Compact analytical form for a class of three-loop vacuum Feynman diagrams

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Abstract

We present compact, fully analytical expressions for singular parts of a class of three-loop diagrams which cannot be factorized into lower-loop integrals. As a result of the calculations we obtain the analytical expression for the three-loop effective potential of the massive $O(N)$ $\varphi^4$ model presented recently by J.-M.Chung and B.K.Chung, Phys. Rev.D56, 6508 (1997).

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Though Quantum Field Theory already has a long history and a number of different approaches, Feynman diagrams (FD) are still the main source of its dynamical information. Vacuum (bubble) FD (without external momenta) considered here have several points of applications. First of all, it is the evaluation of the effective potentials (for the recent study see [1] and references therein) and renormalization group characteristics ($\beta$-functions and anomalous dimensions) of quantum field models and specific operators (for example, anomalous dimensions of operators in the Wilson expansion). The second important place of the applicability is the calculations of various processes (essentially in Standard Model), where there is a possibility to neglect most of the masses and momenta and to calculate only some Taylor coefficients of multipoint FD. These coefficients are bubbles having different (sometimes quite big) powers of propagators (i.e. indices of propagators). Using different recurrence relations [2]-[4] it is possible usually to represent these Taylor coefficients as sets of very simple (usually one-loop) bubbles and several so-called master integrals, which cannot be factorized into sums of lower-loop integrals. These recurrence relations are some partial cases of the relation [5, 6] for a general $n$-point (sub)graph with masses of its lines $m_1, m_2, \ldots, m_n$, line momenta $p_1, p_2 = p_1 - p_{12}, p_n = p_1 - p_{1n}$ and indices $j_1, j_2, \ldots, j_n$, respectively,

$$0 = \int d^D p_1 \frac{\partial}{\partial p_i^j} \left( \prod_{i=1}^{n} c_i^{j_i} \right)^{-1}$$

(1)
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developed in the articles [13, 14]2. The following results for their singular parts have been found3:

\[ J(a) = N_2 \left[ \frac{2}{\varepsilon^3} + \frac{23}{3} \frac{1}{\varepsilon^2} + \frac{35}{2} \frac{1}{\varepsilon} + O(1) \right] \]

\[ J(b) = N_2 \left[ \frac{22}{27} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{83}{27} + \frac{7}{9} \log 3 \right) + \frac{1}{\varepsilon} \left( \frac{365}{54} + \frac{55}{18} \log 3 + \frac{1}{2} \log^2 3 \right) + O(1) \right] \]

\[ J(c) = N_2 \left[ \frac{2}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{23}{27} + \frac{2}{3} \log 3 \right) + \frac{1}{\varepsilon} \left( \frac{35}{18} + \frac{23}{9} \log 3 + \log^2 3 \right) + O(1) \right] \]

\[ K(a) = -N_1 \left[ \frac{1}{\varepsilon^3} + \frac{17}{3} \frac{1}{\varepsilon^2} + \frac{67}{3} \frac{1}{\varepsilon} \left( \varepsilon^3 + 6 \varepsilon^2 + 6 \varepsilon + 6 \right) + O(1) \right] \]

\[ K(b) = -N_1 \left[ \frac{7}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{13}{3} + \frac{1}{3} \log 3 \right) + \frac{1}{\varepsilon} \left( \frac{151}{9} + 3A + \frac{1}{3}B + 2 \log 3 \right) + O(1) \right] \]

\[ K(c) = -N_1 \left[ \frac{5}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{29}{9} + \frac{2}{3} \log 3 \right) + \frac{1}{\varepsilon} \left( \frac{37}{3} + \frac{2}{3}B + 4 \log 3 \right) + O(1) \right] \]

\[ K(d) = -N_1 \left[ \frac{5}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left( 3 + \frac{2}{3} \log 3 \right) + \frac{1}{\varepsilon} \left( \frac{101}{9} + \frac{2}{3}B + 4 \log 3 \right) + O(1) \right] \]

\[ \text{where } C_k = p_k^2 + m_k^2 \text{ are the propagators of } n\text{-point (sub)graph. The equation (1) is based on} \]

1. The example of the direct application of the partial differential equation may be found in [8].

2. The singular part of \( M(a), M(b) \) and \( M(c) \) diagrams has been calculated in [10] using DEM [5]. The regular part of \( M(a) \) has been found very recently by Broadhurst [15].

3. Contrary to [12] we use the space \( D = 4 - 2\varepsilon \) and sum the terms \( \gamma^n \) (\( \gamma \) is Euler constant) and \( \zeta^n \) to exponents.
\[
\begin{align*}
L(a) &= N_0 \left[ \frac{1}{3} \epsilon^3 + \frac{2}{3} \epsilon^2 + \frac{1}{\epsilon} \left( \frac{2}{3} + 2A \right) + O(1) \right] \\
L(b) &= N_0 \left[ \frac{1}{3} \epsilon^3 + \frac{2}{3} \epsilon^2 + \frac{1}{\epsilon} \left( \frac{2}{3} + A + C \right) + O(1) \right] \\
L(c) &= N_0 \left[ \frac{1}{3} \epsilon^3 + \frac{1}{2} \epsilon^2 + \frac{1}{\epsilon} \left( \frac{2}{3} + 2B + 2 \log 3 + \frac{3}{2} \log^2 3 \right) + O(1) \right] \\
L(d) &= N_0 \left[ \frac{1}{3} \epsilon^3 + \frac{2}{3} \epsilon^2 + \frac{1}{\epsilon} \left( \frac{2}{3} + 2C \right) + O(1) \right] \\
M(a) = M(b) = M(c) &= N_0 \left[ \frac{2}{\epsilon} \zeta_3 + O(1) \right],
\end{align*}
\]

where the normalization factor

\[ N_k = \frac{m_{2k}}{(4\pi)^6} \left( \frac{m^2}{\mu^2} \right)^{-3\epsilon} \exp \left( \frac{3}{2} \zeta_{2\epsilon^2} \right) \quad \text{and} \quad \mu^2 = (4\pi \mu^2) e^\gamma \]

and \( \zeta_n \) are Euler \( \zeta \)-functions.

The constants \( A, B \) and \( C \) have been represented [12] in the form:

\[ A = f(1,1), \quad B = f(1,3) \quad \text{and} \quad C = f\left(\frac{1}{3}, \frac{1}{3}\right), \]

where

\[
f(a, b) = \int_0^1 dx \int_0^{1-x} dy \left( -\frac{\log (1-y)}{y} - \frac{z \log z}{1-z} \right), \quad z = \frac{ax + b(1-x)}{x(1-x)} \quad (3)
\]

The purpose of this short letter is to calculate analytically \( A, B \) and \( C \) constants and, thus, to obtain exact results for the singular parts of the FD presented in Fig. 1.

Before evaluations we would like to stress that the knowledge of the exact values for the singular parts of diagrams is very important, because they determine effective potentials and renormalization functions of the quantum field models. Moreover, for various physical processes, the regular parts of diagrams may be evaluated numerically (sometimes with rather good quality) but their singular parts should be known analytically because many types of them should be canceled in the end of calculations.

2. To evaluate \( A = f(1,1) \) we represent the r.h.s. of Eq.(3) as the sum of two terms \( A_1 \) and \( A_2 \).

We introduce the new variable \( s = (1-x)/2 \) and represent \( A_2 \) as

\[
A_2 \equiv - \int_0^1 dx \frac{z \log z}{1-z} = -4 \int_0^1 \frac{ds}{3 + s^2} \log \left( \frac{1 - s^2}{4} \right)
\]

The term \( A_1 \) may be rewritten in the form \((y = 1-t)\)

\[
A_1 \equiv \int_0^1 dx \int_0^{1-x} dy \frac{\log (1-y)}{y} = \int_0^1 ds \int_1^{4/(1-s^2)} dt \frac{\log t}{1-t} \quad (4)
\]

Changing the order of integration in the r.h.s. of (4), we have

\[ A_1 = 6 \int_0^1 \frac{ds}{3 + s^2} \log \left( \frac{1 - s^2}{4} \right) \]
Thus, the evaluation of
\[ A = 4 \int_0^1 \frac{ds}{3 + s^2} \left[ \log (1 - s) + \log (1 + s) - \log 4 \right] \]
is very simple. Integrating by part, we obtain the final result
\[ A = -\frac{2}{\sqrt{3}} \text{Cl}_2 \left( \frac{\pi}{3} \right), \tag{5} \]
where \( \text{Cl}_2(\theta) \) is Clausen integral \([16]\)
\[ \text{Cl}_2(\theta) = \int_0^\theta \log (\sin \theta') d\theta' \]

Repeating above calculations, we have
\[ B = -\frac{4}{\sqrt{3}} \text{Cl}_2 \left( \frac{\pi}{3} \right) \quad \text{and} \quad C = -\frac{1}{2} \log^2 3 + \frac{4}{3\sqrt{3}} \text{Cl}_2 \left( \frac{\pi}{3} \right) \tag{6} \]

3. We have obtained results for the singular parts of a FD class in closed analytical form. These analytical results are important in the calculations of physical processes of Standard Model, where many terms may be involved and the verification and the evaluation of the singularities is very important problem. Moreover, the constant \( A \) is only one numerical factor in the recent calculation \([1]\) of three-loop correction to the effective potential of \( O(N) \varphi^4 \) model:
\[
-\frac{(4\pi)^6}{\lambda^2} \cdot V_{eff}^{(3)}(\varphi_c) = \frac{\lambda^2 \varphi_c^4}{4} \left( \frac{1129}{192} + \frac{A}{8} \right) + \frac{m^2 \lambda \varphi_c^2}{2} \left( \frac{25}{96} - \frac{A}{4} \right) + \left[ \frac{\lambda^2 \varphi_c^4}{4} \left( -\frac{629}{96} - \frac{3A}{4} - \zeta_3 \right) \right]
+ m^2 \lambda \varphi_c^2 \log \left( \frac{1 + \lambda \varphi_c^2}{2m^2} \right) + \left[ \frac{\lambda^2 \varphi_c^4}{4} \cdot \frac{143}{48} \right]
+ \frac{m^2 \lambda \varphi_c^2}{2} \left( \frac{17}{6} + m^4 \cdot \frac{11}{48} \right) \cdot \log^2 \left( \frac{1 + \lambda \varphi_c^2}{2m^2} \right) - \left[ \frac{\lambda^2 \varphi_c^4}{4} \cdot \frac{9}{16} \right]
+ \frac{m^2 \lambda \varphi_c^2}{2} \left( \frac{7}{12} + m^4 \cdot \frac{5}{48} \right) \cdot \log^3 \left( \frac{1 + \lambda \varphi_c^2}{2m^2} \right) \]

After above-mentioned calculation of \( A \) (i.e. Eq.(5)), this three-loop correction becomes known fully analytically.

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References


