Geometric Construction of
\[ N = 2 \] Gauge Theories\(^1\)

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In this lecture we give an elementary introduction to the natural realization of non-perturbative \( N = 2 \) quantum field theories as a low energy limit of *classical* string theory. We review a systematic construction of six, five, and four dimensional gauge theories using geometrical data, which provides the exact, non-perturbative solution via mirror symmetry. This construction has lead to the exact solution of a large class of gravity free gauge theories, including Super Yang Mills (SYM) theories as well as non-conventional quantum field theories without a known Lagrangian description.

June 1998

\(^1\) Lecture given at the conference *Quantum aspects of gauge theories, supersymmetry and unification*, Neuchatel University, 18-23 September 1997.
1. Introduction: a sketch of the geometric idea

In the past few years, the understanding of non-perturbative aspects of field and string theory has been improved drastically. Two of the most outstanding developments have been the exact solution of $N = 2$ SYM theories by Seiberg and Witten [1] and the understanding of D-branes as charged, solitonic degrees of freedom of string theories [2]. The subject of this lecture deals with the relation of these two important works: the realization, derivation and generalization of the field theory results [1] from type II strings compactifications in connection with D-brane configurations [3,4,5,6]. From the point of string theory it is rather satisfying to see that the exact solution of field theory delivers a geometric object - the Seiberg-Witten torus $\Sigma$ - whose appearance is somehow obscure from the point of field theory but has a very transparent meaning in terms of type II brane geometries. In particular, $\Sigma \times M_4$ is to be identified with the world volume of a type IIA five-brane [4]. Thus string theory provides a deeper understanding, and as we will see, also an improvement and generalization of field theory aspects.

We start with a short outline of the geometrical realization of $N = 2$ field theories in terms of type II strings. Let us begin with a type II string compactification to six dimensions on a K3 manifold. Part of the six-dimensional low energy physics will be described by the dimensional reduction of the ten-dimensional supergravity action. However there can be additional light degrees of freedom arising from D-branes wrapped on n-dimensional supersymmetric cycles $C_n$ of the compactification geometry [7]. These states are BPS saturated, with masses (or more generally tensions) depending on the (appropriately defined) volume of the wrapped D-brane geometry.

In the following we will concentrate mostly on supersymmetric cycles of complex dimension one, that is projective spaces $\mathbb{P}^1$ or, equivalently in terms of real geometry, two-dimensional spheres $S^2$. For such a two-cycle $C_2$ inside the K3, the condition to be a supersymmetric cycle is simply that it is holomorphic in one of the possible complex structures [8]. Depending on whether we compactify type IIA or type IIB string on this local geometry, we obtain two different theories in six dimensions. This is shown in fig.1.
More specifically, we obtain a point-like state from the D2-brane wrapping of the type IIA string theory. This kind of D-brane geometry gives rise to a pure $SU(2)$ SYM theory in six dimensions, for reasons which will become clear later on. In the type IIB theory we get a one-dimensional object, that is a string, from wrapping a D3-brane [9]. This non-critical string is of course not the same as the fundamental string we started with. The mass/tension of the particle/string is proportional to the volume of the two-cycle $C_2$.

Although we get very different theories in six dimensions from compactification of type IIA vs. type IIB on the same geometry, there is a new relation on further compactification to four dimensions. In particular, in the compactification on a two-torus $T^2$ with $N = 4$ supersymmetry, the type IIA and type IIB theories are related by T-duality acting on the extra torus. A similar relation holds for more general compactification geometries with $N = 2$ supersymmetry, which are not simply products of a K3 compactification to six dimensions and a torus compactification. In the absence of adjoint matter representations, the geometry that replaces the $T^2$ of the compactification from six to four dimensions is again a $\mathbb{P}^1$ and the Calabi–Yau condition requires the total manifold to have a non-trivial fibration structure rather than being a simple product. This means that the K3 fiber $X_2$, whose two-cycles carry the wrapped D-brane states, varies holomorphically over the points on a new $\mathbb{P}^1$, $X_2 = X_2(z)$, where $z$ denotes the parameter of the $\mathbb{P}^1$. The new $\mathbb{P}^1$, which more generally can be replaced by a collection of intersecting $\mathbb{P}^1$’s, is called the base of the fibration. The total space of the K3 fiber together with the base builds up the Calabi–Yau threefold $X_3$ on which the type IIA theory is compactified to four dimensions. The relation, which identifies this four-dimensional type IIA compactification with a type
IIB compactification on a different Calabi–Yau three-fold $X_3^*$ is called mirror symmetry and plays the key role in the exact solution of the $N = 2$ theory obtained in this way.

*Interactions from intersections:* To construct more interesting kind of theories with various kinds of gauge groups and/or matter content we need interacting D-brane states. This corresponds to a compactification geometry with intersecting two-spheres:

![Diagram of intersecting two-spheres](image)

Fig. 2: Interactions from intersecting two-spheres.

*Dynkin diagrams:* Instead of drawing pictures of intersecting two-cycles as in fig.2, there is a much more convenient representation of the type IIA D-brane geometry which makes at the same time apparent the amazingly close relation to group theory: if we draw a node for each $\mathbb{P}^1$ and a link for each intersection (possibly weighted by an integer number representing multiple intersections), we get from fig.2 the diagram shown in fig.3:

![Diagram of Dynkin diagram](image)

Fig. 3: Dynkin diagram for the geometry in fig.2.

This is the Dynkin diagram of $A_2$. This is no coincidence: in fact the type IIA D-brane geometry in fig.2 gives rise to a $SU(3)$ gauge system in six dimensions. This relation between the geometry of two-cycles and the gauge group of the type IIA theory in six dimensions will hold for all simply laced groups, that is $A_n, D_n$ and $E_n$.

*Matter and product gauge groups:* For reasons that will become clear in a moment, the pure gauge theories with ADE gauge groups are the only possibilities in six dimensions on purely geometrical grounds (assuming the absence of RR background fields). However if we compactify further to four dimensions, we have in addition the possibility to add matter, possibly charged under more than one gauge group. As explained later on, matter arises from an extra localized $\mathbb{P}^1$ over a point of the base $\mathbb{P}^1$ as shown in fig.4.
Fig. 4: Matter from localized enhancement of the singularity.

More general configurations: The relation between geometry of intersecting two-cycles and Dynkin diagrams leads to the natural question of what does it mean if we have much more general configurations of intersecting two-cycles:

\[
\begin{array}{c}
\circ - \circ - \circ - \cdots - \circ - \circ - A_n \\
\circ - \circ - \circ - \cdots - \circ - \circ - D_n \\
\hat{A}_2 - \hat{D}_n - \cdots
\end{array}
\]

Fig. 5: Diagrammatic representation of more general two-cycle configurations.

The general answer to this question is not known. The fact that the two-cycles come as part of a Calabi–Yau geometry reduces this question to the analysis of type IIA compactifications on general singularities of Calabi–Yau three-folds. However there is a very special subclass of $N = 2$ theories arising from type IIA compactifications which we will consider in the following:

- We will restrict to a class of Calabi–Yau geometries, which generalize K3 fibrations and can therefore be interpreted as a six-dimensional compactification followed by a further position dependent compactification to four dimensions. The generalization is in the following sense: instead of considering a global K3 geometry, we consider only a local neighborhood of the geometry of intersecting two-cycles. These geometries are described by non-compact ALE spaces with ADE type singularities at the origin. The

\footnote{The fact that we have to consider \textit{singular} Calabi–Yau spaces is related to the decoupling of gravity as will be explained below.}
two-cycles can be understood as the blow up spheres of the resolution of the ADE singularity. The total space will be therefore a non-compact Calabi–Yau threefold of the form of a two complex dimensional ALE space fibered over a one-dimensional base geometry, which is itself a collection of intersecting two-cycles.

Furthermore we consider geometries leading to $N = 2$ theories which can be consistently decoupled from gravity. This conditions restricts the possible base geometries as well as the possible kinds of fibrations.

The second condition does not necessarily mean that these theories will be conventional $N = 2$ gauge theories in four dimensions, however. Different kind of geometries may also give rise to quantum theories without a (known) Lagrangian formulation, such as interacting conformal field theories or theories involving non-critical strings.

Of course we are not only interested in constructing these theories as type IIA D-brane configurations, but also to solve them exactly. Amazingly, this exact solution is immediately obtained using a *classical* symmetry of string theory, namely mirror symmetry!

The present subject is related to the topics presented in Philip Candelas lecture at this conference on gauge symmetries from toric polyhedra in the context of F-theory/heterotic string duality [10][11][12]. The type IIA D-brane configuration that we discuss provides the microscopic explanation for the observations on the relation between toric polyhedra and the gauge groups of the dual heterotic theory [13]. However note that we do not need any non-perturbative string duality (and in particular no heterotic description) but only classical type II string theory for the understanding of the gauge theory. Moreover the intrinsic objects are D2-branes wrapped on configurations of intersecting two-cycles, no matter how we realize this geometry; in particular this is also true for geometries which cannot be represented in toric geometry. For previous lectures on the subject see [14].

2. Basic Concepts

2.1. *From exact* $N = 2$ *SYM theory to string theory*

In 1994, Seiberg and Witten achieved to determine the exact effective action up to two derivatives of $N = 2$ $SU(2)$ SYM theory [1]. More specifically, the $N = 2$ theory has generically Higgs branches as well as Coulomb branches. The Higgs branch is parametrized by scalar fields in hypermultiplets with flat directions and can be computed using the classical
Lagrangian of gauge theory. On the other hand the Coulomb branch is parametrized by the scalar fields in vector multiplets and the exact effective action is affected by corrections from non-perturbative point-like instantons. The main physical object on the Coulomb branch is the effective gauge coupling $\tau_{\text{eff}} = \theta/\pi + 8\pi i/g^2$ depending on the modulus on the $U(1)$ Coulomb branch, namely the scalar component $a$ of the neutral part of the $SU(2)$ vector multiplet $\phi$. $\tau_{\text{eff}}$ appears as the second derivative of a holomorphic prepotential $\mathcal{F}(a)$:

$$
\tau_{\text{eff}}(a) = \frac{\partial^2}{\partial a^2} \mathcal{F}(a) = \frac{\partial_u a D}{\partial_u a} .
$$

In the second expression, $a$ and $a_D$ denote the two period integrals of a certain meromorphic one form $\lambda$ on a complex torus $\Sigma$:

$$
a = \int_{\alpha_1} \lambda, \quad a_D = \int_{\alpha_2} \lambda ,
$$

where $\alpha_i$, $i = 1, 2$ are a basis of one-cycles on $\Sigma$. Moreover, $u$ is the Weyl invariant modulus $u = tr\phi^2$ parameterizing the complex structure of the torus. There is an important formula for the mass of a BPS state with electric/magnetic quantum numbers $(q_e, q_m)$ in terms of the periods $a$ and $a_D$:

$$
m(q_e, q_m) = \sqrt{2}|q_e \cdot a + q_m \cdot a_D| .
$$

Let us recall the logic of the approach of ref.[1]. Holomorphicity of the $N = 2$ gauge coupling ensures that the exact non-perturbative gauge coupling $\tau_{\text{eff}}$ is determined by a finite set of data, namely the singularities in the moduli space parameterized by the scalar vev together with the local behavior at these singularities. For asymptotic free theories, the local behavior for large values of the Coulomb parameter is known from the perturbative spectrum. Imposing positivity of the gauge coupling, it was possible to collect sufficiently enough information about the extra singularities at strong coupling to determine the exact solution $\tau_{\text{eff}}(a)$, including all non-perturbative instanton corrections: the mathematical answer to the problem is that $\tau_{\text{eff}}$ is the period matrix of the torus $\Sigma$, as described by eqs. (2.1),(2.2).
One of one's first thoughts concerning this result is: "is there life on the torus"? Does it have a physical meaning apart from its mathematical usefulness? Is there a realization of other physical quantities of the $N = 2$ field theory in terms of the torus? The answer to this question has been found soon to be "yes" [3], with a result which appears to be quite a strong hint in favor of string theory:

\[ \Sigma \] is the compactification geometry of a type II string.

We will give a much more precise statement of this relation, including the generalization of $\Sigma$ to other gauge groups later on; in particular the mathematical answer to field theory that replaces the torus of $SU(2)$ in the case of more complicated gauge theories are three complex dimensional Calabi–Yau manifolds, precisely as expected from string theory.

Moreover the quantum effects of the $N = 2$ gauge theory are classical effects from the point of string theory! This correspondence maps instantons of the gauge theory to geometrical objects of a type II compactification, which can be done [15] thanks to the power of mirror symmetry, a symmetry of classical type II string theory [16].

Historically, the use of mirror symmetry for the calculation of space-time instanton effects has been started in the context of type IIA/heterotic duality [11,12,3]. However it is important to note that we do not need the non-perturbative, heterotic picture: all we need is classical type II string theory including the charged RR states from D-brane wrappings [7].

Let us mention some advantages of the string understanding of the torus $\Sigma$ as compared to the field theory point of view. Firstly, we have a concrete physical meaning for the surprising appearance of $\Sigma$ in the exact solution of the SYM theory. Secondly, string theory provides the framework to define additional quantities of the SYM theory starting from the torus: $\Sigma$ appears as part of the target geometry of the string sigma model and there is a well-defined framework to calculate corrections, such as higher derivative terms or gravitational corrections. Note that the exact string theory solution obtained from mirror symmetry already includes all gravitational corrections. In fact the decoupling of gravity is one of the non-trivial steps to obtain the exact solution of globally supersymmetric SYM theory [3].

Moreover, string theory provides also new insights to field theory itself, which will be discussed in the following: the representation of BPS states as windings of self-dual strings on $\Sigma$ and the determination of the stable BPS spectrum using this picture [4], the systematic
generation of many new solutions with arbitrary gauge groups from the classification of geometric singularities, the S-duality groups of these theories and a physical interpretation of this symmetry [6].

2.2. Three higher-dimensional string theory embeddings of $N = 2$ SYM

There are three T-dual string theory embeddings of four-dimensional $N = 2$ SYM which we will discuss in this lecture; all of them are related by some kind of T-duality (fig.6).

<table>
<thead>
<tr>
<th>type IIA</th>
<th>type IIB</th>
<th>type IIA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_3$</td>
<td>$X'_3$</td>
<td>five brane $\Sigma \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>classical string theory</td>
<td>classical string theory</td>
<td>classical five brane world volume theory</td>
</tr>
<tr>
<td>classical SYM</td>
<td>non-perturbative SYM</td>
<td>classical SYM</td>
</tr>
</tbody>
</table>

**Fig. 6:** Three T-dual type II compactifications.

The starting point of the geometric construction is type IIA theory on a Calabi-Yau three-fold $X_3$. The geometry of $X_3$ contains a local patch with a collection of intersecting two-cycles that support the light states which are relevant for the SYM theory in an appropriate region of the moduli space. The wrapped D2-brane states together with the massless fundamental string excitations provide the perturbative degrees of freedom of the four-dimensional gauge theory. The classical\(^3\) string theory answer agrees with the classical gauge theory answer after decoupling gravity.

Mirror symmetry maps the type IIA theory on $X_3$ to a type IIB theory on the mirror manifold $X_3^*$. This symmetry has been interpreted in [17] as a T-duality transformation; for the special geometries that we will consider the relation to T-duality will be quite explicit. The important point for the solution of the perturbative theory constructed in the first step is that the classical string theory answer for type IIB on $X_3^*$ is already the full exact result.

A third representation of the same theory is obtained from this theory by a different T-duality transformation, which maps type IIB on the $A_n$ singularity to type IIA on $n$ symmetric five-branes [18]. The $N = 2$ SYM theory appears as the world volume theory

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\(^3\) Both, in the space-time as well as string worldsheet sense.
of the five-brane, which has the geometry $\Sigma \times \mathbb{R}^4$, with $\Sigma$ the “Seiberg-Witten geometry” as before. We will give a short discussion of this representation in sect.2.9.

What the three representations have in common is that the charged states of the SYM theory are represented by D-brane states. The perturbative definition of the $N = 2$ SYM theory, determined by the root lattice corresponding to a gauge group $G$ and a weight lattice of matter representations $R_i(G)$ of $G$, is in one-to-one correspondence with the D-brane geometry determined by the homology lattice of two-cycles. This close link between the perturbative definition of a $N = 2$ theory and the geometrical data offers an interesting approach to study a large class of quantum field theories: as we will see in a moment, the relevant geometries of two-cycles are a special kind of singularities which are well studied mathematically.

<table>
<thead>
<tr>
<th>perturbative spectrum:</th>
<th>classification of N=2 QFT's</th>
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<tbody>
<tr>
<td>charges</td>
<td>1-1</td>
</tr>
<tr>
<td>2-cycle geometry</td>
<td>singularities</td>
</tr>
<tr>
<td></td>
<td>classification of geometric singularities</td>
</tr>
</tbody>
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2.3. The starting point: $N = 2$ in six dimensions

The four-dimensional theories which we will consider can be understood as certain compactifications of six-dimensional $N = 2$ theories. It is useful to keep in mind this distinction between the compactification to six dimensions followed by a further compactification to four dimensions in this construction: the former will determine the gauge group $G$ of the four-dimensional theory, whereas the second step will contain the information about the matter representations $R_i(G)$.

$N = 2$ theories in six dimensions arise from type IIA compactification on a K3 manifold $X_2$ or a non-compact geometry with the same holonomy properties. The relevant data of the K3 geometry is the homology $H_2(X_2)$ of two-cycles, which is also dual to the cohomology of 2-forms $H^2(X_2)$. There are two sources of particle states in the six-dimensional theory: uncharged fields arise by dimensional reduction of the ten-dimensional RR 3-form $A^{(3)}$: $A^{(3)} \rightarrow A^a \wedge \omega^a$, 

\begin{equation} \tag{2.4} \end{equation}
where \( \omega^a \in H^2(X_2) \) is a basis of 2-forms and \( A^a \) corresponds to a neutral four-dimensional vector multiplet. The charged fields arise from D2-branes wrapped on the two-cycle \( C_2^a \) dual to \( \omega^a \); the charge arises from the worldsheet coupling

\[
\int_{C_2^a} d^3\sigma A^{(3)} \rightarrow \int d^7\tau A^a .
\]  

(2.5)

As an example consider the simple geometry with one two-cycle \( C_2 \) in fig.7. It gives rise to a \( SU(2) \) gauge theory in six dimensions:

![Fig. 7: Geometry for the six-dimensional SU(2) gauge theory.]

The bosonic components of the \( Z \) vector multiplet are the vector field \( A_\mu \) from the 2-form dual to \( C_2 \) and the scalar component \( a \) which measures the volume of \( C_2 \) as defined by the Kähler form. The \( W^\pm \) vector multiplet arises from the D2-brane wrapped on \( C_2 \) with the two possible orientations. The mass of these vector bosons is proportional to the volume of \( C_2 \), that is proportional to the Coulomb parameter \( a \), in agreement with field theory.

More generally, the single two-cycle \( C_2 \) is replaced by a collection of intersecting two-cycles \( C_2^a \) contained in a local piece of the K3 manifold (fig.8).

![Fig. 8: Local geometry of K3 singularities.]

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Let us first explain why we have to consider *local singularities*. Since we want to decouple gravity, we have to take a limit where $m_{\text{pl}} \to \infty$. However, at the same time, we want to keep finite the masses of the wrapped D2-brane states, e.g. the $W^\pm$ bosons, which are proportional to the volume of the two-cycle, $m_{W^\pm} \sim \text{Vol}(C_2)$ in units of $m_{\text{pl}}$. In other words, we have to consider *very small volumes* for the two-cycles which in turn means to consider some sort of singular geometries. Luckily, all singularities of a polarized K3 manifold at finite distance in the moduli space are well-known. The homology of small two-cycles consists of collections of $\mathbb{P}^1$'s which intersect according to the Dynkin diagrams of the simply laced groups A,D and E [19]. These singularities are called the ADE singularities of K3.

The reason that we have to consider only the *local* geometry of the singularity is then obvious for the same reason: all other homology cycles of the global geometry stay at a generic volume and give rise to corrections from super-massive states which vanish in the limit $m_{\text{pl}} \to \infty$. As mentioned already, the local geometry of the K3 ADE singularities is captured by the so-called ALE spaces of ADE type. These are the geometries we have to consider in the following.

There is also a meaning to the extended Dynkin diagrams corresponding to the affine versions of A,D and E: if the ADE singularity arises from the collision of singular fibers of an elliptic fibrations as classified by Kodaira [21], there is an additional two-cycle class (the class of a generic fiber) which corresponds to the extended node of the Dynkin diagram. These extended Dynkin diagrams play a very special role for the superconformal four-dimensional theories considered in [6]. However, shrinking this extra two-cycle is at an infinite distance of the K3 moduli space and is therefore not relevant for the six-dimensional gauge groups.

### 2.4. Four-dimensional theories from compactification

To obtain four-dimensional theories we consider a further compactification on a one complex dimensional base:

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4 For recent results on non-simply laced groups from K3 manifolds see [20].
\[ \text{type IIA} \hspace{2cm} \text{Fiber} \]
\[ \text{ALE} \hspace{2cm} 6d \]
\[ \text{Base} \hspace{2cm} 4d \]

*N = 4 Torus compactification*: The simplest example is a compactification where the base geometry is that of a torus. In this case we get extra scalars from decomposing the six-dimensional vector fields \( A \rightarrow \omega' \wedge \phi \), where \( \omega' \) is a harmonic one-form on the torus. An important relation, which exists independently of the special base geometry, is the relation of the four-dimensional gauge coupling to the volume of the base, as obtained from dimensional reduction:

\[
\frac{1}{g_6^2} \cdot \text{Vol(Base)} = \frac{1}{g_4^2}
\]

This means in particular, that the Montonen-Olive duality \( g_4 \rightarrow 1/g_4 \) arises from the geometric T-duality \( V \rightarrow 1/V \), a classical string symmetry [22,23,24].

\[ \mathbb{P}^1 \text{ as the base: } N = 2 \text{ in four dimensions} \]: To reduce the supersymmetry in the compactification from six to four dimensions (and therefore to end up with \( N = 2 \) supersymmetry) we have to choose a base geometry with non-trivial curvature. The simplest case is to take again a two-sphere, or \( \mathbb{P}^1 \). The curvature of \( \mathbb{P}^1 \) kills the extra scalars from the vector multiplets; in particular there are no harmonic one-forms, \( h^{1,0}(\mathbb{P}^1) = 0 \). However to preserve the Calabi–Yau condition, the total geometry can no longer be of a simple product form; the ALE space has to vary over different points on the base \( \mathbb{P}^1 \). This geometric structure is called a fibration, more precisely, in this case we have a fibration of an ALE space over a base \( \mathbb{P}^1 \).

*Gauged coupling constants*: There is an interesting aspect of the geometric construction, which will play a role later on in the case of four-dimensional theories with vanishing beta-function. Consider the simplest configuration of a single \( \mathbb{P}^1_{\text{Fiber}} \) fibered over a base \( \mathbb{P}^1_{\text{Base}} \):
As mentioned previously, we have still the relation \( 1/g_4^2 \sim Vol(P^1_{\text{Base}}) \). However note that the coupling \( g_4 \) appears as the scalar component of a full vector multiplet. Moreover, from wrapping the D2-brane over the base \( P^1_{\text{Base}} \), we get charged \( W^\pm \) vector multiplets as well. These states are the light degrees of freedom of a different \( SU(2)_{\text{Base}} \) theory [25], which however is restored at infinite coupling of the six dimensional \( SU(2)_{\text{Fiber}} \) from the fiber \( P^1_{\text{Fiber}} \), since its Coulomb parameter is related to the gauge coupling of the fiber theory by

\[
a(SU(2)_{\text{Base}}) \sim \frac{1}{g^2_{\text{Fiber}}}.
\]

In general, this gauge theory will decouple in the \( m_{pl} \to \infty \) limit which we take in going from string theory to the field theory limit with gauge group \( SU(2)_{\text{Fiber}} \). However this is not true in the case of vanishing beta-function for the fiber gauge group. This will lead to interesting new theories discussed later on.

**Incorporation of matter:** Let us next explain in more detail the appearance of matter representation in the four-dimensional \( N = 2 \) theory. Recall the logic of the geometric construction: the perturbative gauge theory is defined by the charge lattice generated by the roots (determining the gauge system) and now in addition weights for the matter representations. If this lattice is realized geometrically as the lattice of homology two-cycles, the D-brane wrappings of the type IIA theory will generate the appropriate physical states in the string compactification. So to add matter, all we have to do is to add two-cycles which intersect in the appropriate way. The simplest example is again that of a \( SU(2) \) theory, now with \( N_f = 1 \) matter. To obtain a matter multiplet, we simply add a new two-cycle, a \( P^1 \) which intersects the first \( P^1 \) of the \( SU(2) \) gauge theory.

The D-brane wrappings on the first \( P^1 \) still generate the \( W^\pm \) bosons of the \( SU(2) \) gauge theory, while the wrapping on the second, intersecting \( P^1 \) should correspond to the
matter multiplet. However such a configuration reminds very much of the geometry of a $SU(3)$ theory as in fig. 2. In fact this is true up to a small subtlety: the new $P^1$ which provides the matter is localized on the base $P^1$. In more detail this means that over the generic point $z$ of the base $P^1$, there is only a single two-cycle in the fiber which supports the gauge bosons, while for a special point on the base $P^1$, say $z = 0$, the fiber contains an extra two-cycle class that supports the matter. This geometry is shown in fig. 4.

The fact that this geometry is similar to the geometry of a $SU(3)$ theory is related to the fact that the matter content of geometrically constructed $N = 2$ theories can be understood in terms of adjoint breaking [26]. Consider the breaking of the $N = 2$ gauge theory in six dimensions by vev's of the adjoint scalar fields. The idea is to consider fibrations, where the scalar field of a $U(1)$ subgroup of the gauge group $G \supset H \times U(1)$ is identified with the fibration parameter $z$.

The surviving gauge group in the lower-dimensional theory is $H$. Over a general point on the base, the $G$ singularity of the fiber is resolved to an $H$ singularity and the two-cycle classes of the latter support the vector bosons corresponding to the roots of $H$. However at the special point $z = 0$, the singularity is still of type $G$ and the extra, localized two-cycles give rise to additional states in a representation $R'$ of $H$. Here $R'$ denotes the representation obtained by the decomposition of adj($G$) according to the breaking $G \supset H \times U(1)$. E.g., in the above example we have

$$SU(3) \supset SU(2) \times U(1) : 8 \rightarrow 2 + 2 \cdot 2 + 1,$$

that is $R' = 2$, in agreement with the appearance of a fundamental hyper multiplet.

The Lorentz quantum numbers of the states wrapped on the generic or special $P^1$'s follow from the quantization of the collective coordinates corresponding to the moduli space of the two-cycles [27, 26]. Note that the moduli space of a generic $P^1$ is a $P^1$ (the base) and that of a special $P^1$ is a single point. A heuristic explanation follows from a brane picture using open strings, either in F-theory [26] or from the T-dual configuration [28] of flat branes. E.g., in the latter case it is well-known, that parallel branes lead to enhanced non-abelian gauge symmetries, while intersecting branes generate matter [29]. Note that the gauge bosons arise from open strings which can move freely along all directions of the parallel branes whereas the open string between intersecting branes is localized in the directions which are not common to both branes. In this case the determination of the Lorentz quantum numbers, arising from a quantization of fermionic zero modes, is identical to a simple orbifold calculation [29].
Product gauge groups with bi-fundamental matter: As is clear from the relation to adjoint breaking, the above example is actually a simple subcase of a more general class of geometries which give rise to product gauge groups with matter representations determined by adjoint breaking [26]. E.g. instead of $SU(N) \supset SU(N-1) \times U(1)$, we can consider a breaking $SU(N) \supset SU(K) \times SU(N-K) \times U(1)$. In more general terms we consider collisions of any ADE singularities on the base manifold. Specifically, consider the case where we have a curve of $A_N$ singularities (that is a base $\mathbb{P}^1$ above which the fiber has a singularity of type $A_N$) intersecting with a curve of $A_M$ singularities. Note that the base consists now of two $\mathbb{P}^1$ factors, one for the $A_N$ singularity and one for the $A_M$ singularity. These two $\mathbb{P}^1$'s intersect at a point. The "Dynkin diagram" of the base geometry is therefore that of an $A_2$ singularity.

![Fig. 10: Intersection of two $A$ type singularities in the fiber with an $A_2$ base geometry.](image)

A general mathematical result assures that at the intersection point, the fiber singularity is of type $A_{N+M+1}$. In other words, at the intersection point, there is an extra two-cycle class corresponding to the $+1$. This is the localized $\mathbb{P}^1$ which carries a matter multiplet in the $(N+1, M+1)_{SU(1)}$ representation of the $SU(N+1) \times SU(M+1) \times U(1)$ gauge group (fig.11).

![Fig. 11: Localized bi-fundamental from enhancement of the singularity.](image)
Note that the bare mass of the matter multiplet corresponds to the Coulomb parameter of the extra $U(1)$ factor. Similarly as in the case of the gauge coupling, this bare mass is part of a full vector multiplet and is therefore gauged.

Degenerate factors: fundamental matter: It is now easy to construct geometries leading to $SU(N)$ factors with $M$ fundamental matter multiplets: recall that the gauge coupling of the $SU(M+1)$ theory in the previous paragraph is given by the volume of the corresponding base $\mathbb{P}^1$, $1/g_{SU(M+1)}^2 \sim Vol(\mathbb{P}^1_{Base})$. We can decouple the $SU(M+1)$ factor by making the second base $\mathbb{P}^1$ very large and therefore $g_{SU(M+1)} \rightarrow 0$. In this limit the vector multiplets decouple, but the matter multiplets do not. What remains geometrically is a single compact two-cycle in the base with one extra special point, the former intersection point, above which there are $M+1$ extra two-cycle classes carrying the $M+1$ fundamental matter multiplets of the $SU(N+1)$ gauge theory.

General base geometries: As we mentioned already, the ADE singularities are the only possible fiber geometries as a consequence of the classification of K3 singularities. This does not say anything about the base geometry, however. In the absence of adjoint matter, the homology of two-cycles of the base geometry is again generated by a collection of intersecting $\mathbb{P}^1$s which can be again characterized by their intersections summarized in ”generalized Dynkin diagrams” as in fig.5. If we add the information about the fiber, we consider collisions of ADE fiber singularities, described by these intersections of the base $\mathbb{P}^1$s as in fig.11. At the intersection points we should get matter representations charged under the gauge group factors corresponding to the intersecting fiber singularities.

![Intersections of fiber singularities and dual "Dynkin diagrams".](image)
However not all of these intersections make sense in terms of four-dimensional field theories, as is clear from the fact that not all combinations of group factors meeting at the intersection points can be obtained from adjoint breaking of a larger group. Only the two left diagrams in fig.12 make sense in this class. Even if one gives up the constraint to obtain a conventional field theory, not all possible collisions will lead to theories which allow for a decoupling of gravity. The classification of geometries corresponding to these two classes of $N = 2$ theories is still an open question. However there is a nice result for the subclass of field theories with gauge group an arbitrary product of $SU(n)$ factors and asymptotic free bi-fundamental and fundamental matter representations (corresponding to geometries with only $A$ type singularities in the fiber): the only possible base geometries are configurations of 2-cycles which intersect according to affine ADE Dynkin diagrams.

2.5. Instanton corrections

Given an appropriate geometry with a homology lattice of two-cycles, the fundamental type IIA string together with the D brane states will give rise to the physical states of an $N = 2$ theory. We are interested now in getting the exact instanton corrected effective action of this theory.

The determination of (an infinite number of) instanton corrections to field theory is a very hard question. As mentioned already, the case of $N = 2$ supersymmetry can be often solved starting from the knowledge of the perturbative theory and requiring consistency of the solution with holomorphicity and positivity of the gauge coupling. String theory provides an alternative framework, which is probably the most systematic and most physical one: the use of mirror symmetry, a symmetry of classical string theory.

That classical type II string theory can provide the exact solution is due to the following two facts:

-o there are no space-time instanton corrections to the vector multiplet moduli space.
-o there are worldsheet instanton corrections which are non-perturbative from the string sigma model point of view. However these instantons can be determined by mirror symmetry.

Space-time instanton corrections: Let us first recall briefly, why we do not have to bother about space-time instanton corrections (from the point of string perturbation theory).

\footnote{For criteria about the existence of such a limit, see [30].}
What we are interested in is the exact moduli space of the scalar components $a_i$ of the neutral vector multiplets, which parametrize the flat directions on the Coulomb branch of the $N = 2$ theory. To obtain the exact gauge coupling $\tau_{\text{eff}}(a_i)$ of the theory we do not have to care about scalars in hyper multiplets, since there are no neutral couplings between hyper multiplets and vector multiplets in the $N = 2$ supersymmetric theory [31].

This decoupling between hyper and vector multiplets is precisely the reason for the absence of space-time instanton corrections to the vector multiplet moduli: for the type II string compactifications on Calabi–Yau manifolds which we consider, the string coupling constant $g_{\text{string}}$ appears as a real scalar in the dilaton hyper multiplet. Therefore the gauge coupling constant on the Coulomb branch does not depend on the string coupling constant at all. Note that the vector multiplets arise from the RR sector of the theory (the anti-symmetric one and three-form potentials in ten dimensions). There are no fundamental states of type II string theory which are charged under the RR gauge fields. The absence of fundamental charged states, which have masses and couplings that depend on $g_{\text{string}}$, may serve as a heuristic, physical reasoning for the absence of space-time instantons.

The fact that there are no space-time instanton corrections to the vector multiplet moduli space in the string theory makes the determination of the exact result of course much easier: all we have to calculate is the tree-level string theory answer.

**Geometrical instantons**: Given the generically infinite series of instanton contributions to $N = 2$ SYM field theory however, it is clear that there has to be some source for these non-trivial corrections in string theory - if its answer succeeds to reproduce field theory in the appropriate point-particle limit. These corrections arise from worldsheet instanton corrections, that is corrections which are non-perturbative from the string sigma model point of view. Again they can be understood as wrappings of supersymmetric two-cycles of the compactification geometry; this time however we consider euclidean wrappings of the $1 + 1$ dimensional fundamental string worldsheet rather than wrappings of D2-branes.

Consider again the simplest situation, where we have a two-sphere $P^1_{\text{Fiber}}$ fibered over another two-sphere $P^1_{\text{Base}}$. Moreover we consider a euclidean string worldsheet, which is wrapped $k$ times on the fiber $P^1$ and $m$ times on the base $P^1$. 

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If $B$ denotes the class of the base $\mathbb{P}^1$ in $H_2(X_3)$ and $F$ that of the fiber $\mathbb{P}^1$, then the class of $C_2$ is $C_2 = k \cdot B + m \cdot F$ and the instanton action of this wrapping is

$$S \sim Vol(C_2) = k \cdot Vol(\mathbb{P}^1_{\text{Base}}) + m \cdot Vol(\mathbb{P}^1_{\text{Fiber}}) = \text{const.} \frac{k}{g^2} + m \hat{a}, \quad (2.9)$$

where we have used eq.(2.6) which identifies $Vol(\mathbb{P}^1_{\text{Base}})$ with the gauge coupling and moreover $\hat{a}$ denotes the scalar field which measures the volume of the fiber $\mathbb{P}^1$ and is related to the Coulomb scalar of the field theory vector multiplet by a holomorphic redefinition. Thus a worldsheet instanton wrapped $k$ times on the base has an action with the same dependence on the gauge coupling as a $k$ space-time instanton from the point of the gauge theory.

Now the direct calculation of the contributions of infinitely many different worldsheet wrappings would be similarly hopeless as the calculation of the space-time instantons directly in field theory. This is the point where mirror symmetry comes to help [15]. It maps the worldsheet instanton corrected type IIA theory, which we used to generate the perturbative spectrum via D2-brane wrappings, to a type IIB compactification on a different manifold. In the latter theory, the worldsheet instantons do not correct the vector moduli space. To explain this step, let us recall some facts about the moduli space of Calabi–Yau compactifications.

**Calabi–Yau moduli spaces:** A Calabi–Yau three-fold $X_3$ is a three complex dimensional Kähler manifold with vanishing first Chern class. The latter condition implies the existence of a covariantly constant spinor field which gives rise to an $N = 2$ supersymmetry in the type II compactification to four dimensions. The generic holonomy group is $SU(3)$; if the holonomy group is further reduced, as in the case of $K3 \times T^2$ or $T^6$, there are more than one covariantly constant spinors and the four-dimensional supersymmetry algebra is extended to $N = 4$ and $N = 8$, respectively [32].

There are two types of parameters which describe the geometry of $X_3$:
Kähler moduli (KM) $t_a$ are defined in terms of volumes of holomorphic two-cycles $C_a^2 \in H_2(X_3)$. If $J$ is the Kähler form on $X_3$, then the value of $t_a$ is given by $t_a = \int_{C_a^2} J$. These parameters can be understood as measuring the sizes of $X_3$ and analytic submanifolds in $X_3$.

Complex structure moduli (CSM) $z_i$ are defined in terms of the volume of three-cycles $C_i^3 \in H_3(X_3)$. The volume form is given by the unique holomorphic tree-form $\Omega$; a convenient parameterization of the complex structure is in terms of the period integrals of $\Omega$

$$z_i = \int_{C_i^3} \Omega.$$ (2.10)

In type II string compactifications, these parameters appear as the scalar components of vector or hyper multiplets in the following way:

<table>
<thead>
<tr>
<th>vector multiplets</th>
<th>type IIA</th>
<th>type IIB</th>
</tr>
</thead>
<tbody>
<tr>
<td>complex structure</td>
<td>Kähler structure</td>
<td>complex structure</td>
</tr>
<tr>
<td>$g_{string}$</td>
<td>$g_{string}$</td>
<td>$g_{string}$</td>
</tr>
</tbody>
</table>

From the above table it is now obvious, that there are no worldsheet instanton corrections to the vector multiplet in the type IIB theory for the same reason that there are also no space-time instantons\(^6\): the worldsheet instanton action depends on the volume of the relevant two-cycle $S \sim \text{Vol}(C_2)$ as in (2.9) which corresponds to a scalar field in the Kähler moduli of $X_3^*$. However the supersymmetric multiplet of the Kähler moduli is the hyper multiplet in the type IIB theory. Therefore a dependence of the vector multiplet moduli space on the worldsheet instantons would contradict the decoupling of hyper and vector multiplets.

Type IIB (D brane-) geometry: Mirror symmetry maps the type IIA compactification on a Calabi–Yau threefold $X_3$ to a type IIB compactification on the mirror manifold $X_3^*$ and the moduli space $\mathcal{M}(X_3)$ to the moduli space $\mathcal{M}(X_3^*)$ with Kähler and complex structure moduli exchanged. Moreover it maps the D brane states of the type IIA theory to D brane states of a type IIB theory.

\(^6\) The string coupling constant is again part of a hyper multiplet in type IIB; thus there are no space-time instantons corrections to the type IIB vector multiplet moduli either.
The even-dimensional D branes of type IIA theory give rise to point particles when wrapped on supersymmetric d=0, 2, 4, 6 cycles in $X_3$. The states which carry perturbative charges with respect to the gauge fields $A^a$ obtained from the decomposition $A^{(3)} = A^a \wedge \omega^a$ arise from the D2-branes wrapped on two-cycles $C_2^a$. This is the reason why we could concentrate on the geometry of two-cycles in the previous discussions. The magnetic-electric dual states arise from the dual homology cycles, that is D4 branes wrapped on 4-cycles\(^7\).

Mirror symmetry maps all these even dimensional branes of type IIA to D3 branes of type IIB wrapped on three-cycles. A hint that the type IIB description is appropriate for the analysis of non-perturbative effects comes from the fact that now electric and magnetic states are described by the same object, a wrapped D3 brane. In fact it is also easy to see from the D brane point of view, why the type IIB theory is free of worldsheet instanton corrections and the classical geometrical answer will agree with the exact result: the scalars of the vector multiplets parameterize now volumes of three-cycles rather than volumes of two-cycles. An instanton correction from wrapping an extended object will be proportional to the volume of these three-cycles. However there are no appropriate two dimensional branes in the type IIB theory which could be wrapped on the three-cycles (the fundamental string worldsheet wrappings give again rise to corrections of the Kähler moduli space, which is now parameterized by the hyper multiplets, however).

Now recall that the vector multiplets correspond to the CSM in type IIB which in turn are parameterized by the period integrals of $\Omega$ (2.10). Since there are neither perturbative nor instanton corrections at all, the classical period integrals of $X_3^*$ describe the exact vector multiplet moduli space of the $N = 2$ theory in four dimensions. This is of course very similar to the situation observed in [1], were the exact solution of $N = 2$ $SU(2)$ theory has been found to be given in terms of period integrals on a torus (2.2). We will explore this relation in more detail below with the result that

*The general solution to a $N = 2$ SYM theory is given in terms of period integrals $\int_{C_3^\Omega} \Omega$ on a local Calabi–Yau geometry $S \subset X_3^*$.*

Here $\Omega$ is again the unique holomorphic three-form of $X_3^*$. In special cases, such as the $SU(2)$ example, it can be natural to do part of the three dimensional integration and to obtain in this way a description in terms of period integrals on a one-dimensional geometry, the Seiberg–Witten torus $\Sigma$.

\(^7\) In addition there is the universal vector multiplet which exists already in ten dimensions, corresponding to the universal 0/6 cycle on $X_3$. 

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2.6. Four-dimensional field theories from the six-dimensional string

We want now to take more serious the point of view to consider the four-dimensional theory obtained from type II on a Calabi–Yau three-fold as a six-dimensional compactification followed by a further compactification to four dimensions. This viewpoint is particularly interesting from the point of the type IIB theory.

Before we explain in more detail the geometry of the type IIB compactification on $X^*_3$, which is mirror to the type IIA compactification on $X_3$ discussed above in detail, let us see what one expects intuitively. In the type IIB theory we start with the self-dual string in six dimensions, which is obtained from wrapping the D3 brane on one of the two-cycles $C^a_2$ of a K3 or a non-compact Calabi–Yau two-fold with $SU(2)$ holonomy. In the four-dimensional $N = 4$ compactification on $K3 \times T^2$, the winding states of these strings on the torus give rise to point like degrees of freedom. More precisely, if $\alpha_i, \; i = 1, 2$ is a standard basis of 1-cycles on the extra $T^2$, we get electric (magnetic) states from wrapping the string around $\alpha_1$ ($\alpha_2$). Moreover, the T-duality transformation $Vol(T^2) \rightarrow 1/Vol(T^2)$ of the torus interchanges $\alpha_1$ and $\alpha_2$. This is of course consistent with the fact that this T-duality represents an electric-magnetic duality transformation in the $N = 4$ field theory. Note that the type IIB D3 brane is now wrapped on a three-cycle $C^a_2 \times \alpha_i$.

In the $N = 2$ case the situation is very similar. Similarly as before, we can consider a D3 brane wrapping on three-cycles $C^a_3$, which are roughly speaking composed of a D3 brane wrapped on a two-cycle in the ALE fiber and a winding of the resulting string on the base (which will turn out to be the Seiberg–Witten geometry $\Sigma$ for $G = A_n$). Again these winding states give rise to the charged states of the four-dimensional theory. Note that the gauge bosons and monopoles are treated on equal footing similarly as in the $N = 4$ case.

Mirror geometry of ALE fibrations: To get a more complete picture, let us describe qualitatively the action of mirror symmetry on the geometry of the type II compactifications.

In the type IIA theory, we start with a six-dimensional compactification on the ALE space of ADE type. In the compactification to four-dimensions on a further $\mathbb{P}^1$, the total space can not be a simple product in order to satisfy the Calabi–Yau condition. Rather the complex structure of the ALE fiber varies over the points on the base $\mathbb{P}^1$. However the volumes of supersymmetric two-cycles, which are holomorphic in the complex structure of the three-fold $X_3$, do not vary. This geometric structure is of the form of a
holomorphic fibration (the ALE fiber corresponds to a certain divisor class of $X_3$). Each of the holomorphic two-cycles which stems from the fiber, supports a D2-brane state of six dimensional origin in the four-dimensional $N = 2$ theory.

Note that we could have been more general in choosing the base geometry and in particular we can choose the two-cycle homology from the base to generate the intersection lattice of an ADE singularity corresponding to any (affine) ADE Dynkin diagram\(^8\).

This will be important due to the following fact: as is well known, the K3 manifold is self-mirrored. In fact, since we are interested in singularities of K3, we can use a local version of this statement: the mirror of a deformation of an ADE singularity of type $G$ is again a deformation of an ADE singularity of type $G$. More precisely, a Kähler deformation is mapped to a deformation in complex structure.

Since our local fiber geometry is of the form of an ALE space with ADE singularity and the base geometry will correspond to the resolution of an ADE singularity as well, mirror symmetry acts somehow trivially on both the fiber and the base. Only the fibration, that is the variation of the fiber over the base, is affected non-trivially by mirror symmetry.

In fact the “fibration” structure of the type IIB geometry $X_3^\ast$ is rather different from that of $X_3$ and in general is not of the type of a holomorphic fibration. In the type IIB geometry, though there is still an ALE space over each point on the base geometry, the size of the relevant two-cycles varies with the point on the base. We will continue to use the notation of an ALE fiber and the base for the type IIB geometry though it is not a fibration in the usual sense. In particular there are special points on the base where the volume of two-cycles $C_2^a$ in the fiber vanishes. So although the naive mirror of the base geometry, which is of the form of a resolved ADE singularity, is again a resolution of an ADE singularity of the same type, there are now extra points $p_a$ on this base geometry above which a special two-cycle $C_2^a$ vanishes in the ALE fiber space.

This is important due to the fact that vanishing cycles in the ALE fiber will be associated with non-trivial monodromies. Let us explain this effect by a simple example. Given a complex torus $\Sigma$, one can define a basis of one-cycles $\alpha_i$, $i = 1, 2$. The complex structure $\tau$ of $\Sigma$ can then be expressed in terms of the period integrals $\Pi_i = \int_{\alpha_i} \omega$ as $\tau = \Pi_1 / \Pi_2$. Here $\omega$ is the holomorphic one-form on $\Sigma$. However if we move to a point in the moduli space where one of the two-cycles, say $\alpha_1$, vanishes, the definition of the basis becomes ambiguous since we can add multiples of $\alpha_1$ to $\alpha_2$ and switch the sign of

\(^8\) The example of a single base $\mathbb{P}^1$ corresponds to the $A_1$ case.
α₁ without changing the complex structure τ. In general, if we move in the moduli space, that is the τ plane divided by discrete identifications, around a closed path that includes a point where one of the cycles vanishes, the basis will be redefined by additions of the vanishing cycle and possibly minus signs. This effect is called monodromy.

Precisely this situation appears in the type IIB geometry: the two-cycles of the ALE space vary with the position z on the base, which therefore plays the role of the modulus in the above torus example. Moving around a closed path on the base which encircles a point where a two-cycle vanishes in the ALE space, the basis of the homology lattice of two-cycles of the ALE space is affected by a redefinition. Though the totality of two-cycles and their intersection properties do not change of course, individual cycles may be exchanged and redefined. The intersection lattices which we consider correspond to root lattices described by (affine) Dynkin diagrams of ADE. These lattices are invariant under the appropriate Weyl group. Since the monodromy has to leave invariant the total lattice, the monodromy transformations act as Weyl transformations on the homology of two-cycles of the ALE space.

To recap, though the mirror transformation acts somehow trivially on the fiber and the base as a consequence of the self-mirror property of the ADE singularity, the fibration structure changes. In particular there are now points on the base where some two-cycle volumes in the ALE fiber vanish. These points are associated with monodromies which take values in the Weyl group of the ADE fiber singularity.

Note that the base B is now described by a collection of intersecting P¹’s with extra points around which there are monodromies. Alternatively, we could consider a multiple cover Ŝ of B such that a closed path on Ŝ has trivial monodromy. This is the definition of a Riemann surface. In fact, in the case of Aₙ these Riemann surfaces are precisely of the form which has been obtained in field theory from consistency reasonings [1,33].

![Diagram](image_url)

**Fig. 14:** Riemann surface Σ as a multiple covering of the z plane.
2.7. Decoupling limit and the dimension of the Seiberg–Witten geometry

The exact vector moduli space obtained from the type IIB theory on Calabi–Yau three-fold $X_3$ contains the information about all gravity and string effects in the $N = 2$ theory in four dimensions. Let us discuss in more detail the geometrical limit which decouples these effects.

As an example consider the asymptotic free $SU(2)$ SYM theory. Physicwise, we want to decouple the massive string and gravity states by sending the string scale $m_{\text{string}}$ to infinity, however at fixed strong coupling scale $\Lambda$ and fixed vector boson mass, $m_{W \pm}$.

$$\alpha' \sim m_{\text{string}}^{-2} \to 0, \quad \Lambda \sim \alpha'^{-1/2} e^{-\frac{1}{\alpha' \pi}} \sim \text{const.}, \quad m_{W \pm} / m_{\text{string}} \sim \alpha'^{1/2} \to 0.$$ (2.11)

From eqs.(2.11),(2.6) we learn that the appropriate limit for the volumes of the fiber and the base is

$$\text{Vol}(\mathbb{P}^1_{\text{base}}) \sim -\frac{b}{2} \ln \alpha' \to \infty, \quad \text{Vol}(\mathbb{P}^1_{\text{fiber}}) \sim \alpha'^{1/2} \to 0.$$ (2.12)

Therefore we have to consider a geometrical limit of large base and small fiber. Note that taking a large base corresponds to a very small gauge coupling constant $g$ at the string scale $m_{\text{string}}$. This is necessary to keep the strong coupling scale $\Lambda$ fixed, taking into account the running of the coupling constant from $m_{\text{string}}$ to the low energy scale of the field theory. Note also that the large base limit at the same time freezes the dynamics of the gauge theory from the base described below fig.9; in particular the mass of the D-brane wrappings on the base diverge in this limit.

A very special situation arises for $N = 2$ theories with vanishing beta-function coefficient $b = 0$. In this case there is no need to take a large base limit to keep the gauge theory coupling finite in the limit $m_{\text{string}} \to \infty$. If we keep the base volume of the order of the fiber volume, the D-brane states wrapped on the base become equally relevant as the states from wrapping the fiber two-cycle. This gives rise to an interesting kind of interacting conformal $N = 2$ theories in four dimensions [6].

The dimension of the mirror geometry $S$: Taking the geometrical limit described above, we obtain the effective action of a gravity free $N = 2$ gauge theory in terms of periods of the holomorphic three-form on a local Calabi–Yau three-fold $S \subset X_3$. However the original solution of Seiberg and Witten is presented in terms of period integrals of a one-form on
a Riemann surface. What is the explanation for this reduction of dimension of the mirror geometry?

The answer is that in general the "Seiberg-Witten geometry" \( S \) is a Calabi-Yau three-fold. Only for sufficiently simple gauge groups such as \( SU(n) \) is it natural to integrate out two of the dimensions to obtain a representation in terms of period integrals on Riemann surfaces \( \Sigma \). Note that the reduction of the dimension is not a consequence of the \( m_{\text{string}} \to \infty \) limit. Rather we get still local three-fold singularities of vanishing three-cycles as the mirror geometry. In the general case, such as for a \( E_8 \) gauge group or product gauge groups, this is the most natural answer.

![Diagram](image.png)

**Fig. 15:** Geometrical limits of the mirror geometry.

The usefulness of representing the exact solution in terms of period integrals on a one complex dimensional geometry rather than in terms of Calabi-Yau three-fold periods is quite limited. Note that the Calabi-Yau representation gives an equally suited description even for the case of \( G = A_n \). The concept to represent the Weyl transformations generated by the monodromies of the ADE singularity in terms of an appropriate multiple cover is less suited for other gauge groups. This is reflected in the properties of the one complex dimensional geometries which have been proposed to describe exact solutions from field theory arguments: for \( G = B_n, C_n, D_n \), one obtains Riemann surfaces with a higher genus (and therefore a larger number of period integrals) than expected from the rank of the gauge group [34,35]. In the case of pure \( E_6 \) gauge theory the situation is even more complicated [36]. A systematic expression for the differential \( \lambda \) and a canonical Prym sub-variety of the Jacobian of correct rank have been given in [35] from the connection to integrable systems.

The increasing difficulties to represent the field theory solution in terms of period integrals on a Riemann surface can be considered as the price for reducing the dimension
of the geometry from three to one complex dimension. Note that, differently than the Riemann surfaces which have been suggested in some cases \( G \neq A_n \), the Calabi-Yau three-fold geometry \( S \) has by construction always the correct dimension of the homology lattice and the periods are in 1-1 correspondence with the 2 \( \text{rk}(G) \) scalar vev's of the gauge theory. Even more importantly, the differential form is canonically given by the unique holomorphic three-form \( \Omega \) on \( X_3^2 \). The somehow awkward and unnecessary complications in the case of more general gauge groups are the reason why we will concentrate on the \( G = A_n \) case in the following section, where we derive the meromorphic one-form and the stable BPS spectrum from reducing the Calabi-Yau three-fold geometry \( S \) to a Riemann surface \( \Sigma \) [4]. It should be evident that the string theory formulation provides the appropriate framework to study the other gauge groups \( G \) as well.

2.8. Meromorphic form and BPS states on the Riemann surface \( \Sigma \)

The meromorphic one-form \( \lambda \) on \( \Sigma \): Let us sketch the string theory derivation of the meromorphic one-form \( \lambda \) which enters the effective field theory solution in terms of periods on a Riemann surface \( \Sigma \) as in (2.2). The starting point will be the unique holomorphic three-form \( \Omega \) which enters the definition of the period integrals (2.10). As an important application we will derive the stable BPS spectrum from the string point of view in the next paragraph.

As a concrete example consider the case of pure \( SU(n) \) gauge theory. The local geometry of the mirror manifold \( X_3^2 \) is defined by the zero of a polynomial \( W \):  

\[
W = \frac{\Lambda^{2n}}{z^2} + z + 2 \prod_{i=1}^{n}(x - r_i(u_k)) + w^2 + y^2 \\
= 2 \prod_{i=1}^{n}(x - \hat{a}_i(z, u_k)) + \text{quadratic terms},
\]

where \( u_k, \ k = 1, \ldots, n - 1 \) are the moduli on the Coulomb branch of the \( SU(n) \) theory and \( z \) denotes the coordinate on the base \( \mathbb{P}^1 \). The polynomial \( W \) describes an ALE space with \( A_n \) singularity varying over the compactified \( z \) plane. Note that at those points \( z_{ij} \), where two of the roots of \( W \) coincide,

\[
\hat{a}_i(z_{ij}, u_k) = \hat{a}_j(z_{ij}, u_k),
\]

the ALE space develops a vanishing two-sphere described by the local equation \( x'^2 + y^2 + z^2 = 0 \). More generally, each pair \( (\hat{a}_i(z, u_k), \hat{a}_j(z, u_k)) \) defines a two-cycle \( C_{ij} \) in the ALE.
fiber with a volume that depends on the position $z$ on the base. There are $\text{rk}(G) = n - 1$ independent two-cycles of this type. The points $z = e^\pm_{ij}$ where (2.14) holds, are at the same time the branch points of the projection to the $z$ plane and thus the special points associated with monodromies. Moreover the same two-cycle $C_{ij}$ vanishes above the two points $e^\pm_{ij}$ on the $z$ plane. These pairs of points are related by the symmetry $z \to \Lambda^{2n}/z$ of the polynomial $W$. Note that in the second expression in (2.13), the equation for the Calabi-Yau is well defined but the functions $\hat{a}_i(z, u_k)$ are not single-valued as functions of $z$; only the product $\prod_{i=1}^n(x - \hat{a}_i(z, u_k))$ is well defined over $z$. As we move around in the $z$-plane, the set of $\hat{a}_i(z, u_k)$ comes back to itself, but the individual $\hat{a}_i(z, u_k)$ do not necessarily come back to themselves. In general they are permuted by an element of $S_n$, the Weyl group of $A_{n-1}$. Since each vanishing cycle is associated with a pair of $\hat{a}_i(z, u_k)$, the behavior of the $\hat{a}_i(z, u_k)$ determines the monodromy action on the vanishing cycles.

The unnormalized unique holomorphic three-form can be represented by

$$\Omega = \frac{dz}{z} \frac{dz \, dy}{\partial u} \tilde{W}.$$  \hspace{1cm} (2.15)

Integrating $\Omega$ over the two-cycle $C_{ij}$ yields a one-form $\lambda(z)_{ij}$

$$\lambda(z)_{ij} = (\hat{a}_i(z, u_k) - \hat{a}_j(z, u_k)) \frac{dz}{z}.$$  \hspace{1cm} (2.16)

Note that the difference $(\hat{a}_i(z, u_k) - \hat{a}_j(z, u_k))$ is a measure for the volume of the two-cycle $C_{ij}$ above the point $z$. The one-forms $\lambda$ are defined on the Riemann surface $\Sigma$ given by the vanishing of

$$W' = \prod_{i=1}^n(x - \hat{a}_i(z, u_k)).$$

To complete the integral over a three-cycle in $X_3$, we have to integrate a one-form $\lambda$ on a path $\gamma(z)$ in the $z$ plane. There are two different types of such paths which correspond to the image of a three-cycle of $X^*_3$ in the $z$ plane (fig.16):

- If we transport a two-cycle $C_{ij}$ in the ALE space along a non-contractible closed path on the base, we obtain a three-cycle $C_3$ of the topology $S^2 \times S^1$. A D3 brane wrapped on $C_3$ will give rise to a vector multiplet.

- A three-cycle $\tilde{C}_3$ of topology $S^3$ is obtained by starting from a point $e^\pm_{ij}$ on the base where a two-cycle $C_{ij}$ vanishes in the ALE space and follow a path to the different point $e^\pm_{ij}$, where the same two-cycle $C_{ij}$ vanishes. A D3 brane wrapping on $\tilde{C}_3$ gives rise to a matter multiplet.

Note that we have obtained in this way a map $\phi : H_3(X^*_3) \to H_1(\Sigma)$ which has the property that integrating a one-form $\lambda$ along $\gamma(z) = \phi(C_3)$ is equivalent to the period integral of the three-form $\Omega$ over $C_3$.  

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Self-dual strings windings and the stable BPS spectrum: Let us follow now the BPS states of the type II string, namely D3 branes wrapped on three-cycles of $S \subset X_3^*$, through the above reduction to the Riemann surface $\Sigma$. The D3 brane wrapped on $C_3$ gives rise to a string stretched on a curve $\gamma(z)$ on $\Sigma$. There are $n-1$ different strings of this kind corresponding to the $n-1$ independent choices of a two-cycle $C_{ij}$ of the ALE space. The tension of these strings is proportional to the volume of $C_{ij}$, which depends on the position $z$ on the base.

The mass of a particle in the four-dimensional theory which arises from the string with tension $T(z)$, stretched on a particular path $\gamma(z)$ is

$$m(\gamma(z)) = \left| \int_{\gamma(z)} T(z) \frac{dz}{z} \right| = \left| \int_{\gamma(z)} \lambda(z) \right|,$$

(2.17)

where in the last step we have used the fact that the mass of the wrapped D3 brane is given by the period integral (2.10). Note that the charges of the BPS state are fixed by the homology class of the path $\gamma(z)$. Thus the last expression in eq.(2.17) is precisely the formula obtained in field theory, $m(a^i_e, q^i_m) = \sqrt{2} |a^i_e a^i_f + a^i_m a^i_D|$. Furthermore note that the string tension $T(z)$ is identified with the differences $\bar{a}_i(z, u_k) - \bar{a}_j(z, u_k)$, as expected.

Fig. 17: BPS string windings on $\Sigma$ with non-trivial metric $g_{z \bar{z}} \sim \lambda z \bar{\lambda}_z$
Though the BPS mass formula of field theory gives the mass of an allowed BPS state, it does not say anything about its existence - a conceptually quite difficult question in field theory. Here is where the string theory point of view improves on field theory reasonings. What we are interested in is to construct minimal volume three-cycles $C_3$ carrying the D3 brane states. For each point on $\gamma(z)$, there is a two-cycle $C_2$ in the ALE fiber which is already of minimal volume. So to minimize the volume of $C_3$, we have to minimize the mass of the resulting string stretched in the $z$ plane. On the latter there is a non-trivial metric arising from the variation of the string tension with respect to the position $z$. The problem is therefore equivalent to consider geodesics in the metric $g_{zz} = \lambda_z \lambda_{\bar{z}}$. The question of whether a BPS state $\Psi(q^i, q^i_m)$ with given quantum numbers exists as a stable state in the spectrum is therefore reduced to a simple geometric question: the existence of $\Psi(q^i, q^i_m)$ is equivalent to the existence of a primitive geodesic $\gamma(z)$ in the homology class defined by the quantum numbers $(q^i, q^i_m)$. The absence of appropriate geodesics can be visualized as in fig.17: for certain quantum numbers, the geodesics run off to infinity and can not be completed to closed curves in the appropriate homology class.

2.9. $N = 2$ SYM from the type IIA five-brane

Let us now describe briefly the third representation of $N = 2$ SYM theory in terms of the world volume theory of a type IIA five-brane wrapped on a one complex dimensional Riemann surface $\Sigma$ [4].

Starting from type IIA on a ALE fibration we ended up with a type IIB mirror geometry $S$, which can be understood in terms of a ALE space varying over the $z$ plane. We can now use a different T-duality transformation described in ref.[18], which maps type IIB(A) theory in the neighborhood of a $A_n$ singularity of the ALE space to type IIA(B) theory on $n$ symmetric five-branes. In six dimensions, this transformation maps the gauge symmetry enhancement of the type IIA theory on the $A_{n-1}$ singularity to type IIB with $n$ symmetric five-branes, which is mapped by the $SL(2, \mathbb{Z})$ of type IIB to $n$ coincident D5-branes with enhanced $SU(n)$ gauge symmetry. The wrapped D2-branes are mapped to elementary strings stretched between the D5-branes. Similarly, starting with the type IIB theory on the $A_n$ singularity, one ends up with type IIA on $n$ symmetric five-branes. The six-dimensional self-dual string from the D3 brane wrapping on two-cycles corresponds now to the boundary of a D2-brane ending on the type IIA five-brane. Note that we have only to use perturbative string symmetries to reach the five-brane representation.
The same configuration has been rederived in ref.[37] from non-perturbative type IIA/M-theory duality.

Since the space transverse to the ALE space is of the form $\Sigma \times M_4$, we expect that the five-brane world volume is compactified on $\Sigma$ after the T-duality map. Let us see how this works in detail. The type IIA five-brane geometry which is T-dual to the type IIB geometry (2.13) is the following: the $n$ five-branes are described by the equations

$$w = y = 0, \quad x = \hat{a}_i(z,u_k).$$

(2.18)

A collision of five-branes, $\hat{a}_i(z_{ij},u_k) = \hat{a}_j(z_{ij},u_k)$ corresponds to an $A_1$ singularity of the ALE fiber in the type IIB theory, and similarly for the higher singularities. The type IIB string with tension $\sim Vd(C^{ij}(z))$ corresponds to the D2-brane stretched between the five-branes at $x = \hat{a}_i(z,u_k)$ and $x = \hat{a}_j(z,u_k)$.

Now the fact that the $\hat{a}_i(z,u_k)$ vary holomorphically with $z$ implies that the several world volumes of the $n$ five-branes located at the $\hat{a}_i(z,u_k)$ are joined together and combine effectively to a *single* five-brane given by

$$\Sigma \times M_4,$$

where the $M_4$ is the four-dimensional Minkowski space-time.

The resulting four-dimensional gauge theory is obtained from dimensional reduction of the six dimensional world volume theory on $\Sigma$. In six dimensions, the spectrum consists of a self-dual two-form $B$ and five real scalars. On compactification on $\Sigma$, the world volume theory is twisted [8]; from the two-form we get $h^{1,0} = \text{rk}(G)$ vector bosons. Moreover two of the five scalars become one-forms on $\Sigma$ upon twisting and give rise to $2\ \text{rk}(G)$ scalars which combine with the gauge bosons to complete $\text{rk}(G)$ four-dimensional vector multiplets. The remaining three scalars are unaffected by the twist and do not give rise to four-dimensional fields due to the absence of normalizable zero modes on the non-compact Riemann surface $\Sigma$ of infinite volume. Note that the vev's of the scalars in the vector multiplets, which correspond to the volumes of the three-cycles in the type IIB theory, can be identified with the Seiberg–Witten differentials $\lambda$:

$$\langle \phi \rangle = \lambda.$$  

(2.19)

Note that the variation of $\lambda$ with respect to the zero mode of $\phi$ yields harmonic one-forms on $\Sigma$ [1]. The latter give rise to the dynamical scalar degrees of freedom of the vector multiplets, in agreement with the identification made in eq.(2.19).
In summary, we have reached a T-dual representation of the $N = 2$ SYM theory in terms of a type IIA five-brane world volume theory on $\Sigma \times M_4$ embedded in an eight-dimensional space $(x, z, M_4)$. The metric on the $x$-plane is the flat metric and on the $z$-plane the metric is cylindrical, given by $|dz/z|^2$. The BPS states correspond to two-branes ending on the five-branes with boundaries wrapped on non-trivial cycles of $\Sigma$ (fig.18). Moreover the tension is given by $|dxdz/z|$.

Though the five-brane picture is appealing, there are some good reasons to concentrate on the T-dual Calabi-Yau representation. Firstly the special geometry of the Calabi-Yau moduli space represents a strong mathematical framework to determine the exact effective action and physical states: in particular the metric on the moduli space as given by the unique holomorphic three-form, the period integrals as well as a treatment of BPS states as described above. Importantly, there is no restriction on the gauge group $G$, differently then in the five-brane picture. Also, we get for free the exact effective action describing the coupling of the gauge system to gravity, a question which has not been addressed so far in the brane language.

3. Outlook

In short, we have seen that type II string theory can be considered as the natural underlying structure of the Seiberg-Witten theory. It gives a concrete physical meaning to the Riemann surface $\Sigma$ and provides a ratio for the appearance of three-dimensional Calabi-Yau geometries in the exact solution of SYM theories with general gauge groups. It gives a unified description of all magnetic and electric BPS states in terms of self-dual strings which wind on the non-trivial, geodesic homology cycles of $\Sigma$.

9 The situation is somehow different in the case of $N = 1$ theories, to which the five-brane picture has been extended to some extent [38], while a geometric analysis is still lacking.
Even more, the string theory approach provides a powerful tool to generate and study a large class of $N = 2$ theories from a systematic study of geometrical Calabi–Yau singularities together with D-brane technology. This class includes gauge theories in $d \leq 6$ with arbitrary gauge groups, interacting conformal field theories and more exotic theories involving non-critical strings. A systematic study has been started in [6] for the subclass of theories with only $A$ type fiber singularities; some interesting results are

- The exact solution of all asymptotic free $N = 2$ SYM theories with gauge group $\prod_i SU(n_i)$ with bi-fundamental and fundamental matter.
- The classification of superconformal theories in the above class in terms of affine ADE Dynkin diagrams.
- The relation of the gauge coupling space of these theories to the moduli of flat ADE connections on a torus.
- The $S$-duality groups of all these theories in terms of the fundamental group of flat ADE connections on the torus.
- The interpretation of the $S$-duality groups as the effective duality group acting on $\tau_{\text{eff}}(a_i)$ of a different gauge theory.
- A new duality of $d \leq 5$ dimensional $N = 2$ theories, e.g. relating a $SU(n)^m$ theory to a $SU(m + 1)^{n-1}$ theory.
- New exotic $N = 2$ theories containing the coupling constants of a SYM theory as dynamical fields.

It will be interesting to complete this program for other fiber singularities and to analyze the new physics of those theories, which do not correspond to known Lagrangian field theories.

Acknowledgments: The research of the author was supported by NSF grants PHY-95-13835 and PHY-94-07194.

References

[34] For a list of references see e.g. [36].


[38] K. Hori, H. Ooguri and Y. Oz, *Strong coupling dynamics of four-dimensional N=1 gauge theories from M theory five-brane*, hep-th/9706082;