Dirac Fields on Spacelike Hypersurfaces, Their Rest-Frame Description and Dirac Observables

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Abstract

Grassmann-valued Dirac fields together with the electromagnetic field (the pseudoclassical basis of QED) are reformulated on spacelike hypersurfaces in Minkowski spacetime and then restricted to Wigner hyperplanes to get their description in the rest-frame Wigner-covariant instant form of dynamics. The canonical reduction to the Wigner-covariant Coulomb gauge is done in the rest frame. It is shown, on the basis of a geometric inconsistency, that the description of fermions is incomplete, because there is no bosonic carrier of the spin structure describing the trajectory of the electric current in Minkowski spacetime, as it was already emphasized in connection with the first quantization of spinning particles in a previous paper.

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I. INTRODUCTION

In a series of papers [1–4] inspired by Ref. [5] the canonical reduction to a generalized noncovariant Coulomb gauge of the standard SU(3)xSU(2)xU(1) model of elementary particles was obtained by using the Shanmugadhasan canonical transformation [6] [see Refs. [7] for reviews].

In Ref. [8,9] there was the definition of a new type of instant form of dynamics, the rest-frame 1-time Wigner-covariant instant form, which generalizes to special relativity the canonical separation of the center of mass from the relative variables of an isolated system. It required the reformulation of classical isolated systems on arbitrary spacelike hypersurfaces foliating Minkowski spacetime (covariant 3+1 splitting), in a way which is suited to the coupling to the gravitational field. The canonical reduction of tetrad gravity [10] will, then, open the path to get a unified Hamiltonian description of the four interactions.

Therefore, one now has nearly all the technology needed to reduce the standard model to a rest-frame Wigner-covariant generalized Coulomb gauge. This has been done for its bosonic part in Refs. [8,9,11], where the coupling of positive energy charged scalar particles plus the electromagnetic and color Yang-Mills fields was studied in the rest-frame Wigner-covariant instant form.

If one makes the canonical reduction of the gauge degrees of freedom of the isolated system in the rest-frame instant form on the Wigner hyperplane, one gets the rest-frame Wigner-covariant generalized Coulomb gauge in which the universal breaking of covariance is restricted to the decoupled center-of-mass variable. However, as shown in Refs. [1,8,11], the region of spacetime over which this noncovariance is spread, is finite in spacelike directions and identifies a classical intrinsic unit of length, the Møller radius

\[ \rho = \sqrt{W^2/P^2} = |\hat{S}|/\sqrt{P^2}, \]

where \( P^2 > 0 \) and \( W^2 = -P^2\hat{S}^2 \) are the Poincaré Casimirs and \( \hat{S} \) the Thomas rest-frame spin of the isolated system respectively. This unit of length gives rise to a physical intrinsic ultraviolet cutoff at the quantum level in the spirit of Dirac and Yukawa.

To get the description of the standard model on spacelike hypersurfaces one still needs the formulation of fermions on them. This is also needed for treating the fermions in general relativity: given their coupling to tetrad gravity (see for instance Refs. [12,13]) one needs this formulation to arrive at the ADM canonical formalism based on 3+1 splittings of the globally hyperbolic asymptotically flat spacetime. However, as can be seen in Refs. [14], where there is known on the subject, in general one restricts himself to hyperplanes \( x^o = \text{const.} \) and there is no real discussion of the Dirac brackets associated with the second class constraints.

As a first step, in a previous paper [15] there was the study of positive energy charged spinning particles plus the electromagnetic field, to which we refer for a review of the approach and for the problematic concerning the spin structure of these particles.

In this paper we shall study the formulation of Grassmann-valued Dirac fields plus the electromagnetic field on spacelike hypersurfaces in Minkowski spacetime. This is the pseudoclassical basis of QED.

In Section II there is the description of Grassmann-valued Dirac and Maxwell fields on spacelike hypersurfaces in Minkowski spacetime.

In Section III there is their restriction to arbitrary spacelike hyperplanes, while in Section IV there is the rest-frame description on Wigner hyperplanes.
In Section V there is the canonical reduction of the system to the Wigner-covariant Coulomb gauge: a canonical basis of Dirac’s observables is found and the reduced Hamilton equations are determined.

In the Conclusions there are some comments on the incompleteness of the description of fermions and on their quantization.

In Appendix A there is a review on the foliations of Minkowski spacetime with spacelike hypersurfaces.

In Appendix B there is a discussion on the Lagrangian for Dirac’s fields on spacelike hypersurfaces.

In Appendix C there are the transformation properties of Dirac’s fields under Wigner boosts.
II. DIRAC AND MAXWELL FIELDS ON SPACELIKE HYPER_SURFACES.

As in Ref. [8], let us consider a 3+1 splitting of Minkowski spacetime with a family of spacelike hypersurfaces $\Sigma_\tau$, whose points $z^\mu(\tau, \bar{\sigma})$ are labelled by Lorentz-scalar parameters: i) $\tau$ labelling the leaves of the foliation; ii) $\bar{\sigma}$ giving curvilinear coordinates on each leave. The coordinates $z^\mu(\tau, \bar{\sigma})$ will be the fields describing the hypersurface. See Appendix A for a review of notations and concepts, in particular for the flat tetrads $z^A_\mu(\tau, \bar{\sigma})$ and their inverse cotetrads $z^A_\mu(\tau, \bar{\sigma})$, connected to the 3+1 splittings.

On the hypersurface $\Sigma_\tau$, we describe the electromagnetic potential and field strength with Lorentz-scalar variables $A_A(\tau, \bar{\sigma})$ and $F_{\dot{A}\dot{B}}(\tau, \bar{\sigma})$ respectively, defined by

\[
A_A(\tau, \bar{\sigma}) = z^\mu_\Lambda(\tau, \bar{\sigma})A_\mu(z(\tau, \bar{\sigma})),
\]
\[
F_{\dot{A}\dot{B}}(\tau, \bar{\sigma}) = \partial_\Lambda A_{\dot{B}}(\tau, \bar{\sigma}) - \partial_{\dot{B}} A_\Lambda(\tau, \bar{\sigma}) = z^\mu_\Lambda(\tau, \bar{\sigma})z^\nu_{\dot{B}}(\tau, \bar{\sigma})F_{\mu\nu}(z(\tau, \bar{\sigma})),
\]

and knowing the embedding of the spacelike hypersurfaces in Minkowski spacetime by construction.

The Grassmann-valued Dirac field on $\Sigma_\tau$ will be

\[
\tilde{\psi}(\tau, \bar{\sigma}) \equiv \psi(z(\tau, \bar{\sigma})), \quad \tilde{\bar{\psi}}(\tau, \bar{\sigma}) \equiv \bar{\psi}(z(\tau, \bar{\sigma})) = \psi^\dagger(\tau, \bar{\sigma})\gamma^\alpha.
\]

Since the field is Grassmann-valued, its components $\psi_\alpha(\tau, \bar{\sigma})$, [$\alpha = 1, .., 4$ are spinor indices], satisfy

\[
\tilde{\psi}_\alpha(\tau, \bar{\sigma})\tilde{\psi}_\beta(\tau, \bar{\sigma}) + \tilde{\bar{\psi}}_\beta(\tau, \bar{\sigma})\tilde{\bar{\psi}}_\alpha(\tau, \bar{\sigma}) = 0,
\]
\[
\tilde{\psi}_\alpha(\tau, \bar{\sigma})\bar{\psi}_\beta(\tau, \bar{\sigma}) + \bar{\psi}_\beta(\tau, \bar{\sigma})\bar{\psi}_\alpha(\tau, \bar{\sigma}) = 0,
\]
\[
\tilde{\bar{\psi}}_\alpha(\tau, \bar{\sigma})\bar{\psi}_\beta(\tau, \bar{\sigma}) + \bar{\psi}_\beta(\tau, \bar{\sigma})\tilde{\bar{\psi}}_\alpha(\tau, \bar{\sigma}) = 0.
\]

In Appendix B there is a discussion of the coupling of tetrad gravity to fermion fields and of how to reexpress it after a 3+1 splitting of spacetime so to be able to define the Hamiltonian formalism. After a restriction to Minkowski spacetime, with the previous 3+1 splitting, we get the form of the Lagrangian for Dirac fields on spacelike hypersurfaces $\Sigma_\tau$ [see Eq.(B3)]

\[
\mathcal{L}(\tau, \bar{\sigma}) = N(\tau, \bar{\sigma})\sqrt{\gamma(\tau, \bar{\sigma})}[\frac{i}{2}\tilde{\psi}(\tau, \bar{\sigma})\gamma^\mu z^A_\mu(\tau, \bar{\sigma}) (\partial_\Lambda - ieA_\Lambda(\tau, \bar{\sigma})) \tilde{\psi}(\tau, \bar{\sigma}) +
\]
\[
- \frac{i}{2} (\partial_\Lambda + ieA_\Lambda(\tau, \bar{\sigma})) \tilde{\psi}(\tau, \bar{\sigma})z^A_\mu(\tau, \bar{\sigma})\gamma^\mu \tilde{\bar{\psi}}(\tau, \bar{\sigma}) - m\tilde{\psi}(\tau, \bar{\sigma})\tilde{\bar{\psi}}(\tau, \bar{\sigma})] +
\]
\[
- \frac{N(\tau, \bar{\sigma})\sqrt{\gamma(\tau, \bar{\sigma})}}{4}g^{\Lambda\bar{\Lambda}}(\tau, \bar{\sigma})g^{\bar{B}\bar{B}}(\tau, \bar{\sigma})[F_{\dot{A}\dot{B}}(\tau, \bar{\sigma})F_{\bar{C}\bar{D}}(\tau, \bar{\sigma})].
\]

It is convenient to take as Lagrangian variables [since we will not use any more the notation $\psi(z(\tau, \bar{\sigma}))$, no confusion will arise]

\[
\tilde{\psi} \rightarrow \psi = \sqrt{\gamma} \psi,
\]
\[
\tilde{\bar{\psi}} \rightarrow \bar{\psi} = \sqrt{\gamma} \bar{\psi}.
\]
Since on spacelike hyperplanes \([8]\) one has \(\gamma(\tau, \bar{\sigma}) = 1\), there we shall recover \(\psi = \bar{\psi}\).

Eq. (4) becomes [see Appendix A for the definition of the lapse and shift functions \(N, N^\rho\); see also Ref. [11]]

\[
\mathcal{L}(\tau, \bar{\sigma}) = N(\tau, \bar{\sigma}) \left[ \frac{i}{2} \bar{\psi}(\tau, \bar{\sigma}) \gamma^\mu z^A_\mu (\tau, \bar{\sigma}) (\partial_\lambda - ieA_\lambda(\tau, \bar{\sigma})) \psi(\tau, \bar{\sigma}) + 
\right. \\
\left. - \frac{i}{2} (\partial_\lambda + ieA_\lambda(\tau, \bar{\sigma})) \bar{\psi}(\tau, \bar{\sigma}) z^A_\mu (\tau, \bar{\sigma}) \gamma^\mu \psi(\tau, \bar{\sigma}) - m \bar{\psi}(\tau, \bar{\sigma}) \psi(\tau, \bar{\sigma}) \right] + \\
- \frac{N(\tau, \bar{\sigma}) \sqrt{\gamma(\tau, \bar{\sigma})}}{4} g^{\lambda \bar{\nu}} (\tau, \bar{\sigma}) g^{\bar{\beta} \bar{\delta}} (\tau, \bar{\sigma}) F_{\lambda \bar{\nu}}(\tau, \bar{\sigma}) F_{\bar{\beta} \bar{\delta}}(\tau, \bar{\sigma}) = \\
= \int d\tau d^3 \sigma \left[ \frac{i}{2} l_\mu(\bar{\psi} \gamma^\mu(\partial_\tau - ieA_\tau) \psi - (\partial_\tau + ieA_\tau) \bar{\psi} \gamma^\mu \psi \right] + \\
- \frac{i}{2} N^\tau l_\mu(\bar{\psi} \gamma^\mu(\partial_\tau - ieA_\tau) \psi - (\partial_\tau + ieA_\tau) \bar{\psi} \gamma^\mu \psi \right] + \\
\left. + \frac{i}{2} N \gamma^{\bar{\tau}} z_{\bar{\mu}}(\bar{\psi} \gamma^\mu(\partial_\tau - ieA_\tau) \psi - (\partial_\tau + ieA_\tau) \bar{\psi} \gamma^\mu \psi \right] - \\
- (\partial_\tau + ieA_\tau) \bar{\psi} \gamma^\mu \psi - N m \bar{\psi} \psi(\tau, \bar{\sigma}) - \\
- \left[ \frac{\sqrt{\gamma}}{2N} (F_{\tilde{T}} - N^\alpha F_{\bar{\alpha}})(\gamma^{\bar{\tau}}(F_{\tilde{T}} - N^\alpha F_{\bar{\alpha}}) + \\
+ \frac{N \sqrt{\gamma}}{4} \gamma^{\bar{\tau}} \gamma^{\bar{\alpha} \bar{\beta}} F_{\bar{\alpha} \bar{\beta}}(\tau, \bar{\sigma}) \right]. (6)
\]

The canonical momenta are \([E_\tilde{T} = F_{\tilde{T}}\) and \(B_\tau = \frac{1}{2} \epsilon_{\tilde{T} \tilde{\lambda} \tilde{\mu}} F_{\tilde{T} \tilde{\lambda} \tilde{\mu}} \epsilon_{\tilde{T} \tilde{\lambda} \tilde{\mu}} = \epsilon_{\tilde{T} \tilde{\lambda} \tilde{\mu}} \) are the electric and magnetic fields respectively; for \(g_{\tilde{A} \tilde{B}} \rightarrow \eta_{\tilde{A} \tilde{B}}\) one gets \(\pi^\tau = -E_\tilde{T} = E_\tilde{T}\)

\[
\pi_\alpha(\tau, \bar{\sigma}) = \frac{\partial \mathcal{L}(\tau, \bar{\sigma})}{\partial (\partial_\tau \psi_\alpha)} = -\frac{i}{2} \left( \bar{\psi}(\tau, \bar{\sigma}) \gamma^\mu \right)_\alpha l_\mu(\tau, \bar{\sigma}), \\
\pi_\alpha(\tau, \bar{\sigma}) = \frac{\partial \mathcal{L}(\tau, \bar{\sigma})}{\partial (\partial_\bar{\tau} \psi_\alpha)} = -\frac{i}{2} (\gamma^\mu \psi(\tau, \bar{\sigma})) l_\mu(\tau, \bar{\sigma}), \\
\pi_\tau(\tau, \bar{\sigma}) = \frac{\partial \mathcal{L}(\tau, \bar{\sigma})}{\partial (\partial_\tau A_\tau)} = 0, \\
\pi_{\bar{\tau}}(\tau, \bar{\sigma}) = \frac{\partial \mathcal{L}(\tau, \bar{\sigma})}{\partial (\partial_{\bar{\tau}} A_{\bar{\tau}})} = -\frac{\gamma(\tau, \bar{\sigma})}{\sqrt{g(\tau, \bar{\sigma})}} \gamma^{\bar{\tau}}(\tau, \bar{\sigma}) (F_{\bar{\tau}} - g_{\bar{\tau} \bar{\bar{\tau}}} \gamma^\bar{\alpha} F_{\bar{\alpha} \bar{\beta}})(\tau, \bar{\sigma}) = \\
= \frac{\gamma(\tau, \bar{\sigma})}{\sqrt{g(\tau, \bar{\sigma})}} \gamma^{\bar{\tau}}(\tau, \bar{\sigma}) (E_{\bar{\tau}}(\tau, \bar{\sigma}) + g_{\bar{\tau} \bar{\bar{\tau}}} \gamma^\bar{\alpha} (\tau, \bar{\sigma}) \epsilon_{\bar{\alpha} \bar{\beta} \bar{\beta}} B_{\bar{\beta}}(\tau, \bar{\sigma})), \\
\rho_{\mu}(\tau, \bar{\sigma}) = -\frac{\partial \mathcal{L}(\tau, \bar{\sigma})}{\partial \bar{e}_\mu^\tilde{\alpha}} = l_\mu(\tau, \bar{\sigma}) \left\{ -\frac{i}{2} \gamma^{\bar{\tau}}(\tau, \bar{\sigma}) z_{\bar{\nu} \bar{\delta}}(\tau, \bar{\sigma}) \left[ \bar{\psi}(\tau, \bar{\sigma}) \gamma^\nu (\partial_\tau \psi(\tau, \bar{\sigma}) + \\
- \partial_\bar{\delta} \bar{\psi}(\tau, \bar{\sigma}) \gamma^\nu \psi(\tau, \bar{\sigma}) \right] + m \bar{\psi}(\tau, \bar{\sigma}) \psi(\tau, \bar{\sigma}) + \\
- \frac{1}{2\sqrt{\gamma(\tau, \bar{\sigma})}} \pi_\bar{\tau}(\tau, \bar{\sigma}) g^\tilde{\alpha} \pi_{\bar{\alpha}}(\tau, \bar{\sigma}) \pi^\bar{\tau}(\tau, \bar{\sigma}) + \\
+ \frac{\sqrt{\gamma(\tau, \bar{\sigma})}}{4} \gamma^{\bar{\tau}}(\tau, \bar{\sigma}) \gamma^\bar{\alpha} (\tau, \bar{\sigma}) F_{\bar{\alpha} \bar{\beta}}(\tau, \bar{\sigma}) F_{\bar{\bar{\beta} \bar{\beta}}}(\tau, \bar{\sigma}) + \\
- e \gamma^{\bar{\tau}}(\tau, \bar{\sigma}) z_{\bar{\nu} \bar{\delta}}(\tau, \bar{\sigma}) A_{\bar{\nu} \bar{\delta}}(\tau, \bar{\sigma}) \bar{\psi}(\tau, \bar{\sigma}) \gamma^\nu \psi(\tau, \bar{\sigma}) \right\} + 
\]
In the system, while the coefficient of $\mu H$, $\chi - \alpha (\int \int \tau, \sigma $ $+ F_\beta (\tau, \sigma) \pi^\alpha (\tau, \sigma) + e A_\beta (\tau, \sigma) \tilde{\psi} (\tau, \sigma) \gamma^\nu l_\nu (\tau, \sigma) \psi (\tau, \sigma)$).}

They satisfy the Poisson brackets

$$
\{\psi_\alpha (\tau, \sigma), \pi_\beta (\tau, \sigma')\} = \{\pi_\beta (\tau, \sigma'), \psi_\alpha (\tau, \sigma)\} = - \delta_\alpha \beta \delta (\sigma - \sigma'),
\{\tilde{\psi}_\alpha (\tau, \sigma), \bar{\pi}_\beta (\tau, \sigma')\} = \{\bar{\pi}_\beta (\tau, \sigma'), \tilde{\psi}_\alpha (\tau, \sigma)\} = - \delta_\alpha \beta \delta (\bar{\sigma} - \sigma'),
\{\gamma (\tau, \sigma), \rho_\alpha (\tau, \sigma')\} = - \eta_\alpha \beta \delta (\sigma - \sigma'),
\{A_\alpha (\tau, \sigma), \pi^\beta (\tau, \sigma')\} = \eta_\beta \delta (\sigma - \sigma').
$$

The primary constraints are

$$
\chi_\alpha (\tau, \sigma) = \pi_\alpha (\tau, \sigma) + \frac{i}{2} (\tilde{\psi} (\tau, \sigma) \gamma^\mu ) A_\mu (\tau, \sigma) \approx 0,
\bar{\chi}_\alpha (\tau, \sigma) = \bar{\pi}_\alpha (\tau, \sigma) + \frac{i}{2} (\gamma^\mu \psi (\tau, \sigma))_\alpha l_\mu (\tau, \sigma) \approx 0,
\pi^\tau (\tau, \sigma) \approx 0,
H_\mu (\tau, \sigma) = \rho_\mu (\tau, \sigma) - l_\mu (\tau, \sigma) \left\{\frac{- \frac{i}{2} \gamma^\mu (\tau, \sigma) z_{\nu \delta} (\tau, \sigma) [\tilde{\psi} (\tau, \sigma) \gamma^\nu \partial_\nu \psi (\tau, \sigma) + \tau, \sigma \] - \partial_\nu \tilde{\psi} (\tau, \sigma) + m \tilde{\psi} (\tau, \sigma) \psi (\tau, \sigma) + \frac{1}{2} \gamma (\tau, \sigma) g_{\mu \nu} (\tau, \sigma) \tilde{\psi} (\tau, \sigma) \psi (\tau, \sigma) + \frac{\sqrt{\gamma (\tau, \sigma)}}{4} \gamma^\mu (\tau, \sigma) \gamma^\nu (\tau, \sigma) F_\mu (\tau, \sigma) F_\nu (\tau, \sigma) + e \gamma^\mu (\tau, \sigma) z_{\nu \delta} (\tau, \sigma) A_\nu (\tau, \sigma) \tilde{\psi} (\tau, \sigma) \gamma^\nu \psi (\tau, \sigma) \right\} + \gamma^\mu \gamma (\tau, \sigma) z_{\mu \delta} (\tau, \sigma) \left[\frac{i}{2} l_\nu (\tau, \sigma) [\tilde{\psi} (\tau, \sigma) \gamma^\nu \partial_\nu \psi (\tau, \sigma) + \tau, \sigma \] - \partial_\nu \tilde{\psi} (\tau, \sigma) \right] + \bar{F}_\mu (\tau, \sigma) \pi^\mu (\tau, \sigma) + e A_\mu (\tau, \sigma) \tilde{\psi} (\tau, \sigma) \gamma^\nu l_\nu (\tau, \sigma) \psi (\tau, \sigma) \right\} \approx 0.
$$

In $H_\mu (\tau, \sigma) \approx 0$ the coefficient of $l_\mu (\tau, \sigma)$ is the energy density $T_\mu (\tau, \sigma)$ of the isolated system, while the coefficient of $z_{\mu \delta} (\tau, \sigma)$ is the 3-momentum density $T_\mu (\tau, \sigma)$.

The canonical and Dirac Hamiltonians are

$$
H^c = \int d^3 \sigma \left[ - \pi_\alpha (\tau, \sigma) \partial_\tau \psi_\alpha (\tau, \sigma) - \bar{\pi}_\alpha (\tau, \sigma) \partial_\tau \bar{\psi}_\alpha (\tau, \sigma) + \pi^\lambda (\tau, \sigma) \partial_\lambda A_\lambda (\tau, \sigma) + \rho_\mu (\tau, \sigma) z_{\nu \delta} (\tau, \sigma) - L (\tau, \sigma) \right] = - \int d^3 \sigma \Gamma (\tau, \sigma) A_\tau (\tau, \sigma),
H_D = \int d^3 \sigma \left[ - A_\tau (\tau, \sigma) \Gamma (\tau, \sigma) + \lambda^\mu (\tau, \sigma) H_\mu (\tau, \sigma) + \bar{a}_\alpha (\tau, \sigma) \chi_\alpha (\tau, \sigma) + \bar{\chi}_\alpha (\tau, \sigma) a_\alpha (\tau, \sigma) + \mu_\tau (\tau, \sigma) \pi^\tau (\tau, \sigma) \right],
$$
where $\lambda^\mu$, $\mu_\tau$, $a_\alpha$ are Dirac multipliers [$a_\alpha$ are odd multipliers].

The constraints $\chi_\alpha$, $\bar{\chi}_\alpha$, are second class, because they satisfy the Poisson brackets

$$\{\chi_\alpha(\tau, \vec{\sigma}), \bar{\chi}_\beta(\tau, \vec{\sigma}')\} = -i (\gamma^\mu l_\mu(\tau, \vec{\sigma}))_{\beta\alpha} \delta^3(\vec{\sigma} - \vec{\sigma}').$$

(11)

It is not convenient to go to Dirac brackets with respect to them at this stage, because
the fundamental variables would not have any more diagonal Dirac brackets; the
elimination of these constraints will be delayed until when the theory will be restricted to spacelike
hyperplanes.

The constraints $H_\mu(\tau, \vec{\sigma}) \approx 0$ imply that the description is independent from the choice
of the family of spacelike hypersurfaces used to foliate Minkowski spacetime. Since these
constraints do not have zero Poisson brackets with $\chi_\alpha$, $\bar{\chi}_\alpha$

$$\{H_\perp(\tau, \vec{\sigma}), \chi_\alpha(\tau, \vec{\sigma}')\} = i\gamma^\mu z_{\mu\alpha}(\tau, \vec{\sigma})(\partial^\tau \bar{\psi}(\tau, \vec{\sigma})\gamma^\mu)_{\alpha}(\vec{\sigma} - \vec{\sigma}'\rangle +$$

$$+ \frac{i}{2}(\bar{\psi}(\tau, \vec{\sigma})\gamma^\mu_{\alpha}\partial^{\tau}(\gamma^\tau z_{\mu\alpha})(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}')) +$$

$$+ m\psi(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}')) +$$

$$- e\gamma^\tau z_{\mu\alpha}(\tau, \vec{\sigma})A_\tau(\tau, \vec{\sigma})\bar{\psi}(\tau, \vec{\sigma})\gamma^\mu_{\alpha}\delta^3(\vec{\sigma} - \vec{\sigma}'),$$

$$\{H_\perp(\tau, \vec{\sigma}), \bar{\chi}_\alpha(\tau, \vec{\sigma}')\} = i\gamma^\mu z_{\mu\alpha}(\tau, \vec{\sigma})(\gamma^\mu\partial^\tau \psi(\tau, \vec{\sigma}))_{\alpha}\delta^3(\vec{\sigma} - \vec{\sigma}') +$$

$$+ \frac{i}{2}(\gamma^\mu \psi(\tau, \vec{\sigma}))_{\alpha}\partial^\tau(\gamma^\tau z_{\mu\alpha})(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}')) +$$

$$- m\psi(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}')) +$$

$$+ e\gamma^\tau z_{\mu\alpha}(\tau, \vec{\sigma})A_\tau(\tau, \vec{\sigma})(\gamma^\mu \psi(\tau, \vec{\sigma}))_{\alpha}\delta^3(\vec{\sigma} - \vec{\sigma}'),$$

$$\{H_\perp(\tau, \vec{\sigma}), \chi_\alpha(\tau, \vec{\sigma}')\} = -i\partial^{\tau}(\bar{\psi}(\tau, \vec{\sigma})\gamma^\mu)_{\alpha}l_\mu(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}') +$$

$$+ \frac{i}{2}(\bar{\psi}(\tau, \vec{\sigma})\gamma^\mu_{\alpha}l_\mu(\tau, \vec{\sigma}')(\vec{\sigma} - \vec{\sigma}')) +$$

$$+ eA_\tau(\tau, \vec{\sigma})(\bar{\psi}(\tau, \vec{\sigma})\gamma^\mu)_{\alpha}l_\mu(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}'),$$

$$\{H_\perp(\tau, \vec{\sigma}), \bar{\chi}_\alpha(\tau, \vec{\sigma}')\} = -i\partial^{\tau}(\gamma^\mu \psi(\tau, \vec{\sigma}))_{\alpha}l_\mu(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}') +$$

$$+ \frac{i}{2}(\gamma^\mu \psi(\tau, \vec{\sigma}))_{\alpha}l_\mu(\tau, \vec{\sigma})(\vec{\sigma} - \vec{\sigma}')) +$$

$$- eA_\tau(\tau, \vec{\sigma})(\gamma^\mu \psi(\tau, \vec{\sigma}))_{\alpha}l_\mu(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}'),$$

(12)

where

$$H_\perp(\tau, \vec{\sigma}) \equiv l^\mu(\tau, \vec{\sigma})H_\mu(\tau, \vec{\sigma}) \approx 0,$$

$$H_\perp(\tau, \vec{\sigma}) \equiv z^\mu(\tau, \vec{\sigma})H_\mu(\tau, \vec{\sigma}) \approx 0,$$

(13)

it is convenient to introduce the new constraints

$$H'_\mu(\tau, \vec{\sigma}) = H_\mu(\tau, \vec{\sigma}) - \int d^3u \{H_\mu(\tau, \vec{u}), \bar{\chi}_\beta(\tau, \vec{u})\} i(\gamma^\mu l_\mu(\tau, \vec{u}))_{\alpha\beta} \chi_\alpha(\tau, \vec{u}) +$$

$$- \int d^3u \{H_\mu(\tau, \vec{u}), \chi_\beta(\tau, \vec{u})\} i(\gamma^\mu l_\mu(\tau, \vec{u}))_{\beta\alpha} \bar{\chi}_\alpha(\tau, \vec{u}) \approx H_\mu(\tau, \vec{\sigma}) \approx 0,$$

$$\{H'_\mu(\tau, \vec{\sigma}), \chi_\alpha(\tau, \vec{\sigma}')\} \approx 0, \quad \{H'_\mu(\tau, \vec{\sigma}), \bar{\chi}_\alpha(\tau, \vec{\sigma}')\} \approx 0,$$

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Let us define
\[
\{H^*_\mu(\tau, \sigma), H^*_\nu(\tau, \sigma')\} \approx \left\{ l_\mu(\tau, \sigma) z_{\mu\nu}(\tau, \sigma) - l_\nu(\tau, \sigma) z_{\nu\mu}(\tau, \sigma) \right\} \frac{\pi^F(\tau, \sigma)}{\sqrt{\gamma(\tau, \sigma)}} +
- z_{\mu\nu}(\tau, \sigma) \gamma^{\delta\alpha}(\tau, \sigma) F_{\alpha\beta}(\tau, \sigma) \gamma^{\beta\epsilon}(\tau, \sigma) z_{\delta\nu}(\tau, \sigma)
\cdot \Gamma(\tau, \sigma) \delta^3(\sigma - \sigma') \approx 0.
\]  
where \(\Gamma(\tau, \sigma) \approx 0\), the Gauss law constraint, will be defined in the next equation. Therefore the constraints \(H^*_\mu(\tau, \sigma) \approx 0\) are constants of the motion and first class.

The time constancy of \(\pi^\tau(\tau, \sigma) \approx 0\), i.e. \(\partial_\tau \pi^\tau(\tau, \sigma) \overset{\approx}{=} \{\pi^\tau(\tau, \sigma), H_D\} \approx 0 [\overset{\approx}{=} \text{means evaluated on the solution of the equations of motion}]\), gives the Gauss law secondary constraint
\[
\Gamma(\tau, \sigma) = \partial_\tau \pi^F(\tau, \sigma) + e\psi(\tau, \sigma) \gamma^\mu l_\mu(\tau, \sigma) \psi(\tau, \sigma) \approx 0.
\]
Since we have
\[
\{\Gamma(\tau, \sigma), \chi_\alpha(\tau, \sigma')\} = -ie\bar{\psi}_\beta(\tau, \sigma)(\gamma^\mu l_\mu(\tau, \sigma))_{\beta\alpha} \delta^3(\sigma - \sigma'),
\{\Gamma(\tau, \sigma), \bar{\chi}_\alpha(\tau, \sigma')\} = ie\bar{\chi}_\alpha(\tau, \sigma) \delta^3(\sigma - \sigma'),
\{\Gamma(\tau, \sigma), H^*_\mu(\tau, \sigma')\} \approx 0,
\{\Gamma(\tau, \sigma), \pi^\tau(\tau, \sigma')\} \approx 0,
\{\Gamma(\tau, \sigma), \pi^\tau(\tau, \sigma')\} = 0.
\]
Let us define
\[
\Gamma^*(\tau, \sigma) \equiv \Gamma(\tau, \sigma) + ie[\chi_\alpha \bar{\psi}_\alpha + \bar{\psi}_\alpha \bar{\chi}_\alpha](\tau, \sigma) \approx 0.
\]
This new constraint satisfies
\[
\{\Gamma^*(\tau, \sigma), \chi_\alpha(\tau, \sigma')\} = -ie\chi_\alpha(\tau, \sigma) \delta^3(\sigma - \sigma'),
\{\Gamma^*(\tau, \sigma), \bar{\chi}_\alpha(\tau, \sigma')\} = ie\bar{\chi}_\alpha(\tau, \sigma) \delta^3(\sigma - \sigma'),
\{\Gamma^*(\tau, \sigma), H^*_\mu(\tau, \sigma')\} \approx 0,
\{\Gamma^*(\tau, \sigma), \pi^\tau(\tau, \sigma')\} \approx 0,
\{\Gamma^*(\tau, \sigma), \pi^\tau(\tau, \sigma')\} = 0.
\]
Therefore, \(\Gamma^*(\tau, \sigma) \approx 0\) is a constant of motion and a first class constraint like \(\pi^\tau(\tau, \sigma) \approx 0\).

The time constancy of \(\chi_\alpha, \bar{\chi}_\alpha\)
\[
\partial_\tau \chi_\alpha(\tau, \sigma) \overset{\approx}{=} \{\chi_\alpha(\tau, \sigma), H_D\} = -eA_\tau(\tau, \sigma)(\bar{\psi}(\tau, \sigma) \gamma^\mu l_\mu(\tau, \sigma) +
+i\bar{\alpha}_\beta(\tau, \sigma)(\gamma^\mu l_\mu(\tau, \sigma))_{\beta\alpha} \approx 0,
\partial_\tau \bar{\chi}_\alpha(\tau, \sigma) \overset{\approx}{=} \{\bar{\chi}_\alpha(\tau, \sigma), H_D\} = eA_\tau(\tau, \sigma) l_\mu(\tau, \sigma)(\gamma^\mu \psi(\tau, \sigma))_{\alpha} -
-ia_\beta(\tau, \sigma)(\gamma^\mu l_\mu(\tau, \sigma))_{\alpha\beta} \approx 0,
\]
gives
\[
\bar{a}_\alpha(\tau, \sigma) \chi_\alpha(\tau, \sigma) + a_\alpha(\tau, \sigma) \bar{\chi}_\alpha(\tau, \sigma) \approx -ieA_\tau(\tau, \sigma)(\chi_\alpha \bar{\psi}_\alpha + \bar{\psi}_\alpha \bar{\chi}_\alpha)(\tau, \sigma),
\]
so that the final Dirac Hamiltonian is
\[
H_D = \int d^3\sigma \left[ -A_\tau(\tau, \sigma) \Gamma^*(\tau, \sigma) + \lambda^\mu(\tau, \sigma) H^*_\mu(\tau, \sigma) + \mu_\tau(\tau, \sigma) \pi^\tau(\tau, \sigma) \right]
\]
in which only the first class constraints \( \mathcal{H}_\mu^*, \pi^\tau, \Gamma^* \) appear.

One can show that the 10 Poincaré generators

\[
\begin{align*}
p_{s\mu} &= \int d^3 \sigma \rho_\mu(\tau, \sigma), \\
J^{\mu\nu} &= \int d^3 \sigma \left[ z^\mu(\tau, \sigma) \rho^\nu(\tau, \sigma) - z^\nu(\tau, \sigma) \rho^\mu(\tau, \sigma) \right] + \\
&\quad + \frac{i}{2} \int d^3 \sigma \left[ \pi(\tau, \sigma) \sigma^{\mu\nu} \psi(\tau, \sigma) + \bar{\psi}(\tau, \sigma) \sigma^{\mu\nu} \bar{\pi}(\tau, \sigma) \right],
\end{align*}
\]

and the electric charge [see Ref. [1] for the boundary conditions on the fields and for the extraction of this weak charge from the Gauss law first class constraint]

\[
Q = -e \int d^3 \sigma \left[ \bar{\psi}(\tau, \sigma) \gamma^\mu l_\mu(\tau, \sigma) \psi(\tau, \sigma) \right],
\]

are constants of the motion.

Let us note that like \( \gamma^\alpha \) the matrix \( \gamma^\mu l_\mu(\tau, \sigma) \) satisfies \( [\gamma^\mu l_\mu(\tau, \sigma)]^2 = I \) and that we have \( \{z^\mu(\tau, \sigma), p_{s\nu}\} = -\eta^{\mu\nu} \).
III. RESTRICTION TO SPACELIKE HYPERPLANES.

As in Ref. [8], let us restrict ourselves to spacelike hyperplanes $\Sigma_{rH}$ by adding the gauge fixings

\[\zeta^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) - x_s^\mu(\tau) - b_s^\mu(\tau)\sigma^s \cong 0,\]

\[\{\zeta^\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma}')\} = -\eta_\nu^\mu\delta^3(\vec{\sigma} - \vec{\sigma}'),\]

(24)

and the Dirac brackets

\[\{A, B\}^* = \{A, B\} - \int d^3\sigma \left[\{A, \zeta^\mu(\tau, \vec{\sigma})\}\{\mathcal{H}_\mu(\tau, \vec{\sigma}), B\} + \right.\]

\[\left. - \{A, \mathcal{H}_\nu(\tau, \vec{\sigma})\}\{\zeta^\mu(\tau, \vec{\sigma}), B\}\right].\]

(25)

The hyperplane $\Sigma_{rH}$ is described by 10 configuration variables: an origin $x_s^\mu(\tau)$ and the 6 independent degrees of freedom in an orthonormal tetrad $b_s^\mu(\tau)$ $[b_A^\mu \eta_{\mu\nu} b_B^\nu = \eta_{AB}]$ with $b_s^\mu = b^\mu$, where $b^\mu$ is the $\tau$-independent normal to the hyperplane. Now, we have $z_s^\mu(\tau, \vec{\sigma}) \equiv b_s^\mu(\tau)$, $z_s^\mu(\tau, \vec{\sigma}) \equiv x_s^\mu(\tau) + b_s^\mu(\tau)\sigma^s$, $g_{\tau\vec{\sigma}}(\tau, \vec{\sigma}) \equiv -\delta_{\tau\vec{\sigma}}$, $\gamma^\mu(\tau, \vec{\sigma}) = \det g_{\tau\vec{\sigma}}(\tau, \vec{\sigma}) \equiv 1$. The nonvanishing Dirac brackets of the variables $x_s^\mu, p_s^\mu, b_A^\mu, S_s^{\mu\nu}, A_A, \pi_A$, are $[C^{\gamma \delta \alpha \beta} = \eta_\gamma^\nu \eta_\delta^\mu H^{\alpha \beta} + \eta_\gamma^\mu \eta_\delta^\nu H^{\alpha \beta} - \eta_\gamma^\nu \eta_\delta^\mu H^{\alpha \beta} - \eta_\gamma^\mu \eta_\delta^\nu H^{\alpha \beta}]$ are the structure constants of the Lorentz group

\[\{x_s^\mu(\tau), p_s^\nu(\tau)\}^* = -\eta^{\mu\nu},\]

\[\{S_s^{\mu\nu}(\tau), b_A^\mu(\tau)\}^* = \eta^{\rho\nu}b_A^\rho(\tau) - \eta^{\rho\mu}b_A^\rho(\tau),\]

\[\{S_s^{\mu\nu}(\tau), S_A^{\alpha\beta}(\tau)\}^* = C^{\mu\nu\alpha\beta} S_A^{\gamma\delta}(\tau).\]

(26)

While $p_s^\mu$ is the momentum conjugate to $x_s^\mu$, the 6 independent momenta conjugate to the 6 degrees of freedom in the $b_s^\mu$ are hidden in $S_s^{\mu\nu}$, which is a component of the angular momentum tensor

\[J^{\mu\nu} = L_s^{\mu\nu} + S_s^{\mu\nu} + \psi_s^{\mu\nu},\]

\[L_s^{\mu\nu} = x_s^\mu(\tau)p_s^\nu(\tau) - x_s^\nu(\tau)p_s^\mu(\tau),\]

\[S_s^{\mu\nu} = b_s^\rho(\tau) \int d^3\sigma \sigma^\rho(\tau, \vec{\sigma}) - b_s^\rho(\tau) \int d^3\sigma \sigma^\rho(\tau, \vec{\sigma}),\]

\[\psi_s^{\mu\nu} = \frac{i}{2} \int d^3\sigma \left[\pi(\tau, \vec{\sigma})\sigma^{\mu\nu}\psi(\tau, \vec{\sigma}) + \bar{\psi}(\tau, \vec{\sigma})\sigma^{\mu\nu}\bar{\pi}(\tau, \vec{\sigma})\right],\]

\[\{J^{\mu\nu}, J^{\alpha\beta}\}^* = C^{\mu\nu\alpha\beta} J^{\gamma\delta}, \quad \{L_s^{\mu\nu}, L_s^{\alpha\beta}\}^* = C^{\mu\nu\alpha\beta} L_s^{\gamma\delta}, \quad \{S_s^{\mu\nu}, S_s^{\alpha\beta}\}^* = C^{\mu\nu\alpha\beta} S_s^{\gamma\delta}, \quad \{S_s^{\mu\nu}, S_\xi^{\alpha\beta}\}^* = C^{\mu\nu\alpha\beta} S_\xi^{\gamma\delta}.\]

(27)

The relations

\[\{S_s^{\mu\nu}, \psi_\alpha(\tau, \vec{\sigma})\}^* = \frac{i}{2} (\sigma^{\mu\nu}\psi(\tau, \vec{\sigma}))_\alpha,\]

\[\{S_s^{\mu\nu}, \bar{\psi}_\alpha(\tau, \vec{\sigma})\}^* = -\frac{i}{2} (\bar{\psi}(\tau, \vec{\sigma})\sigma^{\mu\nu})_\alpha,\]

(28)
satisfies a Lorentz algebra, since we have brackets. Now only the total spin $b$ and can be eliminated by introducing the Dirac brackets of the variables $\tilde{\lambda}^{\mu}(\tau)$, the Dirac Hamiltonian becomes

$$H_D^F = \tilde{\lambda}^{\mu}(\tau)\tilde{H}_\mu^*(\tau) - \frac{1}{2}\tilde{\lambda}^{\mu\nu}(\tau)\tilde{H}_{\mu\nu}^*(\tau) + \int d^3\sigma \left[ -A_\tau(\tau, \vec{\sigma})\Gamma^\tau(\tau, \vec{\sigma}) + \mu_\tau(\tau, \vec{\sigma})\pi^\tau(\tau, \vec{\sigma}) \right],$$

where

$$\tilde{H}_\mu^*(\tau) = \int d^3\sigma \mathcal{H}_\mu^*(\tau, \vec{\sigma}) \approx 0,$$

$$\tilde{H}_{\mu\nu}^*(\tau) = b_{\mu\tau}(\tau) \int d^3\sigma \sigma^\nu \mathcal{H}_\nu^*(\tau, \vec{\sigma}) - b_{\nu\tau}(\tau) \int d^3\sigma \sigma^\mu \mathcal{H}_\mu^*(\tau, \vec{\sigma}) \approx 0.$$ \hspace{1cm} (30)

On $\Sigma_{rH}$ the second class constraints have the form

$$\chi_\alpha(\tau, \vec{\sigma}) = \pi_\alpha(\tau, \vec{\sigma}) + \frac{i}{2}(\hat{\psi}(\tau, \vec{\sigma})\gamma^\mu)_{\alpha\beta}b_{\beta\tau} \approx 0,$$

$$\bar{\chi}_\alpha(\tau, \vec{\sigma}) = \bar{\pi}_\alpha(\tau, \vec{\sigma}) + \frac{i}{2}(\gamma^\mu\psi(\tau, \vec{\sigma}))_{\alpha\beta}b_{\beta\tau} \approx 0,$$

$$\{\chi_\alpha(\tau, \vec{\sigma}), \bar{\chi}_{\beta}(\tau, \vec{\sigma})\}_* = -i(\gamma^\mu b_{\mu\tau})_{\beta\alpha}\delta^\tau_\beta(\vec{\sigma} - \vec{\sigma}'),$$

$$\left[(\gamma^\mu b_{\mu\tau})^2 = I\right].$$ \hspace{1cm} (31)

and can be eliminated by introducing the Dirac brackets

$$\{A, B\}_D^* = \{A, B\}^* - \int d^3u \{A, \chi_\alpha(\tau, \vec{u})\}_*i(\gamma^\mu b_{\mu\tau})_{\alpha\beta}\{\bar{\chi}_{\beta}(\tau, \vec{u}), B\}_* +$$

$$- \int d^3u \{A, \bar{\chi}_\alpha(\tau, \vec{u})\}_*i(\gamma^\mu b_{\mu\tau})_{\beta\alpha}\{\chi_{\beta}(\tau, \vec{u}), B\}_*.$$ \hspace{1cm} (32)

Now we get $\tilde{H}_\mu^*(\tau) \equiv H_\mu(\tau) = \int d^3\sigma \mathcal{H}_\mu(\tau, \vec{\sigma})$, $\tilde{H}_{\mu\nu}^*(\tau) \equiv \tilde{H}_{\mu\nu}(\tau) = b_{\mu\tau}(\tau) \int d^3\sigma \sigma^\nu \mathcal{H}_{\nu\tau}^*(\tau, \vec{\sigma}) - b_{\nu\tau}(\tau) \int d^3\sigma \sigma^\mu \mathcal{H}_{\mu\tau}^*(\tau, \vec{\sigma})$,

$$S_\psi^{\mu\nu} \equiv \frac{1}{4} \int d^3\sigma b_{\mu\tau}(\tau, \vec{\sigma})[\gamma^\rho, \sigma^{\mu\nu}]_{\rho\tau}\psi(\tau, \vec{\sigma}),$$

and

$$\{\psi_\alpha(\tau, \vec{\sigma}), \bar{\psi}_{\beta}(\tau, \vec{\sigma}')\}^*_D = -i(\gamma^\mu b_{\mu\tau})_{\alpha\beta}\delta^\tau_\beta(\vec{\sigma} - \vec{\sigma}'),$$

while the Dirac brackets of the variables $x_{s\tau}^\mu$, $p_{s\tau}^\mu$, $b_{A\tau}^\mu$, $A_A$, $\pi^{\vec{A}}$ are left unaltered by the new brackets. Now only the total spin

$$S^{\mu\nu} = S_\psi^{\mu\nu} + S_\psi^{\mu\nu},$$ \hspace{1cm} (35)

satisfies a Lorentz algebra, since we have
\{S^{\mu\nu}, S^{\alpha\beta}\}^{\ast}_{D} = C_{\gamma\delta}^{\mu\nu\alpha\beta} S^{\gamma\delta},
\{S^{\mu\nu}, \psi_{\alpha}(\tau, \vec{\sigma})\}^{\ast}_{D} = \frac{i}{2}(\sigma^{\mu\nu}\psi(\tau, \vec{\sigma}))_{\alpha},
\{S^{\mu\nu}, \bar{\psi}_{\alpha}(\tau, \vec{\sigma})\}^{\ast}_{D} = -\frac{i}{2}(\bar{\psi}(\tau, \vec{\sigma})\sigma^{\mu\nu})_{\alpha},
\{L^{\mu\nu}_{s}, L^{\alpha\beta}_{s}\}^{*}_{D} = \{L^{\mu\nu}_{s}, L^{\alpha\beta}_{s}\}^{*} = C_{\gamma\delta}^{\mu\nu\alpha\beta} L^{\gamma\delta}_{s},
\{L^{\mu\nu}_{s}, S^{\alpha\beta}_{s}\}^{*}_{D} = \{L^{\mu\nu}_{s}, S^{\alpha\beta}_{s}\}^{*} = 0,
\{J^{\mu\nu}, J^{\alpha\beta}\}^{*}_{D} = \{J^{\mu\nu}, J^{\alpha\beta}\}^{*} = C_{\gamma\delta}^{\mu\nu\alpha\beta} J^{\gamma\delta}.
(36)

The Dirac Hamiltonian becomes
\[ H_{D} = \tilde{\lambda}^{\alpha}(\tau) \tilde{H}_{\alpha}(\tau) - \frac{1}{2} \tilde{\lambda}^{\mu\nu}(\tau) \tilde{H}_{\mu\nu}(\tau) + \int d^{3}\sigma \left[ - A_{\tau}(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma}) + \mu_{\tau}(\tau, \vec{\sigma}) \pi^{\tau}(\tau, \vec{\sigma}) \right], \]
(37)
and contains all the first class constraints
\[ \Gamma(\tau, \vec{\sigma}) = \partial_{\tau} \pi^{\tau}(\tau, \vec{\sigma}) + e \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\mu} b_{\mu \tau} \psi(\tau, \vec{\sigma}) \approx 0, \]
\[ \pi^{\tau}(\tau, \vec{\sigma}) \approx 0, \]
\[ \tilde{H}_{\mu}(\tau) = \int d^{3}\sigma H_{\mu}(\tau, \vec{\sigma}) = p_{\mu \tau} - b_{\mu \tau} \int d^{3}\sigma \left[ \frac{i}{2} b_{\nu \tau}(\tau) \left( \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} \partial_{\tau} \psi(\tau, \vec{\sigma}) + \right. \right. \]
\[ - \partial_{\tau} \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\tau} \psi(\tau, \vec{\sigma}) \left. \left. \right) + m \bar{\psi}(\tau, \vec{\sigma}) \psi(\tau, \vec{\sigma}) + \frac{\pi^{2}(\tau, \vec{\sigma}) + \bar{B}^{2}(\tau, \vec{\sigma})}{2} + \right. \]
\[ + e b_{\nu \tau}(\tau) A_{\nu}(\tau, \vec{\sigma}) \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} \psi(\tau, \vec{\sigma}) \right] + \]
\[ + b_{\mu \tau}(\tau) \int d^{3}\sigma \left[ \frac{i}{2} b_{\nu \tau}(\tau) \left( \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} \partial_{\tau} \psi(\tau, \vec{\sigma}) + \right. \right. \]
\[ - \partial_{\tau} \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\tau} \psi(\tau, \vec{\sigma}) \left. \left. \right) + \left( \bar{\pi}(\tau, \vec{\sigma}) \wedge \bar{B}(\tau, \vec{\sigma}) \right) \right] + \]
\[ + e A_{\nu}(\tau, \vec{\sigma}) \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} b_{\nu \tau} \psi(\tau, \vec{\sigma}) \right) \approx 0, \]
\[ \tilde{H}^{\mu\nu}(\tau) = \int d^{3}\sigma b_{\mu}(\tau) \sigma^{\tau} H^{\nu}(\tau, \vec{\sigma}) - \int d^{3}\sigma b_{\nu}(\tau) \sigma^{\tau} H^{\mu}(\tau, \vec{\sigma}) = \]
\[ = S^{\mu\nu}(\tau) - S^{\nu\tau}(\tau) + \]
\[ - \left( b_{\mu}(\tau) b_{\nu}(\tau) - b_{\nu}(\tau) b_{\mu}(\tau) \right) \int d^{3}\sigma \sigma^{\tau} \left[ \frac{i}{2} b_{\nu \tau}(\tau) \left( \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} \partial_{\tau} \psi(\tau, \vec{\sigma}) + \right. \right. \]
\[ - \partial_{\tau} \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\tau} \psi(\tau, \vec{\sigma}) \left. \left. \right) + m \bar{\psi}(\tau, \vec{\sigma}) \psi(\tau, \vec{\sigma}) + \frac{\pi^{2}(\tau, \vec{\sigma}) + \bar{B}^{2}(\tau, \vec{\sigma})}{2} + \right. \]
\[ + e b_{\nu \tau}(\tau) A_{\nu}(\tau, \vec{\sigma}) \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} \psi(\tau, \vec{\sigma}) \right] + \]
\[ + \left( b_{\mu}(\tau) b_{\nu}(\tau) - b_{\nu}(\tau) b_{\mu}(\tau) \right) \int d^{3}\sigma \sigma^{\tau} \left[ \frac{i}{2} b_{\nu \tau}(\tau) \left( \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} \partial_{\tau} \psi(\tau, \vec{\sigma}) + \right. \right. \]
\[ - \partial_{\tau} \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\tau} \psi(\tau, \vec{\sigma}) \left. \left. \right) + \left( \bar{\pi}(\tau, \vec{\sigma}) \wedge \bar{B}(\tau, \vec{\sigma}) \right) \right] + \]
\[ + e A_{\nu}(\tau, \vec{\sigma}) \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\nu} b_{\nu \tau} \psi(\tau, \vec{\sigma}) \right) \approx 0, \]
(38)
with [see Eq.(37)]
\{H^*_\mu(\tau, \vec{\sigma}), H^*_\nu(\tau, \vec{\sigma}')\} = \{H^*_\mu(\tau, \vec{\sigma}), H^*_\nu(\tau, \vec{\sigma}')\}^* \approx \{H_\mu(\tau, \vec{\sigma}), H_\nu(\tau, \vec{\sigma}')\}^* D \approx 0. \quad (39)
IV. THE REST-FRAME DESCRIPTION ON WIGNER'S HYPERPLANES.

The next step [8] is to select all the configurations of the isolated system which are timelike, namely with $p_s^2 > 0$. For them we can boost at rest with the standard Wigner boost $L^0_s(\vec{p}_s, p_s)$ for timelike Poincaré orbits all the variables of the noncanonical basis $x_s^\mu(\tau)$, $p_s^\mu$, $b_A^\mu(\tau)$, $S_{\mu\nu}^s(\tau)$, $A_A(\tau, \vec{\sigma})$, $\pi^A(\tau, \vec{\sigma})$, $\psi(\tau, \vec{\sigma})$, $\bar{\psi}(\tau, \vec{\sigma})$ with Lorentz indices (except $p_s^\mu$).

Let us now introduce the new variables [see Ref. [8] and Appendix C; they are obtained with a canonical transformation $e^{(\mathcal{F}(p_s^0))}$, whose generator is given in Eq.(C3)]

$$
\begin{align*}
   b_B^A(\tau) &= \epsilon^A_\mu(u(p_s))b_B^\mu(\tau), \\
   \ddot{x}_s^\mu(\tau) &= x_s^\mu(\tau) - \frac{1}{2} \epsilon^A_\mu(u(p_s)) \frac{\partial \epsilon^B_\nu(u(p_s))}{\partial p_{s\mu}} S^{\nu\rho} = \\
   &= x_s^\mu(\tau) - \frac{1}{\eta_s \sqrt{p_s^2(p^0_s + \eta_s \sqrt{p_s^2})}} \left[ p_{s\nu} S^{\nu\mu} + \eta_s \sqrt{p_s^2} (S^{0\mu} - S^{0\nu} p_{s\nu} p_s^\mu / p_s^2) \right] = \\
   &= x_s^\mu(\tau) - \frac{1}{\eta_s \sqrt{p_s^2(p^0_s + \eta_s \sqrt{p_s^2})}} \left[ 2 \eta_s \sqrt{p_s^2(p^0_s + \eta_s \sqrt{p_s^2})} \ddot{S}^{\mu\nu} + \eta_s \sqrt{p_s^2} (S^{0\mu} - S^{0\nu} p_{s\nu} p_s^\mu / p_s^2) \right], \\
   p_s^\mu &= p_s^\mu, \quad A_A(\tau, \vec{\sigma}) = A_A(\tau, \vec{\sigma}), \quad \pi^A(\tau, \vec{\sigma}) = \pi^A(\tau, \vec{\sigma}), \\
   \ddot{S}^{\mu\nu} &= S^{\mu\nu} + \frac{1}{2} \epsilon^A_\mu(u(p_s)) \epsilon^B_\nu(u(p_s)) \frac{\partial \epsilon^B_\nu(u(p_s))}{\partial p_{s\mu}} = S^{\mu\nu} + \frac{1}{2} \epsilon^A_\mu(u(p_s)) \epsilon^B_\nu(u(p_s)) \frac{\partial \epsilon^B_\nu(u(p_s))}{\partial p_{s\mu}} S^{\nu\rho} = \\
   &= S^{\mu\nu} + \frac{1}{\eta_s \sqrt{p_s^2(p^0_s + \eta_s \sqrt{p_s^2})}} \left[ p_{s\beta} (S^{\beta\mu} p_s^\nu - S^{\beta\nu} p_s^\mu) + \eta_s \sqrt{p_s^2} (S^{0\mu} p_s^\nu - S^{0\nu} p_s^\mu) \right], \\
   \dot{\psi}(\tau, \vec{\sigma}) &= S(L(p_s^0, p_s)) \psi(\tau, \vec{\sigma}), \\
   \ddot{\psi}(\tau, \vec{\sigma}) &= \ddot{\psi}(\tau, \vec{\sigma}) S^{-1}(L(p_s^0, p_s)),
\end{align*}
$$

where $\eta_s = \text{sign } p_s^0$ and

$$
\ddot{S}^{AB} = \epsilon^A_\mu(u(p_s)) \epsilon^B_\nu(u(p_s)) S^{\mu\nu},
$$

the “rest-frame spin tensor” (or “Thomas spin tensor”) [the $\ddot{S}^{AB}$’s satisfy a Lorentz algebra].

Now the Lorentz generators are

$$
\ddot{J}^{\mu\nu} = \ddot{L}^{\mu\nu} + \ddot{S}^{\mu\nu} = \ddot{x}_s^\mu p_s^\nu - \ddot{z}_s^\mu p_s^\nu + \ddot{S}^{\mu\nu}.
$$

The new variables have the following Dirac brackets [see Appendix C]

$$
\begin{align*}
   \{ \ddot{x}_s^\mu, p_s^\nu \}_D &= -\eta^{\mu\nu}, \\
   \{ \ddot{x}_s^\mu, \dot{\psi}(\tau, \vec{\sigma}) \}_D &= \{ \ddot{x}_s^\mu, \ddot{\psi}(\tau, \vec{\sigma}) \}_D = \{ \ddot{x}_s^\mu, \ddot{x}_s^\nu \}_D = \{ \ddot{x}_s^\mu, b_A^\nu \}_D = 0, \\
   \{ \ddot{S}^{0i}, b_A^\nu \}_D &= \frac{\delta^{0i} (p_s^\nu b_A^\mu - p_s^\mu b_A^\nu)}{(p_s^0 + \eta_s \sqrt{p_s^2})},
\end{align*}
$$
\[ \{ \tilde{S}^{ij}, b^A_\alpha \}_D = (\delta^{ir} \delta^{js} - \delta^{is} \delta^{jr}) b^A_i, \]
\[ \{ \tilde{S}^{\mu \nu}, \tilde{S}^{\alpha \beta} \}_D = C_{\mu \nu \alpha \beta} \tilde{S}^{\alpha \beta}, \]
\[ \{ \tilde{x}^\mu_\alpha, \tilde{S}^{\alpha \beta} \}_D = -\frac{1}{(p_\mu^0 + \eta_s \sqrt{p_\mu^2})} \left[ \eta^{\mu \alpha} \tilde{S}^{\beta j} + \frac{(p_\mu^0 + \eta_s \sqrt{p_\mu^2}) \tilde{S}^{ik} p_\nu^k}{\eta_s \sqrt{p_\mu^2}} \right], \]
\[ \{ \tilde{x}^\mu_\alpha, \tilde{x}^\nu_\beta \}_D = 0, \]
\[ \{ \psi^\mu_\alpha (\tau, \vec{s}), \psi^\nu_\beta (\tau, \vec{s'}) \}_D = -ib^A_\alpha (\tau) \eta_{\lambda \sigma} (\gamma^\sigma)_{\alpha \beta} \delta^3 (\vec{s} - \vec{s'}), \]
\[ \{ \tilde{S}^{\mu \nu}, \tilde{\psi}^\alpha (\tau, \vec{s}) \}_D = \frac{i}{2} \left[ L^\mu_{\alpha}(p_s, \vec{p}_s) L^\nu_{\beta}(p_s, \vec{p}_s) - \frac{1}{2} \epsilon^A_\rho (u(p_s)) \eta_{AB} \left( \frac{\partial \epsilon^B_\sigma (u(p_s))}{\partial p_{s\mu}} p_\nu^\sigma + \frac{\partial \epsilon^B_\sigma (u(p_s))}{\partial p_{s\nu}} p_\mu^\sigma - \frac{\partial \epsilon^B_\sigma (u(p_s))}{\partial p_{s\mu}} p_\mu^\sigma \right) \right] \psi^\beta (\tau, \vec{s}), \]
\[ \{ \tilde{S}^{\mu \nu}, \tilde{\psi}^\alpha (\tau, \vec{s}) \}_D = -\frac{i}{2} \tilde{\psi}^\alpha (\tau, \vec{s}) \left[ L^\mu_{\alpha}(p_s, \vec{p}_s) L^\nu_{\beta}(p_s, \vec{p}_s) - \frac{1}{2} \epsilon^A_\rho (u(p_s)) \eta_{AB} \right] \sigma^{\alpha \beta} \tilde{\psi}^\beta (\tau, \vec{s}). \]

Moreover, we have
\[ \{ \tilde{S}^{\alpha \beta}, \tilde{S}^{\gamma \delta} \}_D = C^{\alpha \beta \gamma \delta}_{EF} \tilde{S}^{\gamma \delta}, \]
\[ \{ \tilde{S}^{\alpha \beta}, \tilde{\psi}^\alpha (\tau, \vec{s}) \}_D = \frac{i}{2} \delta^A_\alpha \delta^B_\beta \sigma^{\alpha \beta} \tilde{\psi}^\beta (\tau, \vec{s}), \]
\[ \{ \tilde{S}^{\alpha \beta}, \tilde{\psi}^\gamma (\tau, \vec{s}) \}_D = -\frac{i}{2} \tilde{\psi}^\gamma (\tau, \vec{s}) \delta^A_\alpha \delta^B_\beta \sigma^{\alpha \beta}. \]

The new canonical origin \( \tilde{x}^\mu_\alpha (\tau) \) is not covariant, since under a Poincaré transformation \((a, A)\) it transforms as [8]
\[ \tilde{x}^\mu_\alpha (a, A) \rightarrow \tilde{x}^\mu_\alpha = A_\mu^\nu \left[ \tilde{x}^\nu + \frac{1}{2} \tilde{S}_{rs} R^r_{\mu}(A, p_s) \frac{\partial}{\partial p_{s\nu}} R^s_{\mu}(A, p) \right] + a_\mu. \]

As shown in Ref. [8], we can restrict ourselves to the Wigner hyperplane \( \Sigma_{\tau W} \) with \( l^\mu = u^\mu (p_s) \) [i.e. orthogonal to \( p^\mu_\alpha \)] with the gauge fixings
\[ T^\mu_\alpha (\tau) = b^\mu_\alpha (\tau) - \epsilon^\mu_\alpha (u(p_s)) \approx 0, \]
\[ \Rightarrow b^\mu_\alpha (\tau) = \epsilon^\mu_\alpha (u(p_s)) b^\mu_\alpha (\tau) \approx \eta^\alpha_\alpha, \]
whose time constancy implies \( \tilde{\lambda}^{\mu \nu} (\tau) \approx 0 \). After having introduced the Dirac brackets
\[ \{ A, B \}^*_D = \{ A, B \}_D - \frac{1}{4} \left[ \{ A, \tilde{H}^{\delta \beta} \}_D (\eta_{\gamma \sigma} \epsilon^\beta_\delta (u(p_s)) - \eta_{\delta \sigma} \epsilon^\gamma_\beta (u(p_s))) \{ T^\beta_\gamma, B \}^*_D + \{ A, T^\beta_\gamma \}_D (\eta_{\gamma \sigma} \epsilon^\beta_\delta (u(p_s)) - \eta_{\delta \sigma} \epsilon^\gamma_\beta (u(p_s))) \{ \tilde{H}^{\mu \nu}, B \}^*_D \right], \]
we get \( b^\mu_\alpha (\tau) \equiv L^\mu_\alpha (p_s, \vec{p}_s) \) and \( \tilde{H}^{\mu \nu} (\tau) \equiv 0 \), namely the determination of \( S^{\mu \nu} = S^{\mu \nu} - S^{\mu \nu}_{\psi} \) in terms of the variables of the system. The remaining variables form a canonical basis
\[
\{\tilde{x}^\mu(\tau, p^\nu_w)\}^*_D = -\eta^{\mu\nu}, \\
\{A_A(\tau, \sigma), \pi^B(\tau, \sigma')\}^*_D = \eta^A_B \delta^3(\sigma - \sigma'), \\
\{\psi_\alpha(\tau, \sigma), \psi_\beta(\tau, \sigma')\}^*_D = -i(\gamma^0)_{\alpha\beta} \delta^3(\sigma - \sigma').
\]  

(49)

As shown in Ref. [8], the dependence of the gauge-fixing (59) on \(p^\nu_w\) implies that the Lorentz-scalar indices \(A\) become Wigner indices \(A\): i) \(A_{A=r}(\tau, \sigma)\) is a Lorentz-scalar field; ii) \(A_{A=r}(\tau, \sigma)\), are Wigner spin 1 3-vectors which transform with Wigner rotations under the action of Minkowski Lorentz boosts. In particular, \(\gamma^0 = \gamma^r\) is a Lorentz scalar matrix, while \(\gamma^r\) form a Wigner spin 1 3-vector. Therefore, we have a Wigner-covariant realization of Dirac matrices

\[
\gamma^A = (\gamma^0; \{\gamma^r\}, r = 1, 2, 3), \\
[\gamma^A, \gamma^B]_+ = 2\eta^{AB},
\]

(50)

like in the Chakrabarti representation [16] (see also Ref. [15]). Under a Lorentz transformation \(\Lambda\), the bilinears in the Dirac field transform with the associated Wigner rotation \(R(\Lambda, p_s)\). For instance we have

\[
\overset{0}{\psi}(\tau, \sigma)\gamma^A \overset{0}{\psi}(\tau, \sigma) \overset{A}{\rightarrow} R^A_B(\Lambda, p_s) \overset{0}{\psi}(\tau, \sigma)\gamma^B \overset{0}{\psi}(\tau, \sigma),
\]

\[
\overset{0}{\psi}(\tau, \sigma)\gamma^0 \overset{0}{\psi}(\tau, \sigma) \overset{\Lambda}{\rightarrow} \overset{0}{\psi}(\tau, \sigma)\gamma^0 \overset{0}{\psi}(\tau, \sigma) \quad \text{(scalar)},
\]

\[
\overset{0}{\psi}(\tau, \sigma)\gamma^r \overset{0}{\psi}(\tau, \sigma) \overset{\Lambda}{\rightarrow} R^r_s(\Lambda, p_s) \overset{0}{\psi}(\tau, \sigma)\gamma^s \overset{0}{\psi}(\tau, \sigma) \quad \text{(Wigner 3-vector)},
\]

(51)

and the induced spinorial transformation on Dirac fields will be \([S(\Lambda) \mapsto \overset{\circ}{S}(\Lambda, p_s)]\), which could be evaluated by using the last of the next formulas

\[
\overset{\circ}{\psi}(\tau, \sigma) \overset{\Lambda}{\rightarrow} \overset{\circ}{\psi}'(\tau, \sigma) = \overset{\circ}{S}(\Lambda, p_s) \overset{0}{\psi}(\tau, \sigma),
\]

\[
\overset{\circ}{\psi}(\tau, \sigma) \overset{\Lambda}{\rightarrow} \overset{\circ}{\psi}'(\tau, \sigma) = \overset{\circ}{\psi}'(\tau, \sigma)\overset{\circ}{S}^{-1}(\Lambda, p_s),
\]

\[
\overset{\circ}{S}^{-1}(\Lambda, p_s) \quad \gamma^A \overset{\circ}{S}(\Lambda, p_s) = R^A_B(\Lambda, p_s)\gamma^B.
\]

(52)

The original variables \(z^\mu(\tau, \sigma), \rho_\mu(\tau, \sigma)\), are reduced only to \(\tilde{x}^\mu(\tau), p^\nu_w\), on the Wigner hyperplane \(\Sigma_{rW}\). On it there remain only six first class constraints

\[
\pi^r(\tau, \sigma) \approx 0,
\]

\[
\Gamma(\tau, \sigma) = \partial_\tau \pi^r(\tau, \sigma) + e \overset{0}{\psi}(\tau, \sigma)\gamma^0 \overset{0}{\psi}(\tau, \sigma) \approx 0,
\]

\[
\tilde{H}^\mu(\tau) = p^\mu_s - [u^\mu(p_s)H_{rel}(\tau) + e^\mu_r(u(p_s))H_{pr}(\tau)] =
\]

\[
= u^\mu(p_s)H(\tau) + e^\mu_r(u(p_s))H_{pr}(\tau) \approx 0,
\]

or
\[ H(\tau) = \eta_s \sqrt{p_s^2} - H_{\text{rel}}(\tau) = \]
\[ = \eta_s \sqrt{p_s^2} - \int d^3\sigma \left[ \frac{i}{2} \left( \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) + \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \tilde{\psi}(\tau, \vec{\sigma}) \right) + m \tilde{\psi}(\tau, \vec{\sigma}) \tilde{\psi}(\tau, \vec{\sigma}) + \right. \]
\[ + \frac{1}{2}(\vec{\pi}^2 + \vec{B}^2)(\tau, \vec{\sigma}) + eA_\tau(\tau, \vec{\sigma}) \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \tilde{\psi}(\tau, \vec{\sigma}) \approx 0, \]
\[ H_{pr}(\tau) = \int d^3\sigma \left[ \frac{i}{2} \left( \tilde{\psi}(\tau, \vec{\sigma}) \gamma^\rho \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) - \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) \gamma^\rho \tilde{\psi}(\tau, \vec{\sigma}) \right) + \right. \]
\[ + \left( \vec{\pi} \wedge \vec{B} \right)_\tau(\tau, \vec{\sigma}) + eA_\tau(\tau, \vec{\sigma}) \gamma^\rho \tilde{\psi}(\tau, \vec{\sigma}) \tilde{\psi}(\tau, \vec{\sigma}) \right] \approx 0, \]
\[ \{ \tilde{H}^\mu \tau, \tilde{H}^\nu \tau \}_D^* = \int d^3\sigma \left[ (u^\mu(p_s)e^\nu_s(u(p_s))) - (u^\nu(p_s)e^\mu_s(u(p_s))) \right] \pi^\tau(\tau, \vec{\sigma}) + \]
\[ - \epsilon^\mu_s(u(p_s))F_\tau(\tau, \vec{\sigma}) \epsilon^\nu_s(u(p_s)) \right] \Gamma(\tau, \vec{\sigma}) \approx 0, \]
\[ \{ \Gamma(\tau, \vec{\sigma}), \tilde{H}^\mu \tau \}_D^* = \{ \Gamma(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}') \}_D^* = \{ \Gamma(\tau, \vec{\sigma}), \pi^\tau(\tau, \vec{\sigma}') \}_D^* = \]
\[ = \{ \pi^\tau(\tau, \vec{\sigma}), \tilde{H}^\mu \tau \}_D^* = \{ \pi^\tau(\tau, \vec{\sigma}), \pi^\tau(\tau, \vec{\sigma}') \} = 0. \]

The constraints \( \tilde{H}_p(\tau) \approx 0 \) identify the Wigner hyperplane \( \Sigma_{\tau W} \) with the intrinsic rest frame (vanishing of the total Wigner spin 1 3-momentum of the isolated system) and say that the 3-coordinate \( \vec{\sigma} = \vec{x}_{\text{com}} \) \( [x^\mu_{\text{com}} = z^\mu(\tau, \vec{x}_{\text{com}})] \) of the center of mass of the isolated system on \( \Sigma_{\tau W} \) is a gauge variable, whose natural gauge-fixing is \( \vec{x}_{\text{com}} \approx 0 \) [so that it coincides with the origin of \( \Sigma_{\tau W} \): \( x^\mu_{\text{com}}(\tau) = z^\mu(\tau, \vec{\sigma} = 0) \)]. See Ref. [17] for the definition of \( x^\mu_{\text{com}} \) for the configurations of the Klein-Gordon field. The remaining constraint \( H(\tau) \approx 0 \) identifies \( \epsilon_s = \eta_s \sqrt{p_s^2} \) with the invariant mass \( H_{\text{rel}} \) of the isolated system.

On \( \Sigma_{\tau W} \) the Dirac Hamiltonian becomes
\[ H_D = \lambda(\tau)H(\tau) - \bar{\lambda}(\tau)\tilde{H}_p(\tau) + \int d^3\sigma \left[ \mu_\tau(\tau, \vec{\sigma}) \pi^\tau(\tau, \vec{\sigma}) - A_\tau(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma}) \right], \]
so that \( \tilde{x}^\mu_s \) has a velocity parallel to \( p^\mu_s \) \( \{ \tilde{x}^\mu_s, H_D \}_D^* = -\lambda(\tau)u^\mu(p_s) \), namely it has no classical zitterbewegung.

Since \( \tilde{H}^\mu \tau(\tau) \equiv 0 \) implies
\[ \left( \epsilon^\mu_s(u(p_s))u^\nu(p_s) + \right. \]
\[ = \epsilon^\mu_s(u(p_s))u^\nu(p_s)) \int d^3\sigma \sigma^\rho \left[ \frac{i}{2} \left( \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) + \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \tilde{\psi}(\tau, \vec{\sigma}) \right) + \right. \]
\[ \left. + \frac{1}{2}(\vec{\pi}^2 + \vec{B}^2)(\tau, \vec{\sigma}) + eA_\tau(\tau, \vec{\sigma}) \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \tilde{\psi}(\tau, \vec{\sigma}) \right] + \]
\[ - \left( \epsilon^\mu_s(u(p_s))e^\nu_s(p_s) - \epsilon^\nu_s(u(p_s))e^\mu_s(p_s) \right) \int d^3\sigma \sigma^\rho \left[ \frac{i}{2} \left( \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) + \right. \right. \]
\[ \left. \left. - \partial_\rho \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \tilde{\psi}(\tau, \vec{\sigma}) \right) + \left( \vec{\pi} \wedge \vec{B} \right)_\tau(\tau, \vec{\sigma}) + eA_\tau(\tau, \vec{\sigma}) \tilde{\psi}(\tau, \vec{\sigma}) \gamma_\rho \tilde{\psi}(\tau, \vec{\sigma}) \right], \]
with
we get the following expression for the rest-frame spin tensor

\[
\bar{S}_{\psi}^{AB} = \frac{1}{4} \int d^3 \sigma \bar{\psi}(\tau, \vec{\sigma}) [\gamma^0, \sigma^{AB}]_+ \bar{\psi}(\tau, \vec{\sigma}),
\]

(56)

On \( \Sigma_r \) the Poincaré generators are

\[ p^\mu_s, \quad J^{\mu\nu} = \hat{x}^\mu p^\nu_s - \hat{x}^\nu p^\mu_s + \bar{S}^{\mu\nu}, \]

\[
\bar{S}_{\psi}^{\mu\nu} = \bar{S}_{\psi}^{\mu\nu} + \bar{S}_{\xi}^{\mu\nu},
\]

\[
\bar{S}^{0i} = -\frac{\delta^{ir} \bar{S}^{rs} p^s}{p^0_s + \eta_s \sqrt{p^2_s}}, \quad \bar{S}^{ij} = \delta^{ir} \delta^{js} \bar{S}^{rs},
\]

(59)

because one can express \( \bar{S}_{\psi}^{\mu\nu} \) in terms of \( \bar{S}_{\psi}^{AB} = e^A_{\mu}(u(p_s)) \varepsilon(u(p_s)) S^{\mu\nu} \). Only the Thomas spin \( S_r = \frac{1}{2} \varepsilon_{rst} \bar{S}_{st} \) contributes to them. This is the universal realization of the Poincaré generators associated with the rest-frame Wigner-covariant instant form of the dynamics.

The electric charge takes the form

\[
Q = -e \int d^3 \sigma \bar{\psi}(\tau, \vec{\sigma}) \gamma^0 \bar{\psi}(\tau, \vec{\sigma}),
\]

(60)

and is the weak Noether charge of the conserved 4-current \( j^A(\tau, \vec{\sigma}) = -e \bar{\psi}(\tau, \vec{\sigma}) \gamma^A \bar{\psi}(\tau, \vec{\sigma}) \), \( \partial_A j^A(\tau, \vec{\sigma}) = 0 \).

Therefore, the rest-frame Wigner-covariant instant form of the system Dirac plus electromagnetic fields (pseudoclassical electrodynamics) formally coincides with the standard noncovariant Hamiltonian formulation of the system on the hyperplanes \( z^0(\tau, \vec{\sigma}) = \text{const.} \) only the covariance properties of the objects are different and, moreover, there are the first class constraints \( H_\mu(\tau) \approx 0 \) defining the rest frame.
V. DIRAC’S OBSERVABLES AND EQUATIONS OF MOTION.

As shown in Ref. [1], the Dirac observables of the electromagnetic field are the transverse quantities $\vec{A}(\tau, \vec{\sigma}), \vec{\pi}(\tau, \vec{\sigma})$, defined by the decomposition

\begin{align*}
\vec{A}(\tau, \vec{\sigma}) &= \vec{\partial}_\eta(\tau, \vec{\sigma}) + \vec{A}_\perp(\tau, \vec{\sigma}), \\
\vec{\pi}(\tau, \vec{\sigma}) &= \vec{\pi}_\perp(\tau, \vec{\sigma}) + \frac{\vec{\partial}}{\Delta_\sigma}\left[\Gamma(\tau, \vec{\sigma}) - \sum_{i=1}^{N} Q_i(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))\right], \\
\eta(\tau, \vec{\sigma}) &= -\frac{\vec{\partial}}{\Delta_\sigma} \cdot \vec{A}(\tau, \vec{\sigma}),
\end{align*}

(61)

while the gauge variables are $A_\tau(\tau, \vec{\sigma})$ and $\eta(\tau, \vec{\sigma})$, which are conjugate to the first class constraints $\pi^\tau(\tau, \vec{\sigma}) \approx 0, \Gamma(\tau, \vec{\sigma}) \approx 0$.

Since we have

\begin{align*}
\{\bar{\psi}(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}')\}^{**}_D &= -ie\gamma^\tau \psi(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}'),
\end{align*}

(62)

we see that the Dirac field is not gauge invariant. Like in Ref. [1], it turns out that its Dirac observables are

\begin{align*}
\bar{\psi}(\tau, \vec{\sigma}) &= e^{-ie\eta(\tau, \vec{\sigma})} \bar{\psi}(\tau, \vec{\sigma}), \\
\hat{\psi}(\tau, \vec{\sigma}) &= \bar{\psi}(\tau, \vec{\sigma}) e^{ie\eta(\tau, \vec{\sigma})}, \\
\{\hat{\psi}(\tau, \vec{\sigma}), \bar{\psi}(\tau, \vec{\sigma}')\}^{**}_D &= -i(\gamma^\rho)_{\bar{\psi}}^\rho \delta^3(\vec{\sigma} - \vec{\sigma}'),
\end{align*}

(63)

representing Dirac fields dressed with a Coulomb cloud.

By using Eqs.(61) we get

\begin{align*}
\int d^3\sigma \vec{\pi}^2(\tau, \vec{\sigma}) &= \int d^3\sigma \vec{\pi}_{\perp}^2(\tau, \vec{\sigma}) + \\
&+ e^2 \int d^3\sigma d^3\sigma' \left[\bar{\psi}(\tau, \vec{\sigma}) \gamma^\rho \psi(\tau, \vec{\sigma}) \right] \left[\bar{\psi}(\tau, \vec{\sigma}') \gamma^\rho \psi(\tau, \vec{\sigma}') \right] \frac{1}{4\pi |\vec{\sigma} - \vec{\sigma}'|} = \\
&= \int d^3\sigma \vec{\pi}_{\perp}^2(\tau, \vec{\sigma}) + \\
&+ e^2 \int d^3\sigma d^3\sigma' \left[\bar{\psi}(\tau, \vec{\sigma}) \gamma^\rho \bar{\psi}(\tau, \vec{\sigma}) \right] \left[\bar{\psi}(\tau, \vec{\sigma}') \gamma^\rho \bar{\psi}(\tau, \vec{\sigma}') \right] \frac{1}{4\pi |\vec{\sigma} - \vec{\sigma}'|},
\end{align*}

(64)

so that in the Coulomb gauge, $A_\tau(\tau, \vec{\sigma}) = \pi^\tau(\tau, \vec{\sigma}) = \eta(\tau, \vec{\sigma}) = \Gamma(\tau, \vec{\sigma}) = 0$, we have

\begin{align*}
H(\tau) &= \epsilon_s - H_{rel}(\tau) = \\
&= \epsilon_s \sqrt{p_s^2} - \int d^3\sigma \frac{i}{2} \left(\bar{\psi}(\tau, \vec{\sigma}) \gamma_\nu \partial_\nu \psi(\tau, \vec{\sigma}) + \\
&- \partial_\nu \bar{\psi}(\tau, \vec{\sigma}) \gamma_\nu \bar{\psi}(\tau, \vec{\sigma})\right) + m \bar{\psi}(\tau, \vec{\sigma}) \psi(\tau, \vec{\sigma}) +
\end{align*}
\[
+ \frac{1}{2} (\vec{\pi}_\perp + \vec{B}^2) (\tau, \vec{\sigma}) + eA_{\perp r}(\tau, \vec{\sigma}) \vec{\psi}(\tau, \vec{\sigma}) \gamma_\tau \vec{\psi}(\tau, \vec{\sigma}) + \\
+ \frac{e^2}{2} \vec{\psi}(\tau, \vec{\sigma}) \gamma_0 \vec{\psi}(\tau, \vec{\sigma}) \int d^3 \sigma' \frac{\vec{\psi}(\tau, \vec{\sigma}') \gamma_0 \vec{\psi}(\tau, \vec{\sigma}')}{4\pi |\vec{\sigma} - \vec{\sigma}'|} \approx 0.
\]

(65)

Like in Ref. [1], the last term is the nonrenormalizable Coulomb self interaction of the Dirac field in the rest frame. The constraints identifying the rest frame take the form expected in an instant form of dynamics [they do not depend on the interaction]

\[
H_{pr}(\tau) = \int d^3 \sigma \left[ \frac{1}{2} \left( \vec{\psi}(\tau, \vec{\sigma}) \gamma_0 \partial_\tau \vec{\psi}(\tau, \vec{\sigma}) - \partial_\tau \vec{\psi}(\tau, \vec{\sigma}) \gamma_0 \vec{\psi}(\tau, \vec{\sigma}) \right) + \left( \vec{\pi}_\perp + \vec{B} \right)_r (\tau, \vec{\sigma}) \approx 0. \quad (66)
\]

By doing the canonical transformation [23] from the variables \( \tilde{x}_i(\tau) \) e \( p_i^\mu \), to the new ones

\[
T_s = \frac{p_s \tilde{x}^\mu_s}{\eta_s \sqrt{p_s^2}}, \quad \epsilon_s = \eta_s \sqrt{p_s^2}, \\
\vec{z}_s = \eta_s \sqrt{p_s^2} \left( \vec{x} - \frac{\vec{p}_{\perp} \vec{x}^0}{p^0} \right), \quad \vec{k}_s = \frac{\vec{p}_{\perp}}{\eta_s \sqrt{p_s^2}}, \\
\{T_s, \epsilon_s\}_D^{**} = -1, \quad \{z_s^i, h_s^{ij}\}_D^{**} = \delta^{ij}, \quad (67)
\]

we arrive at a canonical basis with the rest-frame Lorentz-scalar time \( T_s \) and with the canonical noncovariant 3-variable \( \vec{z}_s \) [replacing the origin of \( \Sigma_{rW} \)] with the same covariance of the Newton-Wigner position operator.

By adding the gauge-fixing \( T_s - \tau \approx 0 \) [which identifies the rest-frame time \( T_s \) with the parameter \( \tau \) of the foliation of Minkowski spacetime with the Wigner hyperplanes associated with the isolated system], whose time constancy implies \( \lambda(\tau) = -1 \), we get the Dirac Hamiltonian

\[
\hat{H}_D = H_{rel}(\tau) - \vec{\lambda}(\tau) \cdot \hat{H}_p(\tau), \quad (68)
\]

where

\[
H_{rel} = \int d^3 \sigma \left[ \frac{i}{2} \left( \vec{\psi}(\tau, \vec{\sigma}) \gamma_\tau \partial_\tau \vec{\psi}(\tau, \vec{\sigma}) - \partial_\tau \vec{\psi}(\tau, \vec{\sigma}) \gamma_\tau \vec{\psi}(\tau, \vec{\sigma}) \right) + \\
+ m \vec{\psi}(\tau, \vec{\sigma}) \vec{\psi}(\tau, \vec{\sigma}) + \frac{1}{2} (\vec{\pi}_\perp + \vec{B}^2)(\tau, \vec{\sigma}) \right] + \\
+ \frac{e^2}{2} \int d^3 \sigma d^3 \sigma' \frac{\vec{\psi}(\tau, \vec{\sigma}) \gamma_0 \vec{\psi}(\tau, \vec{\sigma}) \vec{\psi}(\tau, \vec{\sigma}') \gamma_0 \vec{\psi}(\tau, \vec{\sigma})}{4\pi |\vec{\sigma} - \vec{\sigma}'|} + \\
+ e \int d^3 \sigma A_{r\perp}(\tau, \vec{\sigma}) \vec{\psi}(\tau, \vec{\sigma}) \gamma_\tau \vec{\psi}(\tau, \vec{\sigma}). \quad (69)
\]

In the gauge \( \vec{\lambda}(\tau) = 0 \), the Dirac field has the following Hamilton equation
\[ \partial_r \tilde{\psi}(\tau, \vec{\sigma}) \overset{\circ}{=} \{ \tilde{\psi}(\tau, \vec{\sigma}), \hat{H}_D \}_{\hat{D}}^{**} = \gamma^0 \gamma_r \left[ \partial_r - ieA_\perp(\tau, \vec{\sigma}) \right] \tilde{\psi}(\tau, \vec{\sigma}) - im\gamma^0 \tilde{\psi}(\tau, \vec{\sigma}) + \]

\[ - ie^2 \int d^3 \sigma' \frac{\tilde{\psi}(\tau, \vec{\sigma}') \gamma^0 \tilde{\psi}(\tau, \vec{\sigma}')}{4\pi |\vec{\sigma} - \vec{\sigma}'|} \tilde{\psi}(\tau, \vec{\sigma}), \quad (70) \]

which can be rewritten in the standard form

\[ \{ i\gamma^A [\partial_A - ie \tilde{\tilde{A}}_A(\tau, \vec{\sigma})] - m \} \tilde{\psi}(\tau, \vec{\sigma}) \overset{\circ}{=} 0, \quad (71) \]

with

\[ \tilde{\tilde{A}}_r(\tau, \vec{\sigma}) \equiv A_\perp(\tau, \vec{\sigma}), \]

\[ \tilde{\tilde{A}}_\sigma(\tau, \vec{\sigma}) \equiv -e \int d^3 \sigma' \tilde{\psi}(\tau, \vec{\sigma}') \gamma^0 \tilde{\psi}(\tau, \vec{\sigma}') \]

\[ = \tilde{\tilde{A}}_\sigma(\tau, \vec{\sigma}). \quad (72) \]

Analogously we get

\[ \tilde{\tilde{\psi}}(\tau, \vec{\sigma}) \{ -i[\partial_A + ie \tilde{\tilde{A}}_A(\tau, \vec{\sigma})] \gamma^A - m \} \overset{\circ}{=} 0. \quad (73) \]

Eqs.(71) and (72) are nonlocal and nonlinear due to the reduction to the rest-frame Coulomb gauge.

For the transverse electromagnetic fields \( \tilde{\tilde{A}}_\perp(\tau, \vec{\sigma}), \tilde{\pi}_\perp(\tau, \vec{\sigma}) \) we get the Hamilton equations

\[ \partial_r A_\perp^r(\tau, \vec{\sigma}) \overset{\circ}{=} \{ A_\perp^r(\tau, \vec{\sigma}), \hat{H}_D \}_{\hat{D}}^{**} = -\pi_\perp^r(\tau, \vec{\sigma}), \]

\[ \partial_r \pi_\perp^r(\tau, \vec{\sigma}) \overset{\circ}{=} \{ \pi_\perp^r(\tau, \vec{\sigma}), \hat{H}_D \}_{\hat{D}}^{**} = \Delta_\sigma A_\perp^r(\tau, \vec{\sigma}) + \]

\[ + eP^{rs}(\vec{\sigma}) \left[ \tilde{\tilde{\psi}}(\tau, \vec{\sigma}) \gamma^s \tilde{\tilde{\psi}}(\tau, \vec{\sigma}) \right], \quad (74) \]

implying the equation

\[ \square_\sigma A_\perp^r(\tau, \vec{\sigma}) \overset{\circ}{=} P^{rs}(\vec{\sigma})j^s(\tau, \vec{\sigma}) \equiv j^r_\perp(\tau, \vec{\sigma}), \quad (75) \]

with

\[ \square_\sigma \equiv \partial_A \partial^A, \]

\[ P^{rs}(\vec{\sigma}) \equiv \delta^{rs} + \frac{\partial^r \partial^s}{\Delta_\sigma}, \]

\[ j^s(\tau, \vec{\sigma}) = -e\tilde{\tilde{\psi}}(\tau, \vec{\sigma}) \gamma^s \tilde{\tilde{\psi}}(\tau, \vec{\sigma}). \quad (76) \]

The wave equation is actually an integrodifferential equation due to the projector appearing in the transverse fermionic Wigner spin 1\,3-current.
VI. CONCLUSIONS

In this paper we have given the formulation of Grassmann-valued Dirac fields plus the electromagnetic field on spacelike hypersurfaces in Minkowski spacetime. Then, we have done the canonical reduction of this pseudoclassical basis of QED to the rest-frame Wigner-covariant Coulomb gauge, finding a canonical basis of Dirac’s observables in the rest-frame Wigner-covariant instant form of dynamics. What has still to be done is the extension of the theory to include tetrad gravity.

These results are valid for massive Dirac fields and for all the massive \([P^2 > 0]\) configurations of massless Dirac fields. They also hold for the massive configurations of chiral fields simply by replacing everywhere \(\psi\) with \(\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi\). Instead, the massless \([P^2 = 0]\) and infrared \([P^\mu = 0]\) configurations of either massless or chiral fermion fields have to be treated separately, because they require the reformulation of the front form of dynamics in the instant form like for massless spinning particles (this problem will be studied in a future paper).

Therefore, these results open the path to the reformulation and the canonical reduction to a rest-frame Wigner-covariant Coulomb gauge of the SU(3)xSU(2)xU(1) standard model, which will be studied elsewhere.

However, the rest-frame Wigner-covariant instant form evidentiates that there is a nontrivial problem in the description of fermion fields: Eq.(53) shows that the geometrical Minkowski 4-vector \(p^\mu_s\) normal to the Wigner hyperplane \(\Sigma_{rW}\) is the sum of an even “bosonic” part determined by the electromagnetic field plus a “fermionic” part bilinear in the Grassmann-valued Dirac fields. In absence of the electromagnetic field, \(p^\mu_s\) would weakly become a bilinear in Grassmann variables, but this is inconsistent from both a geometrical point of view and from an algebraic one [one can neither define \(\sqrt{p^2_s}\), nor divide by it to get the standard Wigner boost due to the nilpotent character of Grassmann variables]. It seems that the Wigner hyperplanes and the rest-frame description of an isolated pseudoclassical Dirac field is impossible due to the necessity of having it Grassmann-valued as it is required to evaluate the fermionic path integral.

But this does not sound reasonable. Something is missing. In Ref. [15], we have seen that a spinning particle with a definite sign of the energy (but this holds also for the ordinary spinning particles) is described by a fibration: there is a scalar particle (the tracer of the Minkowski worldline, i.e. the path of the electric current if the spinning particle is charged) with a Grassmann fiber over it describing the spin structure and this is particularly evident in its supersymmetric description with the superfield \(X^\mu = x^\mu + \theta \xi^\mu\) [18] [\(\xi^\mu\) are the Grassmann variables for the spin description, which go into the Dirac matrices after quantization]. As said in Ref. [15], one expects that this fibered structure should survive in first quantization: the electron wave function is expected to be some kind of scalar superfield \(\phi_D + \theta \psi_D\), where \(\phi_D\) is a charged Klein-Gordon wave function (restricted to its lower level, with the same degrees of freedom of a scalar particle, to avoid the introduction of spurious levels in bound state spectra) and \(\psi_D\) is the ordinary Dirac wave function [\(\psi_D\) should live in a Clifford fiber over \(\phi_D\)]. This superfield does not describe a supersymmetric multiplet \(\{selectron, electron\}\), but the fibration associated with the spin structure [see Ref. [21] for the description of the pseudoclassical photon as a ray of light with a Grassmann fibration describing the spin structure, namely the light polarization]. The obstacle to arrive at this point of view is that
the first class constraint \( p^2 - m^2 \approx 0 \) has always been quantized in a form which corresponds to the square of the Dirac equation so that it acts on Dirac wave functions. But at the classical level it describes the bosonic scalar particle tracing the worldline so that it has to be quantized so to act on Klein-Gordon wave functions. The double role of this constraint is to give a consistency between what happens on the base (Minkowski spacetime) and what happens on the fiber.

Therefore, one expects that also Dirac fields should be replaced by a fibration \( \varphi + \theta \psi \) describing the spin structure with a pair \( (\varphi, \psi) \), where \( \psi \) is the Grassmann-valued Dirac field and \( \varphi \) a charged Klein-Gordon field suitably restricted.

We shall study this problem elsewhere. A guide to its solution will be to find the way to extract the rest-frame constraints of the spinning particle of Ref. [15] from the fermion field theory, in the same way in which the analogous constraints for a scalar particle were extracted by the rest-frame description of scalar electrodynamics in Ref. [9] by using the Feshbach-Villars transformation [19]; now both Feshbach-Villars and Foldy-Wouthuysen [20] transformations will be needed. The other ingredient will be the individuation of the subspace of Klein-Gordon configurations with the same degrees of freedom of a scalar particle [monopole configurations], by using the technology developed to study the canonical separation of the center-of-mass degrees of freedom from the relative ones for Klein-Gordon fields [17].

Let us remark that the same problems are present in the nonrelativistic description of fermion fields [Grassmann-valued Dirac spinors are replaced by Grassmann-valued Pauli spinors], needed for the theory of Cooper pairs in superconductivity.

The problem of quantization of isolated systems in these rest-frame Wigner-covariant Coulomb gauges is an open problem [see Refs. [26,27] for what is known on the quantization in the Coulomb gauge], which will be the subject of future investigations trying to use the Møller radius \( \rho = \sqrt{-W^2/P^2} = |\vec{S}|/\sqrt{P^2} \) as a physical ultraviolet cutoff [7].
APPENDIX A: FOLIATIONS OF MINKOWSKI SPACETIME WITH FAMILIES OF SPACELIKE HYPERSURFACES

Let us review some preliminary results from Refs. [8,9] needed in the description of physical systems on spacelike hypersurfaces.

Let \( \{ \Sigma_\tau \} \) be a one-parameter family of spacelike hypersurfaces foliating Minkowski spacetime \( M^4 \) and giving a 3+1 decomposition of it. At fixed \( \tau \), let \( z^\mu(\tau, \vec{\sigma}) \) be the coordinates of the points on \( \Sigma_\tau \) in \( M^4 \), \( \{ \vec{\sigma} \} \) a system of coordinates on \( \Sigma_\tau \). If \( \sigma^A = (\tau, \vec{\sigma}) = \{\sigma^\tau\} \) [the notation \( \vec{A} = (\tau, \vec{r}) \) with \( \vec{r} = 1, 2, 3 \) will be used; note that \( \vec{A} = \tau \) and \( \vec{A} = \vec{r} = 1, 2, 3 \) are Lorentz-scalar indices] and \( \partial_A = \partial/\partial \sigma^A \), one can define the vierbeins

\[
\begin{align*}
z'^A(\tau, \vec{\sigma}) &= \partial_A z^\mu(\tau, \vec{\sigma}), \quad \partial_B z'^A - \partial_A z'^B = 0, \\
g_{AB}(\tau, \vec{\sigma}) &= z'^A_{\bar{A}}(\tau, \vec{\sigma}) \eta_{\mu\nu} z'^\nu_{\bar{B}}(\tau, \vec{\sigma}), \quad g_{\tau\tau}(\tau, \vec{\sigma}) > 0,
\end{align*}
\]

so that the metric on \( \Sigma_\tau \)

\[
\begin{align*}
g_{\bar{A}B}(\tau, \vec{\sigma}) &= \frac{\gamma(\tau, \vec{\sigma})}{g(\tau, \vec{\sigma})}, \\
g_{\bar{A}\bar{B}}(\tau, \vec{\sigma}) &= -[\gamma g_{\tau\bar{u}} \gamma^{\bar{u}\bar{v}}](\tau, \vec{\sigma}), \\
g_{\bar{A}\bar{B}}(\tau, \vec{\sigma}) &= \gamma z'^A(\tau, \vec{\sigma}) + \frac{\gamma g_{\tau\bar{u}} g_{\bar{u}\bar{v}} \gamma^{\tau\bar{u}} \gamma^{\bar{u}\bar{v}}}(\tau, \vec{\sigma}),
\end{align*}
\]

(\text{A1})

(\text{A2})

(\text{A3})

If \( \gamma^{\bar{A}\bar{B}}(\tau, \vec{\sigma}) \) is the inverse of the 3-metric \( g_{\bar{A}\bar{B}}(\tau, \vec{\sigma}) \) [\( \gamma^{\bar{A}\bar{B}}(\tau, \vec{\sigma}) g_{\bar{A}\bar{B}}(\tau, \vec{\sigma}) = \delta_\tau^\tau \)], the inverse \( g^{\bar{A}B}(\tau, \vec{\sigma}) \) of \( g_{\bar{A}B}(\tau, \vec{\sigma}) \) \[g^{\bar{A}B}(\tau, \vec{\sigma}) g_{\bar{A}\bar{B}}(\tau, \vec{\sigma}) = \delta_\tau^\tau \] is given by

\[
\begin{align*}
g^{\tau\tau}(\tau, \vec{\sigma}) &= \frac{\gamma(\tau, \vec{\sigma})}{g(\tau, \vec{\sigma})}, \\
g^{\tau\bar{r}}(\tau, \vec{\sigma}) &= -\frac{\gamma g_{\tau\bar{u}} \gamma^{\bar{u}\bar{v}}}(\tau, \vec{\sigma}), \\
g^{\bar{r}\bar{r}}(\tau, \vec{\sigma}) &= \gamma z'^A(\tau, \vec{\sigma}) + \frac{\gamma g_{\tau\bar{u}} g_{\bar{u}\bar{v}} \gamma^{\tau\bar{u}} \gamma^{\bar{u}\bar{v}}}(\tau, \vec{\sigma}),
\end{align*}
\]

(\text{A4})

(\text{A5})

(\text{A6})

We have

\[
\begin{align*}
z'^\mu(\tau, \vec{\sigma}) &= \sqrt{g} l'^\mu(\tau, \vec{\sigma}), \\
\gamma_{\mu\nu} &= \gamma_A g_{AB}(\tau, \vec{\sigma}) z'^B(\tau, \vec{\sigma}) = \\
&= (l'^\mu + \gamma^{\bar{A}\bar{B}} z'^\mu_{\bar{A}} z'^B_{\bar{B}})(\tau, \vec{\sigma}),
\end{align*}
\]

(\text{A7})

where

\[
\begin{align*}
l'^\mu(\tau, \vec{\sigma}) &= \frac{1}{\sqrt{\gamma}} \epsilon^\mu_{\alpha\beta\gamma} z'^\alpha_{\bar{A}} z'^\beta_{\bar{B}} z'^\gamma_{\bar{C}}(\tau, \vec{\sigma}), \\
l^2(\tau, \vec{\sigma}) &= 1, \quad l_{\mu}(\tau, \vec{\sigma}) z'^\mu(\tau, \vec{\sigma}) = 0.
\end{align*}
\]

\[
\begin{align*}
l^2(\tau, \vec{\sigma}) &= 1, \quad l_{\mu}(\tau, \vec{\sigma}) z'^\mu(\tau, \vec{\sigma}) = 0, \quad (A7)
\end{align*}
\]
is the unit (future pointing) normal to $\Sigma_\tau$ at $z^\mu(\tau, \vec{\sigma})$.

For the volume element in Minkowski spacetime we have

$$d^4z = z^\mu(\tau, \vec{\sigma}) d\tau d^3\Sigma_\mu = d\tau [z^\nu(\tau, \vec{\sigma}) l_\mu(\tau, \vec{\sigma})] \sqrt{\gamma(\tau, \vec{\sigma})} d^3\sigma = \sqrt{g(\tau, \vec{\sigma})} d^4z.$$  \hspace{1cm} (A8)

Let us remark that according to the geometrical approach of Ref. [22], one can use Eq. (A5) in the form $z^\nu(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) l^\mu(\tau, \vec{\sigma}) + N^\tau(\tau, \vec{\sigma}) z^\tau(\tau, \vec{\sigma})$, where $N = \sqrt{g/\gamma} = \sqrt{g_{\tau\tau} - \gamma_{\tau\tau} g_{\tau s} g_{s \bar{s}}}$. Let $N^\tau = g_{\tau s} \gamma_{s \bar{s}}$ are the standard lapse and shift functions, so that $g_{\tau\tau} = N^2 + g_{s \bar{s}} N^s N^\bar{s}, g_{s \bar{r}} = g_{sr} N^s, g_{\tau s} = N^2, g_{\bar{s} \bar{r}} = -N^r/N^2, g_{s \bar{s}} = \gamma_{s \bar{s}}$. The standard Wigner boost transforming $p^\mu = \eta \sqrt{p^2}$, $p^2 = p^2$, where $\eta = \text{sign} p^\nu$. The standard Wigner boost transforming $\bar{p}^\mu$ into $p^\mu$ is

$$L^\mu_{\nu}(p, \bar{p}) = c^\mu_{\nu}(u(p)) =$$

$$= \eta^\nu_{\nu} + 2\theta^\nu \bar{p}^\nu - \frac{(p^\mu + \bar{p}^\mu)(p_\nu + \bar{p}_\nu)}{p \cdot \bar{p} + p^2} =$$

$$= \eta^\nu_{\nu} + 2u^\nu(p)u_\nu(\bar{p}) - \frac{u^\nu(p) + u^\nu(\bar{p})}{1 + u^\nu(p)} u_\nu(\bar{p}).$$

$$\quad \nu = 0 \quad c^\mu_\nu(u(p)) = u^\mu(p) = p^\mu/\eta \sqrt{p^2},$$

$$\quad \nu = r \quad c^\mu_\nu(u(p)) = (-u_\nu(p); \delta^\nu_{\nu} - \frac{u^\nu(p) u_\nu(p)}{1 + u^\nu(p)}).$$ \hspace{1cm} (A10)

The inverse of $L^\mu_{\nu}(p, \bar{p})$ is $L_{\nu \mu}(p, \bar{p})$, the standard boost to the rest frame, defined by

$$L^\mu_{\nu}(p, \bar{p}) = L_{\nu \mu}(p, \bar{p}) = L^\mu_{\nu}(p, \bar{p})|_{\bar{p} \rightarrow \bar{p}}.$$  \hspace{1cm} (A11)

Therefore, we can define the following vierbeins $[c^\mu_\nu(u(p))]$'s are also called polarization vectors; the indices $r, s$ will be used for $A=1,2,3$ and $\bar{r}$ for $A=0$]

$$c^\mu_A(u(p)) = L^\mu_A(p, \bar{p}),$$

$$c^\mu_A(u(p)) = L_A^\mu(p, \bar{p}) = \eta^{AB} \eta_{\mu \nu} c^\nu_B(u(p)),$$

$$\delta^\mu_{\nu}(u(p)) = \eta_{\mu \nu} c^\nu_{\bar{r}}(u(p)) = u_\mu(p),$$

$$\delta^\mu_{\nu}(u(p)) = -\delta^\nu\Sigma_{\mu \nu} c^\nu_{\bar{r}}(u(p)) = (\delta^\nu u_\nu(p); \delta^\nu_{\nu} - \delta^\nu r_{\nu h} \frac{u^h(p) u_\nu(p)}{1 + u^\nu(p)}),$$

$$\delta^\mu_{\nu}(u(p)) = u_A(p),$$ \hspace{1cm} (A12)
which satisfy

\[ \epsilon^A_\mu(u(p))\epsilon^A_\nu(u(p)) = \eta^\nu_\mu, \]
\[ \epsilon^B_\mu(u(p))\epsilon^B_\nu(u(p)) = \eta^A_\mu, \]
\[ \eta^{\mu\nu} = \epsilon^A_\mu(u(p))\eta^{AB} \epsilon^B_\nu(u(p)) = u^\mu(p)u^\nu(p) - \sum_{r=1}^3 \epsilon^r_\mu(u(p))\epsilon^r_\nu(u(p)), \]
\[ \eta_{AB} = \epsilon^A_\mu(u(p))\eta_{\mu\nu} \epsilon^B_\nu(u(p)), \]
\[ p_\alpha \frac{\partial}{\partial p_\alpha} \epsilon^A_\mu(u(p)) = p_\alpha \frac{\partial}{\partial p_\alpha} \epsilon^A_\mu(u(p)) = 0. \]  

(A13)

The Wigner rotation corresponding to the Lorentz transformation \( \Lambda \) is

\[ R^\nu_\mu(\Lambda, p) = [L(\hat{\Lambda}, p)\Lambda^{-1}L(\Lambda p, \hat{\Lambda})]^\mu_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i_j(\Lambda, p) \end{pmatrix}, \]
\[ R^i_j(\Lambda, p) = (\Lambda^{-1})^i_j - \frac{(\Lambda^{-1})^i_o p_\beta(\Lambda^{-1})^\beta_j}{p^\rho(\Lambda^{-1})^\rho_o + \eta\sqrt{p^2}} - \frac{p^i}{p^o + \eta\sqrt{p^2}}[(\Lambda^{-1})^o_j - \frac{(\Lambda^{-1})^o_o - 1)p_\beta(\Lambda^{-1})^\beta_j}{p^\rho(\Lambda^{-1})^\rho_o + \eta\sqrt{p^2}}]. \]  

(A14)

The polarization vectors transform under the Poincaré transformations \((a, \Lambda)\) in the following way

\[ \epsilon^r_\mu(\Lambda p)) = (R^{-1})^r_s\Lambda^s_\mu \epsilon^s(u(p)). \]  

(A15)
APPENDIX B: LAGRANGIAN FOR DIRAC FIELDS ON SPACELIKE HYPERSURFACES.

In tetrad gravity, given a 3+1 splitting of a globally hyperbolic spacetime $M^4$ with metric $g_{\mu\nu}$ [$\mu$ are world indices, $(\mu)$ are flat rectangular Minkowski indices], one writes the following action for Dirac fields $\bar{\psi}(\tau, \bar{\sigma}) = \psi(\tau, \bar{\sigma})$ (see Refs. [13,12])

$$S = \int d\tau d^3\sigma N(\tau, \bar{\sigma}) \sqrt{\gamma(\tau, \bar{\sigma})} \left( \frac{i}{2} \bar{\psi} \gamma^{(\mu)} (\gamma^{\mu} A^A_\mu) \bar{D}_A - \bar{D}_A \bar{E}^A_{(\mu)} (\gamma^{(\mu)} \bar{\psi} - m \bar{\psi} \gamma^0 \bar{\psi}) \right) \bigg|_{\Sigma} - m \bar{\psi} \gamma^0 \bar{\psi},$$

where

i) $4E^A_{(\mu)}(\tau, \bar{\sigma}) = b^A_\mu(\tau, \bar{\sigma}) 4E^\nu_{(\mu)}(\tau, \bar{\sigma})$ are cotetrads in coordinates $\sigma^A = (\tau, \bar{\sigma})$ adapted to the spacelike hypersurfaces $\Sigma$, leaves of a foliation giving a 3+1 splitting of $M^4$ [$4\bar{E}^A_{\nu}(\tau, \bar{\sigma}) 4E^B_{(\nu)} = 4\gamma^{AB}$ ($^{(4)}\gamma^{(\mu)(\nu)}$ is the flat Minkowski inverse metric) with $4\gamma^{AB}(\tau, \bar{\sigma})$ inverse of the metric $4g_{\mu\nu}(\tau, \bar{\sigma}) = b^A_\mu b^B_\nu 4g_{\mu\nu} = 4E^A_{(\mu)} 4E^{B(\nu)}$, where $4E^A_{(\mu)}(\tau, \bar{\sigma}) = b^A_\mu(\tau, \bar{\sigma}) 4E^{\mu}_{(\nu)}(\tau, \bar{\sigma})$ are tetrads, $4E^A_{(\mu)} 4E^{B}_{(\nu)} = \delta^{(\mu)}_{(\nu)}$, $4E^A_{(\nu)} E^{B}_{(\nu)} = \delta^B_A$;  

ii) $\gamma^{(\mu)}$ are flat Dirac matrices: $\left[ \gamma^{(\mu)}, \gamma^{(\nu)} \right] = \gamma^{(\mu)\nu} + \gamma^{(\nu)\mu} = 2^{(4)}\eta^{(\mu)(\nu)}$, $\sigma^{(\mu)(\nu)} = \frac{i}{2} [\gamma^{(\mu)}, \gamma^{(\nu)}]$;  

iii) $\bar{D}_A = \partial_A - \frac{i}{4} 4\omega^A_{(\mu)(\nu)} \sigma^{(\mu)(\nu)}$ is the spinor covariant derivative ($\partial_A = \partial/\partial \sigma^A$ and $4\omega^A_{(\mu)(\nu)}$ is the 4-spin connection [see Ref. [13] for its expression in terms of cotetrads];  

iv) $N(\tau, \bar{\sigma})$ is the lapse function and $\gamma(\tau, \bar{\sigma}) = \text{det} \frac{3}{2} g_{\bar{r}\bar{s}}(\tau, \bar{\sigma}) \frac{3}{4} g_{r\bar{s}} = - \frac{3}{4} g_{r\bar{s}}$.

See Ref. [14] for the 3+1 splitting of the 4-spin connection in terms of the 3-spin connection [function of cotriads $^{(3)}e^A_\nu$ on $\Sigma$,] and lapse and shift functions. In the Hamiltonian version of tetrad gravity cotriads, lapse and shift functions are the independent variables. In this case the tetrads $4E^\mu_{(\mu)}(\tau, \bar{\sigma})$ correspond to a $\Sigma$-adapted anholonomic basis of coordinates in the spacetime $M^4$.

When we consider a 3+1 splitting of Minkowski spacetime with a family of spacelike hypersurfaces $\Sigma_{\tau}$ the tetrads $4E^\mu_{(\mu)}(\tau, \bar{\sigma})$ and cotetrad $4E^A_{(\mu)}(\tau, \bar{\sigma})$ go into the flat tetrads $z^A_{(\mu)}(\tau, \bar{\sigma}) = \partial z^{(\mu)}(\tau, \bar{\sigma})/\partial \sigma^A$ and cotetrad $z^A_{(\mu)}(\tau, \bar{\sigma}) = \partial \sigma^A(z)/\partial z^{(\mu)}(\tau, \bar{\sigma})$ respectively corresponding to a holonomic basis for Minkowski spacetime. Since in this paper we consider only Minkowski spacetime, from now on we shall drop the distinction between flat and world indices, $z^A_{(\mu)}(\tau, \bar{\sigma}) = z^A(\tau, \bar{\sigma})$.

Since in the holonomic basis for Minkowski spacetime the 4-spin connection vanishes, $4\omega^\mu_{A\nu}(\tau, \bar{\sigma}) = \eta_{\mu\nu} 4\omega^\nu_{A\mu}(\tau, \bar{\sigma}) = - 4\omega^\nu_{A\mu}(\tau, \bar{\sigma}) = 0,$

$$4\Gamma^B_{AC}(\tau, \bar{\sigma}) = \frac{1}{2} 4g^{BD}(\tau, \bar{\sigma}) \left[ \partial_A 4g_{CD}(\tau, \bar{\sigma}) + \partial_C 4g_{AD}(\tau, \bar{\sigma}) - \partial_D 4g_{AC}(\tau, \bar{\sigma}) \right] = 0,$$

$$4\Gamma^B_{AC}(\tau, \bar{\sigma}) = \frac{1}{2} 4g^{BD}(\tau, \bar{\sigma}) \left[ \partial_A z^B_{(\nu)}(\tau, \bar{\sigma}) + 4\Gamma^B_{AC}(\tau, \bar{\sigma}) z^C_{(\nu)}(\tau, \bar{\sigma}) \right] = 0,$$  

(B2)

the action for Dirac fields becomes

$$S = \int d\tau d^3\sigma N(\tau, \bar{\sigma}) \sqrt{\gamma(\tau, \bar{\sigma})} \left( \frac{i}{2} \bar{\psi} \gamma^{(\mu)} (\gamma^{\mu} z^A_{A\mu} - \partial_A z^A_{\mu} \gamma^{\mu}) \bar{\psi} - m \bar{\psi} \gamma^0 \bar{\psi} \right).$$  

(B3)
The normal \( l^\mu(\tau, \sigma) = [\frac{1}{\sqrt{-g}} \varepsilon^\mu_{\alpha\beta\gamma} z^\alpha_1 z^\beta_1 z^\gamma_1](\tau, \sigma) \) to \( \Sigma_\tau \) is timelike. so that \( \gamma^\mu = (1; 0) \), one has \( l^\mu(\tau, \sigma) = L^\mu(\tau, \sigma), l \) by using the corresponding Wigner boost. One can then define new flat (nonholonomic) tetrads \( \overset{o}{z}^A_\mu(\tau, \sigma) \) through \( \overset{o}{z}_A^\mu(\tau, \sigma) = L^\mu(\tau, \sigma), l \overset{o}{z}^A_\mu(\tau, \sigma) \) and the corresponding cotetrad through \( \overset{c}{z}_\mu^A(\tau, \sigma) = \overset{c}{z}_\nu^A(\tau, \sigma) L^\nu(\tau, \sigma), l \). One finds

\[
\begin{align*}
\overset{o}{z}^\mu(\tau, \sigma) &= \left( N(\tau, \sigma); N^\tau(\tau, \sigma), 3 e^\mu_1(\tau, \sigma) \right), \\
\overset{o}{z}_\mu^A(\tau, \sigma) &= \left( 0; 3 e^\mu_1(\tau, \sigma) \equiv \frac{l^i(\tau, \sigma)}{1 + l^0(\tau, \sigma)} z^i(\tau, \sigma) \right), \\
\overset{o}{z}_\mu^A(\tau, \sigma) &= \left( \frac{1}{N(\tau, \sigma)}; 0 \right), \\
\overset{o}{z}_\mu^f(\tau, \sigma) &= \left( -\frac{N^f}{N}(\tau, \sigma), 3 e^f_3(\tau, \sigma) = -\gamma^{f_3}(\tau, \sigma) 3 e^f_3(\tau, \sigma) \right),
\end{align*}
\]  

(B4)

where now \( N = \sqrt{g/\gamma}, N^f = g_{f_3} \gamma^{f_3} \) (see Appendix A).

The metric becomes

\[
\begin{align*}
g_{AB}(\tau, \sigma) &= \overset{o}{z}^\mu(\tau, \sigma) \eta_{\mu\nu} \overset{o}{z}^\nu_B(\tau, \sigma) = \overset{o}{z}^\mu_A(\tau, \sigma) \eta_{\mu\nu} \overset{o}{z}^\nu_B(\tau, \sigma), \\
3 g_{f_3}(\tau, \sigma) &= \overset{o}{z}_\mu^f(\tau, \sigma) \eta_{\mu\nu} \overset{o}{z}_\nu^f(\tau, \sigma) = -3 e^f_3(\tau, \sigma) 3 e^f_3(\tau, \sigma), \\
g^{AB}(\tau, \sigma) &= \overset{o}{z}_\mu^A(\tau, \sigma) \eta^{\mu\nu} \overset{o}{z}_\nu^B(\tau, \sigma) = \overset{o}{z}_\mu^A(\tau, \sigma) \eta^{\mu\nu} \overset{o}{z}_\nu^B(\tau, \sigma).
\end{align*}
\]  

(B5)

In this case cotriads \( 3 e^f_3(\tau, \sigma), triads \( 3 e^f_3(\tau, \sigma) \)

[\[3 e^f_3(\tau, \sigma) 3 e^f_3(\tau, \sigma) = \delta^f_3, \quad 3 e^f_3(\tau, \sigma) 3 e^f_3(\tau, \sigma) = \delta^f_3]] \), lapse \( N(\tau, \sigma) \) and shifts \( N^f(\tau, \sigma) \) [whose expression is given in Appendix A] are all functionals of the independent variables \( z^\mu(\tau, \sigma) \) [which do not exist in general relativity, which has no global holonomic basis], the coordinates describing the embedding of 3-surfaces as spacelike hypersurfaces \( \Sigma_\tau \) in Minkowski spacetime.

Let us now consider point dependent Lorentz transformations

\[
\begin{align*}
\psi(\tau, \sigma) &\to \psi'(\tau, \sigma) = S(\Lambda(\tau, \sigma)) \psi(\tau, \sigma), \\
\bar{\psi}(\tau, \sigma) &\to \bar{\psi}'(\tau, \sigma) = \bar{\psi}(\tau, \sigma) S^{-1}(\Lambda(\tau, \sigma)).
\end{align*}
\]  

(B6)

and their action on the 4-spin connection

\[
4 \omega^A_\Lambda(\tau, \sigma) \to S(\Lambda(\tau, \sigma)) 4 \omega^A_\Lambda(\tau, \sigma) S^{-1}(\Lambda(\tau, \sigma)) - \partial_\Lambda S(\Lambda(\tau, \sigma)) S^{-1}(\Lambda(\tau, \sigma)).
\]  

(B7)

for \( \Lambda(\tau, \sigma) = L(l(\tau, \sigma), l) \) [see Appendix C for the associated Lorentz transformation]. Then, by using

\[
\bar{\psi}(\tau, \sigma) = S^{-1}(L) \bar{\psi},
\]
we get the following form for the action for Dirac fields

\[
S = \int d^{4}\sigma N(\tau, \vec{\sigma}) \sqrt{\gamma(\tau, \vec{\sigma})} \frac{i}{2} \bar{\psi}(\gamma^{\mu} \bar{z}_{\mu}^{A} - \bar{D}_{A} \bar{\psi}) + m \bar{\psi}(\gamma_{\mu} \bar{z}_{\mu}^{A} \bar{D}_{A} \bar{\psi}) \psi(\tau, \vec{\sigma}) +
\]

An analogous action could be written in tetrad gravity by using the expression of the 4-spin connection in terms of cotriads, lapse and shifts and this is the form of action to be used for defining the Hamiltonian description of fermion fields in tetrad gravity [one could also use Eq.(B1) with the 4-spin connection expressed in terms of cotriads, lapse and shifts].

In Minkowski spacetime it is more convenient to use Eq.(B3) [namely Eq.(B1) in a holonomic basis] rather than either Eq.(B9) or Eq.(B1) in other bases.

For instance in pseudoclassical electrodynamics in an arbitrary basis, Eq.(B1) becomes [for the sake of simplicity we write \( \psi \) and \( \bar{\psi} \) for \( \psi^1 \gamma^0 \)]

\[
L(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) \sqrt{\gamma(\tau, \vec{\sigma})} \left\{ \frac{i}{2} \bar{\psi}(\tau, \vec{\sigma}) \gamma^{\mu} \bar{z}_{\mu}^{A} (\tau, \vec{\sigma}) (\bar{D}_{A} - \bar{D}_{A}) \psi(\tau, \vec{\sigma}) + m \bar{\psi}(\tau, \vec{\sigma}) \right\}
\]

\[
\]
\[ -\frac{i}{8} \mathcal{F}_{ij}(\tau, \sigma) \gamma^{0} \gamma^{j} \psi(\tau, \sigma) - \frac{i}{2} \left( Tr^{3} K(\tau, \sigma) \right) \gamma^{0} \psi(\tau, \sigma) \}
+ N(\tau, \sigma) \gamma(\tau, \sigma) \left\{ \frac{i}{2} \gamma(\tau, \sigma) e^{i} \sigma^{j} \psi(\tau, \sigma) \gamma^{j} \right\} + \frac{i}{2} \left( \partial_{\tau} \gamma^{0} \psi(\tau, \sigma) + \frac{1}{4} \omega_{\tau j k}(\tau, \sigma) \gamma^{j} \gamma^{k} \psi(\tau, \sigma) \right) +
- \frac{i}{2} \left( \partial_{\tau} \gamma^{0} \psi(\tau, \sigma) - \frac{1}{4} \omega_{\tau j k}(\tau, \sigma) \psi(\tau, \sigma) \gamma^{j} \right) e^{i} \sigma^{j} \psi(\tau, \sigma) - m \gamma^{0} \psi(\tau, \sigma) \psi(\tau, \sigma) \right\},\]

(B11)

where \([3K_{ij}] \) is the extrinsic curvature of \( \Sigma_{\tau} \) embedded in Minkowski spacetime

\[ \mathcal{F}_{ij}(\tau, \sigma) \equiv \left( 3 e^{i} \sigma^{j}(\tau, \sigma) + 3 e^{j} \sigma^{i}(\tau, \sigma) + 3 e^{i} \sigma^{j}(\tau, \sigma) \right) \partial_{\sigma} N^{\tau}(\tau, \sigma) +
- N^{\tau}(\tau, \sigma) \left( 3 e^{i} \sigma^{j}(\tau, \sigma) \partial_{\sigma} e^{j}(\tau, \sigma) + 3 e^{j} \sigma^{i}(\tau, \sigma) \partial_{\sigma} e^{i}(\tau, \sigma) \right) +
+ 3 e^{i} \sigma^{j}(\tau, \sigma) \partial_{\tau} e^{j}(\tau, \sigma) + 3 e^{j} \sigma^{i}(\tau, \sigma) \partial_{\tau} e^{i}(\tau, \sigma), \]

\[ 3 K_{\tau \delta}(\tau, \sigma) = \frac{1}{2N(\tau, \sigma)} \left[ \partial_{\delta} N_{\tau}(\tau, \sigma) - \partial_{\tau} N_{\delta}(\tau, \sigma) - \partial_{\tau} g_{\delta \tau}(\tau, \sigma) \right], \]

\[ Tr^{3} K(\tau, \sigma) \equiv -3 e^{i}(\tau, \sigma) e^{i}(\tau, \sigma) \partial_{\tau} N^{\tau}(\tau, \sigma) + N^{\tau}(\tau, \sigma) e^{i}(\tau, \sigma) \partial_{\tau} e^{i}(\tau, \sigma) +
- 3 e^{i}(\tau, \sigma) \partial_{\tau} e^{i}(\tau, \sigma). \] (B12)
APPENDIX C: TRANSFORMATION PROPERTIES OF THE DIRAC FIELD UNDER WIGNER BOOSTS.

We can give an exponential form [23] of the Wigner boost of Eq. (A9)

\[ L_{\mu \nu}(p, \tilde{p}) = \exp [\omega(p)I(p)]^{\mu \nu} = \]

\[ = \left[ \cosh (\omega(p)I(p)) + \sinh (\omega(p)I(p)) \right]^{\mu \nu} = \]

\[ = \left[ I - I^2(p) + I^2(p) \cosh \omega(p) + I(p) \sinh \omega(p) \right]^{\mu \nu}, \]

\[ L_{\mu \nu}(\tilde{p}, p) = \exp \left[ -\omega(p)I(p) \right]^{\mu \nu}, \]  

(C1)

where

\[ \cosh \omega(p) = \frac{\eta p_0}{\sqrt{p^2}}, \quad \sinh \omega(p) = \eta \frac{p}{\sqrt{p^2}}, \]

\[ I(p) \equiv \| I(p) \| = \left( \begin{array}{cc} 0 & -\frac{p^0}{\sqrt{p^2}} \\ \frac{p^0}{\sqrt{p^2}} & 0 \end{array} \right), \]

\[ I_{\mu \nu}(p) = -I_{\nu \mu}(p), \quad I^3(p) = I(p). \]  

(C2)

If we consider the generating function of the canonical transformation of Eq.(40)

\[ F(p_s) = \frac{1}{2} \omega(p_s)I_{\mu \nu}(p_s)S^{\mu \nu}, \]  

(C3)

with \( S^{\mu \nu} \) the total spin of Eq.(35) [and not only \( S^{\mu \nu}_s \) as one would have for scalar fields], we get

\[ \tilde{\psi}(\tau, \bar{\sigma}) = \exp \{ F, \}_{D} \tilde{\psi}(\tau, \bar{\sigma}) \equiv \]

\[ = \psi(\tau, \bar{\sigma}) + \{ F, \psi(\tau, \bar{\sigma}) \}_{D}^* + \frac{1}{2} \{ F, \{ F, \psi(\tau, \bar{\sigma}) \} \}_{D}^* + ... = \]

\[ = \psi(\tau, \bar{\sigma}) + \frac{i}{4} \omega(p_s)I_{\mu \nu}(p_s)\sigma^{\mu \nu}\psi(\tau, \bar{\sigma}) + ... = \]

\[ = \exp \left[ \frac{i}{4} \omega(p_s)I_{\mu \nu}(p_s)\sigma^{\mu \nu} \right] \psi(\tau, \bar{\sigma}). \]  

(C4)

This shows that this canonical transformation implements on the Dirac fields the action of Wigner boosts realized by using the standard representation of Lorentz transformations in terms of Dirac matrices [24]

\[ S(L(\tilde{p}_s, p_s)) = \exp \left[ \frac{i}{4} \omega(p_s)I_{\mu \nu}(p_s)\sigma^{\mu \nu} \right]. \]  

(C5)

Since we have from Section IV

\[ \{ \tilde{x}_s^\mu, \tilde{\psi}(\tau, \bar{\sigma}) \}_{D}^* = \{ \tilde{x}_s^\mu, S(L(\tilde{p}_s, p_s))\psi(\tau, \bar{\sigma}) \}_{D}^* = \]

\[ = -\frac{\partial S(L(\tilde{p}_s, p_s))}{\partial p^\mu_{s \mu}} S^{-1}(L(\tilde{p}_s, p_s)) \tilde{\psi}(\tau, \bar{\sigma}) + \]

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\[
- \frac{i}{4} e^A(u(p_s)) \frac{\partial e^B(u(p_s))}{\partial p_{\mu\nu}} S(L(\hat{p}_s, p_s)) \sigma^{\nu\rho} S^{-1}(L(\hat{p}_s, p_s)) \partial_\lambda \psi (\tau, \vec{\sigma}) =
\]
\[
- \frac{\partial S(L(\hat{p}_s, p_s))}{\partial p_{\mu\nu}} S^{-1}(L(\hat{p}_s, p_s)) +
\]
\[
- \frac{i}{4} \eta^{\sigma B} \frac{\partial p_{\mu\rho}(u(p_s))}{\partial p_{\mu\nu}} \lambda \eta (p_s, \hat{p}_s) \sigma^{\sigma \eta} \right) \partial_\lambda \psi (\tau, \vec{\sigma}),
\]

(C6)

to verify \( \{ \hat{x}_s^\mu, \psi (\tau, \vec{\sigma}) \} = 0 \), see Eq.(43), we need the evaluation of \( \frac{\partial S(L(\hat{p}_s, p_s))}{\partial p_{\mu\nu}} \).

In Ref. [25], there is the following formula
\[
\frac{\partial e^{B(\lambda)}}{\partial \lambda} = \int_0^1 dx e^{x B(\lambda)} \frac{\partial B(\lambda)}{\partial \lambda} e^{-x B(\lambda)} e^{B(\lambda)},
\]
(C7)
giving the derivative with respect to a continuous parameter \( \lambda \) of the exponential of an operator \( B(\lambda) \). If we put
\[
A(p_s) \equiv \frac{i}{4} \omega(p_s) I_{\mu\nu} p_s \sigma^{\mu\nu} = -\frac{i}{2} \frac{\omega(p_s)}{|p_s|} p_s \sigma^0,
\]
\[
S(L(\hat{p}_s, p_s)) = e^{A(p_s)},
\]
(C8)
and if we suppose \( p_s^\mu = p_s^\mu(\lambda) \), we have
\[
a) \frac{\partial e^{A(p_s(\lambda))}}{\partial \lambda} = \frac{\partial p_{\mu\nu}(\lambda)}{\partial \lambda} \frac{\partial e^{A(p_s(\lambda))}}{\partial p_{\mu\nu}},
\]
\[
b) \frac{\partial e^{A(p_s(\lambda))}}{\partial \lambda} = \int_0^1 dx e^{x A(p_s(\lambda))} \frac{\partial A(p_s(\lambda))}{\partial \lambda} e^{x A(p_s(\lambda))} e^{A(p_s(\lambda))} =
\]
\[
= \frac{\partial p_{\mu\nu}(\lambda)}{\partial \lambda} \int_0^1 dx e^{x A(p_s(\lambda))} \frac{\partial A(p_s(\lambda))}{\partial p_{\mu\nu}} e^{x A(p_s(\lambda))} e^{A(p_s(\lambda))}.
\]
(C9)

This implies
\[
\frac{\partial S(L(\hat{p}_s, p_s))}{\partial p_{\mu\nu}} = \frac{\partial e^{A(p_s)}}{\partial p_{\mu\nu}} = \int_0^1 dx e^{x A(p_s)} \frac{\partial A(p_s)}{\partial p_{\mu\nu}} e^{x A(p_s)} e^{A(p_s)}.
\]
(C10)

Following Ref. [25], the solution of this equation is
\[
\frac{\partial e^{A(p_s)}}{\partial p_{\mu\nu}} = \sum_{n=0}^{\infty} \left[ A^n(p_s), \frac{\partial A(p_s)}{\partial p_{\mu\nu}} \right] e^{A(p_s)},
\]
(C11)
where \( [A^n, B] \) means
\[
[A^0, B] = B, \quad [A^1, B] = AB - BA,
\]
\[
[A^{n+1}, B] = [A, [A^n, B]].
\]
(C12)

From the following commutators of Dirac matrices
\[\left[\gamma^\mu, \gamma^\nu\right] = -2i\sigma^{\mu\nu},\]
\[\left[\sigma^{\mu\nu}, \gamma^\rho\right] = 2i(\gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}),\]
\[\left[\sigma^{\mu\nu}, \sigma^{\alpha\beta}\right] = 2iC^{\mu\nu\alpha\beta}\sigma^{\gamma\delta},\]  
\hspace{1cm} (C13)

and using Eq.(C2), we get

\[
\frac{\partial S(L(p_s^0, p_s))}{\partial p_s^{\mu}} = \left[ \frac{i}{2} \frac{p_s i \sigma^{0i}}{\epsilon_s^2 (p_s + \epsilon_s^2)} \left( p_s^\mu + 2\epsilon_s \eta^\mu_0 \right) - \frac{i}{2} \frac{\sigma^{0\mu}}{\epsilon_s} + \right. \\
\left. + \frac{i}{2} \frac{p_s i \sigma^{i0}}{\epsilon_s (p_s + \epsilon_s)} \right] S(L(p_s^0, p_s)) = \\
\left[ -\frac{i}{4} \eta_{\epsilon B} \frac{\partial \epsilon_B}{\partial p_s^{\mu}} (u(p_s)) L_{\eta}^\rho (p_s, p_s^0) \sigma^{\eta} \right] S(L(p_s^0, p_s)).\]  
\hspace{1cm} (C14)
REFERENCES

P.A.M.Dirac, in “Recent Development in General relativity” (Pergamon, 1962).


