is roughly linear with a maximum at \( \chi = 1 \); the slope depends on \( \epsilon^h \beta_c \). Thus, at \( \chi < 1 \), condition eq. (27) or condition eq. (26) are somewhat relaxed by a factor of order unity.

The critical condition for whistlers present in a collision dominated hydrogen plasma (cf. the last line of eq. (19)) is found from eq. (23) to be:
\[
\epsilon^h \beta > 2 \times 10^{-4},
\]
(28)
provided that \( \beta \) is in the range given by eqs. (27), (26). There are a few more conditions for the validity of eq. (28). One of them is
\[
\epsilon^h \lesssim 10^{-2} \Rightarrow \beta > 10^{-2},
\]
(29)
and it stems from the validity of the Knudsen expansion. The latter constraint is so weak that it can be ignored.

The last condition to be considered is merely a formal one and it stems from the validity of eq. (15) namely:
\[
\zeta \equiv \frac{\max(\gamma^w_{\text{coll}})}{\omega^w} < 1
\]
(30)
This condition is rather strong and it requires that \( \epsilon^h \beta_c < 10 \). However, even if this is not the case if when whistlers dominate the plasma, the value of \( \nu_{w,s} \) is sufficiently high and \( \zeta \) can be maintained much smaller than unity.

The conclusion of this section is that any hydrodynamical model in which \( \beta > 0.01 \) and \( 10^{-4} < \epsilon^h < 10^{-2} \) gives rise to whistlers that exceed an excess of perpendicular momentum in the hotward velocity hemisphere. The unstable on-axis whistlers propagate in the coldward direction.

Note that the hotward part of the distribution function, which has excess perpendicular momentum is not the part that carries the heat. The coldward particles, which carry the heat from hot to cold, have an excess of parallel momentum, and are stable to all on-axis whistlers.

Thus, the heat carrying particles cannot resonantly excite on-axis whistlers that propagate parallel to the field lines and on-axis whistlers that propagate parallel to the field lines do not significantly inhibit heat flux. Whistlers that propagate off the axis, on the other hand, can inhibit heat flux. These off-axis waves have right-handed elliptical polarity which can be represented as a superposition of left-handed and right-handed circular polarity. The left-handed circular polarity allows for whistler emission in perpendicular momentum deficient regions in phase space. Although at marginal stability of a collision dominated plasma only on-axis whistlers are excited, it is incorrect to proceed under the assumption that these on-axis whistlers can lead to heat flux inhibition. Rather, we argue, any on-axis waves merely modify the distribution function in such a way that the marginal stability is transferred to off-axis waves.

### 4.3 Condition for Whistler Dominated Plasma and Energy Density of Whistlers

The questions are related as to when the plasma would become whistler dominated and how much energy density would be stored in the whistlers, and we consider them together. As was pointed out above, a whistler dominated plasma does not necessarily imply heat-flux inhibition. The plasma becomes whistler dominated when
\[
\frac{\rho^w}{\rho_{\text{coll}}} \gg 1,
\]
(31)
whereas the whistler dominance leads to heat flux inhibition when
\[
\frac{\rho^w}{\rho_{\text{coll}}} > 1.
\]
(32)

Using eq. (3) and the first line of eq. (8) in eq. (32) we find that heat flux inhibition is obtained provided that:
\[
\frac{\rho^w}{\rho_{\text{coll}}} = \frac{9}{8} \beta_c^2 \frac{N}{\log(\Lambda_{\text{cool}})} \frac{1}{c} \int_{1-\xi}^{1} \bar{W}(\tilde{k}_R, \chi)(1 - \chi^2) \frac{d\chi}{\chi} \gtrsim 1.
\]
(33)
Writing the spectrum in the following general way:
\[
\bar{W} = \Sigma_i K^i(\tilde{k}_||) \Xi^i(\chi),
\]
(34)
where \( K^i \) are arbitrary functions of \( \tilde{k}_|| \), and \( \Xi^i(\chi) \) are arbitrary functions of \( \chi \) we have:
\[
\int_{1-\xi}^{1} \bar{W}(\tilde{k}_R, \chi)(1 - \chi^2) \frac{d\chi}{\chi} = \Sigma_i K^i(\tilde{k}_||) \int_{1-\xi}^{1} \Xi^i(\chi)(1 - \chi^2) \frac{d\chi}{\chi}.
\]
(35)
Thus, if \( \xi \) is sufficiently big we have heat flux inhibition. Assuming that this is the case, we write \( \bar{W}(\tilde{k}_R, \chi) = \text{Const} \bar{W}(\tilde{k}) \) and assume \( \text{Const} \) is of order unity. We can now invert the above ratio and obtain the constraint on the energy density in the waves which reads:
\[
\bar{W}(\tilde{k}) = N_D^{-1} \frac{\rho^w}{\rho_{\text{coll}}} \beta_c^2 \log(\Lambda_{\text{cool}}) \left( \frac{\nu_w}{c} \right)^{-1}.
\]
(36)
The energy density in the whistlers is a function of the inhibition factor, and inversely proportional to it. Since the inverse of the Debye number is tiny, \( 10^{-16} \) for the ISM and \( 10^{-23} \) for the ICM, inhibition factors can be extremely large. Thus, even an inhibition of the heat flux by a factor of \( 10^{10} \) leaves the energy density in the whistlers small compared to the energy density in the magnetic field. Therefore, one can quite safely assume that the energy density stored in the whistlers is invariably much less than the energy density of the background magnetic field.
4.4 Whistler Dominated Plasma

We have shown that once the condition is satisfied, the plasma becomes whistler dominated. We now proceed under the assumption that eq. is satisfied by a large margin, i.e. that

\[ e^{th} \beta \gg 2 \times 10^{-4}. \]  

(37)

Under these conditions whistlers grow as \( \gamma_{\text{w}} \gg 0 \), for at least the same mode, and we need to solve the wave equation eq. \[k\].

Thus, we need to determine \( W \), and \( \vec{\nu} \) self-consistently. We consider quasi-linear saturation i.e. \( \gamma = 0 \), for the most unstable mode. In particular we need to know whether quasi-linear saturations sets in before or after eq. \[k\] becomes relevant, as only the latter leads to heat flux inhibition. Eq. \[k\] though implying a whistler dominated plasma, merely implies a reduction of the heat-flux only by a factor of two.

Advection by waves is negligible (cf. § 4) and the wave equation eq. \[k\] with eq. \[k\] reduces to

\[ \gamma = \Gamma_w. \]  

(38)

Quasi-linear saturation is defined as the vanishing of \( \gamma \) and hence of \( \Gamma_w \). If the wave spectrum achieves quasi-linear saturation to a good approximation, i.e. for every \( \vec{k} \) there is a critical angle \( \chi_{\text{crit}} \) such that:

\[ \gamma[\vec{k}, \chi_{\text{crit}}(\vec{k})] = 0, \]  

(39)

and for all other angles the growth rate \( \gamma \) is negative. Thus, \n
\[ W \propto \delta(\chi - \chi_{\text{crit}}). \]  

(40)

The picture is that the wave spectrum is highly peaked near \( \chi_{\text{crit}} \), where the condition of marginal stability is met. Once this has been established, the condition \( \zeta \ll 1 \) (cf. eq. \[k\]) is \textit{a priori} fulfilled.

The assumption of quasi-linear saturation is exact if \( \chi_{\text{crit}} \) is unique and does not depend on \( \vec{k} \). Otherwise the assumption \( \Gamma_w = 0 \) is formally invalid (Mode coupling does not exist if all the waves propagate at the same angle). However, the non-linear Landau damping rate \( \Gamma^{LD} \) is of order \( \beta_w \) compared to the mode coupling term, and it can be shown (Levinson 1992; cf. § 4) that if the parent wave and the resulting waves have energies of the same order, then the non-linear damping is less than \( \gamma_{\text{w}} / \gamma_{\text{coll}} \) \( \eta \) is defined in eq. \[k\]). Thus, it is of order the suppression factor times \( \eta \) and can quite safely be ignored for the suppression factors that we are interested in.

Using eq. \[k\] in eq. \[k\] we obtain:

\[ e^{th} \beta \gamma \pi \frac{1}{\vec{k}} \langle \chi_{\text{crit}} \rangle P^{-1}(\vec{k}), \chi_{\text{crit}} \rangle - 1 = 0 \]  

(41)

We now seek solutions of eq. \[k\] which will yield a value of \( \chi_{\text{crit}} \) by exploiting the fact that \( Q(\vec{k}, \chi_{\text{crit}})P^{-1}(\vec{k}, \chi_{\text{crit}}) \) has a maximum of \( \chi \) at \( \chi_{\text{crit}} \). To start this procedure we evaluate \( Q \), which is defined in eq. \[k\]. In particular we note the \( Q \) dependence on the ratios \( \gamma_{\text{w}} / \gamma_{\text{coll}} \) and \( \gamma_{\text{w}} / \gamma_{\text{coll}} \) at \( \vec{v} = \vec{v_c} \). Using eq. \[k\] in eq. \[k\] and \( \vec{v}_{\text{coll}} = \vec{v_c} \) we find:

\[ \frac{\partial \gamma_{\text{w}}}{\partial \chi_{\text{crit}}} \bigg|_{\vec{v} = \vec{v_c}} = \mathcal{N} \left( \frac{k_0}{(k_0^\mu)^2} \right) \Sigma_i K(\vec{k}) A_i \]  

\[ \frac{\partial \gamma_{\text{w}}}{\partial \chi_{\text{crit}}} \bigg|_{\vec{v} = \vec{v_c}} = \mathcal{N} \left( \frac{k_0}{(k_0^\mu)^2} \right) \Sigma_i K(\vec{k}) S_i \]  

(42)

where

\[ \mathcal{N} \equiv \left( \frac{4N_i B_i^2 \vec{B}_i}{4\pi m_i c^3} \right) \sqrt{\frac{m_i}{\mu}} \left( \frac{m_i}{c^2} \right) \geq 1 \]  

(43)

\[ A_i \equiv 4 \int_0^1 d\chi \frac{\mathcal{N}(\chi)}{\chi^2} A_i \]  

\[ S_i \equiv \int_0^1 d\chi \frac{\mathcal{N}(\chi)}{\chi^2} (1 - \chi)^2. \]  

We consider now eq. \[k\] which gives the relevant case. The elimination of the case given by eq. \[k\] is considered further below.

A whistler dominated plasma under the condition eq. \[k\] implies that \( \mathcal{N} \mathcal{N}_A K(\vec{k}) \gg 1 \) and \( \mathcal{N} \mathcal{N}_S K(\vec{k}) \gg 1 \) for the dominant \( i \). Substituting eq. \[k\] into eq. \[k\] we find

\[ Q(\vec{k}, \chi) = \int_0^1 d\mu (1 - \mu^2) \frac{k_0}{(k_0^\mu)^2} \left[ \frac{1}{(k_0^\mu)^2} - \frac{1}{\chi^2} \right] \left[ (1 + \chi^2) + (1 - \chi)^2 \frac{k_0^2}{(k_0^\mu)^2} \right]. \]  

(44)

Using the last equation in eq. \[k\] we obtain:

\[ \left( \chi_{\text{crit}} + \frac{k_0^2}{(k_0^\mu)^2} \right) \mathcal{N} A_i \frac{\mathcal{N}(\chi_{\text{crit}}(1 - \chi_{\text{crit}})^2)}{1 + \chi_{\text{crit}}} \approx 1 \]  

(45)

\[ G^w(\vec{k}) = \frac{1}{k_0^2} \int_0^1 d\mu (1 - \mu^2) \frac{1}{(k_0^\mu)^2} - \frac{1}{\chi^2} \left[ (1 + \chi^2) + (1 - \chi)^2 \frac{k_0^2}{(k_0^\mu)^2} \right]. \]  

(46)

is a rational function of \( \vec{k} \), and

\[ G^w = \frac{1}{2} \frac{k_0^2}{(k_0^\mu)^2} \left[ \vec{k}^2 + \vec{k}^2 \left( \frac{1 - \chi^2}{\chi^2} \right) \right]. \]  

(47)

Substituting eq. \[k\] into eq. \[k\] one finds:

\[ e^{th} \beta \gamma \pi \frac{1}{\vec{k}} \langle \chi_{\text{crit}} \rangle P^{-1}(\vec{k}), \chi_{\text{crit}} \rangle - 1 = 0 \]  

(48)

\[ \frac{1}{\mathcal{N} A_i K(\vec{k}) \Sigma_i (S_i + A_i)} \left( \chi_{\text{crit}} + \frac{\mathcal{N} A_i K(\vec{k}) \Sigma_i (S_i + A_i)}{1 + \chi_{\text{crit}}} \right) \approx 1 \]  

If we proceed under the assumption that only a single value of \( i \) contributes to the sum, we find:

\[ e^{th} \beta \gamma \pi \frac{1}{\vec{k}} \langle \chi_{\text{crit}} \rangle P^{-1}(\vec{k}), \chi_{\text{crit}} \rangle - 1 = 0 \]  

(49)

\[ \frac{1}{\mathcal{N} A_i K(\vec{k}) \Sigma_i (S_i + A_i)} \left( \chi_{\text{crit}} + \mathcal{R} \chi_{\text{crit}}(1 - \chi_{\text{crit}})^2 \right) \approx 1 \]  

where

\[ \mathcal{R} = \frac{A_i}{S_i} \]  

(50)

We can now obtain the critical angle for whistlers propagation at marginal quasi-linear stability. Note that the function given in the parenthesis might have a local maximum in \( \chi_{\text{crit}} \). One finds this maximum from the condition:

\[ \frac{1}{\mathcal{N} A_i K(\vec{k}) \Sigma_i (S_i + A_i)} \left( \chi_{\text{crit}} + \frac{1 + \chi_{\text{crit}}}{1 + \chi_{\text{crit}}^2} \mathcal{R} \chi_{\text{crit}}(1 - \chi_{\text{crit}})^2 \right) = 0 \]  

(51)

Thus, the location of the maximum for the value of \( \chi_{\text{crit}} \) depends on the value \( \mathcal{R} \) only. Using eq. \[k\] in eq. \[k\] to estimate the value of \( \mathcal{R} \) we find:

\[ \mathcal{R} \approx \frac{4 \chi_{\text{crit}}}{(1 - \chi_{\text{crit}})^2}. \]  

(52)
The cubic equation eq. 51 has two complex roots and one real root. Substituting into the expression for the real root the value of $R$ from eq. 52, we find an equation for $\chi_{crit}$. We solve this equation numerically, and find that

$$R = 4.8,$$

(53)

and

$$\chi_{crit} = 0.414214 \Rightarrow \theta \approx 64^\circ. \quad (54)$$

We now substitute the values of $\chi_{crit}$ and $R$ into eq. 49 and find:

$$\epsilon^h \beta_e \left( \frac{1}{k^2} + 2 - \frac{Y^2}{Y^2} \right) \frac{5.88}{2R\beta(k_{||})} \approx 1. \quad (55)$$

One can now solve this equation for the value of $NK(\bar{k}_{||})S$. Bearing in mind that only one $i$ contributes to the sum in eq. 52 and substituting the value of $NK(k_{||})S$ into the last line of eq. 12 we find:

$$\bar{\nu}^{e,a} \approx \frac{3\epsilon^h \beta_e}{\nu \bar{v}} \left( (\bar{v} \mu)^2 + 2 - \frac{Y^2}{Y^2} \right),$$

(56)

and with the numerical value of $R$ we have from the first line of eq. 12

$$\bar{\nu}^{e,a} \approx \frac{14\epsilon^h \beta_e}{\nu \bar{v}} \left( (\bar{v} \mu)^2 + 2 - \frac{Y^2}{Y^2} \right).$$

(57)

which implies that the heat flux along field lines is roughly proportional to $\beta_e^{-1}$, and can be determined from observable parameters. The solution procedure continues by finding the value of the ratio $Y^2/Y^1$. This problem can be considered separately, and we do so in § 14. Thus, with eqs. 57 54 and eq. 15 we have:

$$\beta_e^{-1} \left( \frac{\partial}{\partial \mu} \right) \approx \frac{\epsilon^2}{\nu \bar{v}^2 \left( (\mu \bar{v})^2 + 2 - \frac{Y^2}{Y^2} \right)} f_{MB},$$

(58)

and from § 10 we find $Y^2/Y^1 \approx 0.18$. Thus, the distribution function reads:

$$\beta_8^{-1} \left( \frac{\partial}{\partial \mu} \right) \approx \frac{\epsilon^2}{\nu \bar{v}^2 \left( (\mu \bar{v})^2 + 2 - \frac{Y^2}{Y^2} \right)} f_{MB},$$

(59)

and a self consistent distribution function at quasilinear saturation of off-axis waves has been found.

Although in a collision dominated plasma whistlers are least stable along the axis we now show that at quasilinear (QL) saturation this is no longer the case: If the waves at QL saturation had propagated in a narrow cone along the axis this would have implied that $\mathcal{A}_1 K(\bar{k}_{||}) \gg 1$ for some $i$ and that $S_i \ll \mathcal{A}_i$ for every $i$. Substituting this assumption into eq. 43 we find:

$$\epsilon^h \beta_e \left( \chi_{crit} \frac{G^5(\bar{k}_{||}) \bar{k}_{||}^4}{\mathcal{A}_i} + G^6(\bar{k}_{||}) \chi_{crit} (1 - \chi_{crit})^2 \frac{1 + \chi_{crit}}{1 + \chi_{crit}} \right) \approx 1(60)$$

Now $G^5$ and $G^6$ are functions of the same order of magnitude. The function in the parenthesis has a local maximum at $\chi_{crit}$. Provided we neglect the term of order $1/\epsilon$ the location of the maximum does not depend on $\bar{k}_{||}$ or $\epsilon^h \beta$ and it is found at $\chi_{crit} \approx 0.3$, in contradiction with our assumption that $\chi_{crit} \approx 1$ that $S_i$ is of order $\mathcal{A}_i$. Since the corrections to the location of the maximum would only be of order $1/\epsilon$, we conclude that it is very difficult to support a whistler dominated plasma that does not generate off-axis waves without imposing strict fine tuning.

5 THE HEAT-FLUX ALONG FIELD LINES

The heat flux vector is, after integration by parts,

$$q_i = -\pi m_e \int_0^\infty d\nu \int_{-1}^1 d\mu (1 - \mu^2) \frac{\partial f}{\partial \mu} \left( \frac{\partial^2}{\partial \nu^2} \right), \quad (61)$$

When eq. 59 is used in eq. 60 one gets:

$$q_i = -\pi^2 m_e v^2 n_e e^{\epsilon h}(\beta_e e^{\epsilon h})^{-1} \int_0^\infty d\nu e^{-\nu^2} \left( \bar{v}^2 - 0.18 \right) \int_0^1 d\mu (1 - \mu^2) \mu \beta_e^{-1} (\bar{v} \mu)^2 + 1.82^2)^{-1}.$$

Carrying the integration over the angles yields

$$q_i \approx \frac{14}{4\pi} \frac{\epsilon^2}{\nu \bar{v}^2} m_e v^2 n_e e^{\epsilon h}(\beta_e e^{\epsilon h})^{-1} \int_0^\infty d\nu e^{-\nu^2} \left[ 1.82 \left( 1 + \frac{\bar{v}^2}{1.82} \right) \log \left( 1 + \frac{\bar{v}^2}{1.82} \right) - \bar{v}^2 \right]. \quad (63)$$

Carrying the integral over the velocities we find:

$$q_i \approx 0.13 \pi^2 m_e v^2 n_e e^{\epsilon h}(\beta_e e^{\epsilon h})^{-1} \left[ (3.5) - 0.18 \Gamma(2.5) + 0.083(-2(1.82) + (1.82)^2 + 1.82^2 - 6e^{1.82} G_{4,0}^4 \frac{\phi, 1, 1}{-1, -1, 3, \phi} + 4e^{1.82} G_{4,0}^4 \frac{\phi, 2, 2}{0, 0, 4, \phi} - e^{1.82} G_{4,0}^4 \frac{\phi, 2, 2}{1, 1, 5, \phi} \right) + 0.25 + 0.18 \left( 1.82 + (1.82)^2 + 1.82^2 \right)^{-1} e^{1.82} G_{4,0}^4 \frac{\phi, 1, 1}{-1, -1, 3, \phi} - e^{1.82} G_{4,0}^4 \frac{\phi, 2, 2}{1, 1, 4, \phi} \right] \approx -0.65 \pi^2 m_e v^2 n_e e^{\epsilon h}(\beta_e e^{\epsilon h})^{-1} \left[ 0.13 \pi^2 m_e v^2 n_e \bar{v} \beta_e^{-1}, \right.$$

where

$$G_{p, q}^{m, n} = \frac{1}{\pi} \int d\nu e^{-\nu^2} \int d\mu (1 - \mu^2) \frac{\partial f}{\partial \nu} \left( \frac{\partial}{\partial \mu} \right) \left[ n_1^{p, q} \gamma_{1, 1}^{m, n} \Gamma(1 - a_i - s) + n_s^{p, q} \gamma_{1, s}^{m, n} \Gamma(1 - b_i - s) \right]. \quad (65)$$

is the Meijer G function (Gradshetein & Ryzhik 1980: p. 897, f. 4), and $\phi$ denotes an empty set. The numerical evaluation of $G$ was carried out by the use of Mathematica.

The last equation shows the heat flux strongly suppressed compared to the Spitzer value when the scattering rate by off-axis waves exceeds the electron-ion collision frequency (Levinson & Eichler 1992). Previous literature for
heat flux inhibiting astrophysical environments used a term \( f_{\text{supp}} \), the so called suppression factor, that multiplied the collision dominated plasma heat flux i.e.

\[
q_l = -f_{\text{supp}}64\epsilon^c(\pi)^{-\frac{1}{2}}m_ev_th^3n_e,
\]

where we used eq. [3] in eq. [2] to obtain eq. [8] and the factor \( f_{\text{supp}} \) was introduced by hand. The value of \( f_{\text{supp}} \) obtained from eq. [5] and [6] is

\[
f_{\text{supp}} = (64/65) \times 10^{-2}(\epsilon^c)^{-1} \approx 1 \times 10^{-2} \epsilon^c
\]

If the distribution function were not to relax to quasilinear magnetic stability then there would be less suppression. However, the non-linear damping terms have been shown to be much smaller than the quasilinear damping term (which in marginal stability is balanced by a growth term) and thus it can safely be argued that the suppression factor computed above is of general applicability (provided that \( \beta \) satisfies the required constraints). Since \( f_{\text{supp}} \) has been calculated under the assumption eq. [12] we suggest the following extrapolation:

\[
f_{\text{supp}} \approx \frac{1}{1 + 50(\epsilon^c/\beta)}.
\]  

Note that the suppression factor has been calculated relative to a Lorentzian plasma. Taking into account that the Spizer-Härm heat flux is about a factor of five higher relative to a Lorentzian plasma, the suppression factor reads:

\[
f_{\text{supp}} \approx \frac{1}{1 + 250(\epsilon^c/\beta)}.
\]

6 DISCUSSION AND CONCLUSIONS

Heat-flux inhibition along field lines obtains when off-axis waves are present in the plasma. We have found within quasilinear theory that whistler instability indeed generates off-axis whistlers. Thus, heat flux inhibition along field lines by whistlers should be a common phenomenon in a weakly magnetized plasma. The basic argument for off-axis whistlers is, in qualitative terms, that if the waves were excited only in a narrow cone along the axis, then the distribution function would evolve to a state in which the growth rate of off-axis waves would generally exceed that of the on-axis ones.

One of our assumptions is that the energy density in the magnetic field of the whistler will not exceed the energy density of the background magnetic field. We find that this is generally the case by a large margin in tenuous astrophysical plasmas.

While Gary & Feldman (1977) find whistlers inefficient for inhibition of heat flux in the solar wind, \( f_{\text{supp}} \sim 1 \), Levinson & Eichler (1992) find a suppression factor of \( 10^{-2} \leq f_{\text{supp}} \leq 10^{-1} \), in the interstellar medium (ISM), sufficient to explain ISM observations. Pistinner et al. (1996) find \( 1 \geq f_{\text{supp}} \geq 10^{-3} \) in the intracluster medium, ICM, which is enough to resolve many of the observational problems there. In these papers, the level of wave turbulence was calculated by assuming a balance between linear growth and non-linear wave removal, and it was noted that the results are suspect when the heat flux so found was below that needed to drive the whistlers beyond quasilinear saturation. Here, on the other hand, we have used the criterion of marginal stability of the least stable whistler wave to derive a relatively simple expression for the heat flux [eq. 12], or, equivalently, the suppression factor by which classical heat conduction is suppressed of eq. [23]. The heat flux that we derive here is at least that found in our previous papers inasmuch as of the geometry of wave - particle interactions has been considered more carefully, i.e. only some particles can interact with some waves, as opposed to all particles with all waves via an oversimplified BGK operator. On the other hand, it will be argued in a subsequent paper that the asymmetry of the scattering in the forward and backward hemispheres is sufficient to allow for a non-resonant firehose instability of the magnetic field. This causes field line tangling, and further inhibits the heat flux. A careful evaluation of the heat flux as a function of plasma parameters will therefore be attempted only in the subsequent paper.

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8 APPENDIX A

Here we briefly outline the nature of the assumptions made in writing the BFPE eq. [1]. The assumption of steady state does not imply that the plasma does not evolve with time. Rather it implies that any microscopic process occurs on a time scale which is much faster than the time scale for macroscopic changes and thus, temporal macroscopic changes may be ignored to first order.

The second assumption implies that energy exchange in a collision is negligibly small and may therefore be ignored. If the electrons collide only with ions it implies that the ions have infinite mass. Thus, electron-electron collisions have been ignored. If the plasma is collision dominated this assumption modifies the collision dominated plasma distribution function by up to a factor of five. A plasma dominated by collective phenomena which leads only to pitch angle scattering (the case considered in this paper) is tantamount to the above approximation.

The third assumption is related to the ratio of the gyro-radius and the large scale magnetic field variation, the so called correlation length (not to be confused with the coherence length). It implies that this ratio can be effectively set to zero. There is a subtle issue involved. If the plasma is pervaded by collective magnetic phenomena (electromagnetic waves) that have \( r_L \approx L_H \) it seems that the approximation which led to eq. [12] is not valid. However, this presents a technical problem that can be circumvented by a standard procedure which allows one to include the small scale magnetic fluctuation into \( \nu \) (cf. Melrose 1980). Observations suggest that the large scale magnetic field variation i.e. the magnetic field correlation length, is comparable to the electron mean free path. Thus, we may define:
\[ \eta \equiv \frac{r_L}{L_B} \approx \frac{\eta_1}{\lambda_e}, \]  
\( (70) \)

where \( \lambda_e \) is the electron mean free path, and \( L_B \) the magnetic field correlation length. Typical values of \( \eta \) are:

(i) \( \eta \approx 10^{-12} \) ISM
(ii) \( \eta \approx 10^{-16} \) ICM.

Assuming that \( \tilde{\eta} \approx \eta \to 0 \) one gets the “adiabatic drift approximation”.

9 APPENDIX B

Consider the advection of whirls. The magnitude of the quasilinear damping term \( \Omega_c \beta e^{-1} k \tilde{\eta} \hat{P}(k, \chi) \) (eq. [19]) compared to the advection term in eq. [18]:

\[ \frac{\chi v_y \frac{\partial \ln(W)}{\partial \eta}}{2} : \Omega_c \beta e^{-1} \sqrt{\frac{\pi}{2}}(1 + \chi)^2 e^{-\frac{(\chi^2)}{2}}. \]  
\( (71) \)

The entire whirl emission process is driven by gradients of the distribution function. Thus, the shortest length scale for the variation of the spectrum is determined by the length scale for the variability of the distribution function and we have:

\[ \frac{\chi v_y \frac{\partial \ln(W)}{\partial \eta}}{2} \approx \Omega_c \beta e^{-1} \chi |\eta|^2 \eta \sim \frac{\eta^2}{\eta_{\text{th}}}. \]  
\( (72) \)

On the axis:

\( \eta_{\text{th}} \approx 10^{-14} : 1. \)  
\( (73) \)

10 APPENDIX C

Levinson (1992) shows that non-linear Landau damping is given by:

\[ \Gamma^{LD} = \frac{\beta}{\sqrt{2\pi}m_e e^2} \int d\chi dk_1 \omega_k \frac{W(k_1, \chi) e^{i(k_1 \chi)}}{\sqrt{1 + \frac{8\pi}{\gamma} (\omega_k)^2 \chi^2}} \]  
\( (74) \)

Casting the last equation into a non-dimensional form we obtain:

\[ \Gamma^{LD} \approx (\Omega_c \beta e^{-1}) \times \frac{14.4\beta e^{-1}}{\omega^2} \int d\chi dk_1 \frac{W(k_1, \chi) e^{i(k_1 \chi)}}{\sqrt{1 + \frac{8\pi}{\gamma} (\omega_k)^2 \chi^2}} \]  
\( (75) \)

\[ \Gamma^{MC} = \frac{1}{16} (\Omega_c \beta e^{-1}) \int d\omega_k d\omega_k' (1 - \chi)^2 \frac{\omega_k' e^{-i(k_1 \chi)}}{\omega^2} \]  
\( (76) \)

Thus,

\[ \Gamma^w = \Gamma^{LD} + \Gamma^{MC} \approx \Gamma^{MC}. \]  
\( (77) \)

Levinson (1992) evaluates the mode coupling term under the assumption that the parent waves and the daughter waves have energy of the same order and finds:

\[ \Gamma^w \approx \Gamma^{MC} \approx (\Omega_c \beta e^{-1}) \frac{\hat{P}^{w,s}}{\eta_{\text{coll}}}. \]  
\( (78) \)

Consider now the ratio of the quasilinear damping term to the non-linear damping term.

\[ \frac{\Omega_c \beta e^{-1}}{\Omega_c \beta e^{-1}} \frac{\hat{P}^{w,s}}{\eta_{\text{coll}}}, \eta \approx 10^{-8} \ll 1 \]  
\( (79) \)

11 APPENDIX D

Here we consider how the exact numerical value of \( Y^2 \) is obtained. Although in principle we deal with an integral equation, it can be reduced by the procedure outlined below. Let

\[ a = 2 - \frac{Y^2}{Y^T}, \]  
\( (80) \)

then substituting, eqs [46] and [57] into eq. [49] we find:

\[ Z(\bar{v}) = (2e^{\beta \bar{v}} - 1) \frac{\theta \bar{v}^{-3}}{a \left( 1 + \frac{\bar{v}^2}{2} \right) \log \left( 1 + \frac{\bar{v}^2}{2} \right) - \bar{v}^2}. \]  
\( (81) \)

One can now proceed and calculate formally the value of \( Y^2 \) and \( Y^T \). Substituting the expression for \( Z(\bar{v}) \) into eq. [49] one finds after some algebra

\[ \frac{\gamma_{\text{th}}^\beta \beta e^{-1}}{Y^2} = 1 - 0.25[a + a^2 + 2e^{\beta e^{-1}}] \left[ a \phi, 1, 1 \right]^T \left[ 0, 0, 3, \phi \right] - \]  
\[ e^{\beta e^{-1}} G_{4,0} (a \phi, 1, 1, 0, 0, 2, \phi) \]  
\( (82) \)

\[ 0.51 e^{\beta \bar{v}} Y^1 = \frac{e^{\beta \bar{v}} - 1}{(1 - e^{\beta e^{-1}}) (a \phi, 1, 1, 0, 0, 2, \phi)} \]  
\( (83) \)

Subtracting from 2 the formal ratio of \( Y^2/Y^1 \) one obtains a transcendental equation for \( a \). This equation reads:

\[ a = 2 + 2 \left[ 1 - 0.25[a + a^2 + 2e^{\beta e^{-1}}] \left[ a \phi, 1, 1 \right]^T \left[ 0, 0, 3, \phi \right] - \]  
\[ e^{\beta e^{-1}} G_{4,0} (a \phi, 1, 1, 0, 0, 2, \phi) \right]^{-1} \]  
\( (83) \)

Solving this equation numerically, we find \( a = 1.82 \).

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