Pursuing Gravitational $S$-Duality

H. García-Compeán$^a$, O. Obregón$^b$† and C. Ramírez$^c$‡

$^a$ Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN

P.O. Box 14-740, 07000, México D.F., México

$^b$ Instituto de Física de la Universidad de Guanajuato

P.O. Box E-143, 37150, León Gto., México

$^c$ Facultad de Ciencias Físico Matemáticas

Universidad Autónoma de Puebla

P.O. Box 1364, 72000, Puebla, México

Abstract

Recently a strong-weak coupling duality in non-abelian non-supersymmetric theories in four dimensions has been found. An analogous procedure is reviewed, which allows to find the ‘dual action’ to the gauge theory of dynamical gravity constructed by the MacDowell-Mansouri model plus the superposition of a $\Theta$ term.

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$^*$Present Address: School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton NJ 08540 USA. E-mail: compean@sns.ias.edu

$^†$E-mail: octavio@ifug3.ugto.mx

$^‡$E-mail: cramirez@fcfm.buap.mx
I. INTRODUCTION

S-Duality

Duality is a notion which in the last years has led to remarkable advances in nonperturbative quantum theory. It is an old known type of symmetry which by interchanging the electric and magnetic fields leaves invariant the vacuum Maxwell equations. It was extended by Dirac to include sources, with the well known price of the prediction of monopoles, which appear as the dual particles to the electrically charged ones and whose existence could not be confirmed up to now. Dirac obtained that the couplings (charges) of the electrical and magnetical charged particles are the inverse of each other, i.e. as the electrical force is ‘weak’, and it can be treated perturbatively, then the magnetic force among monopoles will be ‘strong’.

This duality, called ‘S-duality’, has inspired a great deal of research in the last years. By this means, many non-perturbative exact results have been established. In particular, the exact Wilson effective action of $\mathcal{N} = 2$ supersymmetric gauge theories has been computed, showing the duality symmetries of these effective theories. It turns out that the dual description is quite adequate to address standard non-perturbative problems of Yang-Mills theory, such as confinement, chiral symmetry breaking, etc.

For string theories as well, quite a lot of new interesting results have been obtained. Undoubtedly, one of the most important and exciting results is the ample evidence of the existence of a ‘master’ eleven dimensional non-perturbative quantum theory named M-theory. All five ten-dimensional superstring theories, together with the eleven dimensional supergravity theory, become specific vacua of the moduli space of M-theory. These and other vacua arising by compactifying M-theory to lower dimensions, are related among them by a web of dualities (for recent reviews see [1]).

Roughly speaking, for the effective low energy actions of superstring theories, strong-weak coupling (S-duality) symmetry is realized at the level of the axion and dilaton moduli. The gravitational sector appears dynamical with respect to this symmetry. In fact, it is
well known that the inclusion of gravitational corrections is required in order to test string
duality [2], and to check consistency conditions in $M$-theory [3]. For the heterotic string
theory in ten dimensions, toroidal compactification to four dimensions on $T^6$, gives for its
low energy limit $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^4$. $S$-duality in the four dimensional
theory, arises as a consequence of superstring duality in six-dimensions between the heterotic
theory on $T^4$ and the Type IIA on K3 (for a review see [5]). Dynamical gravity arising in
string theory can be switched off from the four-dimensional theory, i.e. gravity enters only
as an non-dynamical external field. From this theory a twisted topological field theory can
be constructed on a curved four-manifold, which is shown to be $S$-dual (according to the
Montonen-Olive conjecture), by using different formulas of four-manifolds well known by
the topologists [4].

$\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions have vanishing renormalization
group $\beta$-function. Montonen and Olive conjectured that (at the quantum level) these theories
would possess an $\text{SL}(2,\mathbb{Z})$ exact dual symmetry [6]. Many evidences of this fact have been
found, although a rigorous proof does not exist at present. For $\mathcal{N} = 2$ supersymmetric gauge
theories in four dimensions, the $\beta$-function in general does not vanish. So, Montonen-Olive
conjecture cannot be longer valid in the same sense as for $\mathcal{N} = 4$ theories. However, for
theories with $\text{SU}(2)$ gauge group, Seiberg and Witten found that a strong-weak coupling
'effective duality' can be defined on its low energy effective theory for the cases pure and
with matter [7]. The quantum moduli space of the pure theory is identified with a
complex plane, the $u$-plane, with singularities located at the points $u = \pm 1, \infty$. It turns
out that at $u = \pm 1$ the original Yang-Mills theory is strongly coupled, but effective duality
permits the weak coupling description at these points in terms of monopoles or dyons (dual
fields). $\mathcal{N} = 1$ gauge theories are also in the class of theories with non-vanishing $\beta$-function.
More general, for a gauge group $\text{SU}(N_c)$, an effective non-abelian duality is implemented
even when the gauge symmetry is unbroken. It has a non-abelian Coulomb phase. Seiberg
has shown that this non-abelian Coulomb branch is dual to another non-abelian Coulomb
branch of a theory with gauge group $\text{SU}(N_f - N_c)$, where $N_f$ is the number of flavors [8].
$\mathcal{N} = 1$ theories have a rich phase structure [9]. Thus, it seems that in supersymmetric gauge theories strong-weak coupling duality can only be defined for some particular phases.

For non-supersymmetric gauge theories in four dimensions, the subject of duality has been explored recently in the abelian as well as in the non-abelian cases [10–14]. In the abelian case (on a curved compact four-manifold $X$) the $CP$ violating Maxwell theory partition function $Z(\tau)$, transforms as a modular form under a finite index subgroup $\Gamma_0(2)$ of $\text{SL}(2, \mathbb{Z})$ [10,11]. The dependence parameter of the partition function is given by $\tau = \frac{\theta}{2\pi} + \frac{4\pi i e^2}{e^2}$, where $e$ is the abelian coupling constant and $\theta$ is the usual theta angle. In the case of non-abelian non-supersymmetric gauge theories, strong-coupling dual theories can be constructed which results in a kind of dual “massive” non-linear sigma models [12–15]. The starting Yang-Mills theory contains a $CP$-violating $\theta$-term and it turns out to be equivalent to the linear combination of the actions corresponding to the self-dual and anti-self-dual field strengths [10–14].

**Gravitational Duality**

As a matter of fact, string theory constitutes nowadays the only consistent and phenomenologically acceptable way to quantize gravity. It contains in its low energy limit Einstein gravity. Thus, a legitimate question is the one of which is the ‘dual’ theory of gravity or, more precisely, how gravity behaves under duality transformations.

It turns out that, at the quantum level, the description of $\mathcal{N} = 2$ gauge theories on curved manifolds $X$, at the singular points $u = \pm 1$ of the moduli space, is given by the Maxwell theory coupled to non-dynamical gravity,

$$I = I_{M, \theta} + \int_X \left( B(\tau) \text{tr} R \wedge \tilde{R} + C(\tau) \text{tr} R \wedge R \right),$$

in such a way that, as shown by Witten [10], in order to cancel the modular anomaly in effective theories, suitable holomorphic couplings $B(\tau)$ and $C(\tau)$ have to be chosen.

Gravitational analogs of non-perturbative gauge theories were studied several years ago, particularly in the context of gravitational Bogomolny bound [16]. As recently was shown
there are additional non-standard $p$-branes in $D = 10$ type II superstring theory and $D = 11$ $M$-theory, and which are required by U-duality. These branes were named ‘gravitational branes’ (‘G-branes’), because they carry global charges which correspond to the ADM momentum $P_M$ and to its ‘dual’, a $(D - 5)$-form $K_{M_1 \cdots M_{D-5}}$, which is related to the NUT charge. These charges are ‘dual’ in the same sense that the electric and magnetic charges are dual in Maxwell theory, but they appear in the purely gravitational sector of the theory. Last year, Hull has shown in [18] that these global charges $P$ and $K$ arise as central charges of the supersymmetric algebra of type II superstring theory and $M$-theory. Thus the complete spectrum of BPS states should include a gravitational sector.

An attempt to extend $S$-duality of Yang-Mills theory to gravity was discussed in [19]. There, a clear evidence of identical quantum properties of electrically and magnetically charged black-holes was found.

Finally, a different approach to the ‘gravitational duality’ was worked out in [20]. In this paper, some techniques of strong-weak coupling duality for non-supersymmetric Yang-Mills theories were applied to the MacDowell-Mansouri dynamical gravity. These results will be reviewed and discussed here.

\textit{Gauge Theories of the Gravitation}

In the past, in the search for the construction of an unified model for the fundamental interactions, different routes have been followed. A natural one is to consider higher dimensional models of gravity (and supergravity) [21]. The metric, as well as in Einstein gravity, is considered the basic field from which the four dimensional fields that describe the fundamental interactions and gravity itself can be obtained.

In contrast, if one searches for an unified framework of the non-gravitational fundamental interactions in four space-time dimensions, the usual way is that these interactions are described in terms of a connection associated with an internal group leaving space-time nondynamical. However, the evolution of the metric of this space-time should describe just
gravity. Various attempts have been done to construct Yang-Mills type gauge theories of
gravity [22], where the basic fields are the gauge fields of an appropriate group \( G \). The metric (the tetrad) and the Lorentz connection are obtained only as components of these fields, so that standard general relativity is a consequence of the proposed gauge theory. Following this scheme twenty years ago, MacDowell and Mansouri (MM) succeeded in constructing a gauge theory of gravity [23]. Pagels gave an Euclidean formulation [24]. Inspired in the MM work, another approach has been constructed [25], based only on a self-dual spin connection, where duality is defined with respect to the corresponding Lorentz group indices and not, as it is usual, with respect to the space-time indices. This proposal was generalized to include the MM gauge theory of gravity and the Plebański-Ashtekar formulation [26].

Knowing the self-dual formulation of the MM theory [23], it is tempting to search if by combining it linearly with the anti-self-dual part, one gets the MM action plus a \( \Theta \)-term and then try to proceed as in Yang-Mills theories to obtain a dual model. It turns out that all this works, as it will be shown in this paper.

The paper is organized as follows. Sec. II is devoted to review non-supersymmetric non-Abelian duality following [13,14]. In Sec. III we introduce the MM theory and its corresponding theory based on its self-dual sector [25]. Then, in Sec. IV, we describe the gravitational duality in MM theory. Finally in Sec. V we give our final remarks and a glimpse of future research.

II. STRONG-WEAK COUPLING DUALITY IN NON-SUSY YANG-MILLS THEORIES

In Yang-Mills theories, the lack of a generalized Poincaré lemma means that there is no dual theory in the same sense as for abelian theories [27]. However, for generic Yang-Mills theories, one can follow the same procedure as in the abelian case [12–14]. Usually one constructs a parent Lagrangian from which one recuperates the original Lagrangian and its
dual, as different limits.

\[ L = -\alpha G^a_{\mu\rho} G^{\mu\rho}_a + G^a_{\mu\nu} F^a_{\mu\nu}, \]  

(2)

\( \alpha \) is the coupling constant, \( F^a_{\mu\nu} \) is the field strength given by \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu \), and \( G^a_{\mu\nu} \) is a Lie algebra-valued Lagrange multiplier tensor field. If the variables \( G^a_{\mu\nu} \) are integrated out, we get the usual Yang-Mills Lagrangian \( L = \frac{1}{4\alpha} F^a_{\mu\nu} F^{a\mu\nu} \).

Further, the euclidean partition function of (2), after a partial integration of the derivatives of \( A \), can be rewritten as

\[ Z = \sqrt{\pi} \int DG (\det M)^{-1/2} \exp \left( - \int (\alpha G^a_{\mu\rho} G^{\mu\rho}_a - M^{-1ab}_{\mu\nu} \partial_\rho G^a_\nu \partial_\sigma G^{a\sigma}_{\mu}) dx \right), \]

(3)

where \( M^{\mu\nu}_{ab} = f^{c}_{ab} G^{\mu\nu}_c \) is the adjoint transformed of \( G \).

This result represents in some sense the dual of the starting Yang-Mills theory (2). It can be interpreted as a complicated “massive” non-linear sigma model [15] with a complicated metric \( M \). Gannor and Sonnenschein [13] showed how to regain a Yang-Mills theory from this, in such a way that it apparently leads to the dual theory. Following them, let us define \( \bar{A}^a_\mu = -(M^{-1})^{ab}_{\mu\rho} \partial_\rho G^c_\nu \), from which it turns out that \( G \) satisfies the equations of motion \( \partial_\nu G^{\nu\mu}_b + M^{\mu\nu}_{ab} \bar{A}^a_\nu = 0 \). In abelian theories, the Poincaré lemma gives the solution to this equation in terms of a vector potential for the dual of \( G \). However, for non-abelian theories, although \( F^a_{\mu\nu} (\bar{A}) = \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\mu + f^a_{bc} \bar{A}^b_\mu \bar{A}^c_\nu \) is a solution for \( \tilde{G}^a_{\mu\nu} \), it is not the most general solution.

Nevertheless, it can be easily seen that the second term in the exponential of (3) can be rewritten in two different ways, as \( M^{\mu\nu}_{ab} \bar{A}^b_\nu \bar{A}^a_\mu \), or as \( \partial_\rho G^a_{\mu\rho} \bar{A}^a_\mu \) plus a total derivative term. Hence, the partition function can be written as

\[ Z = \sqrt{\pi} \int DG (\det M)^{-1/2} \exp \left( - \int (\alpha G^a_{\mu\rho} G^{\mu\rho}_a - M^{-1ab}_{\mu\nu} \partial_\rho G^a_\nu \partial_\sigma G^{a\sigma}_{\mu}) dx \right), \]

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(3)

where \( M^{\mu\nu}_{ab} = f^{c}_{ab} G^{\mu\nu}_c \) is the adjoint transformed of \( G \).
\[ Z = \sqrt{\pi} \int D G (\det M)^{-1/2} \int D \tilde{A} \exp \left( - \int [\alpha G^a_{\mu \nu} G^a_{\mu \nu} - G^a_{\mu \nu} F^a_{\mu \nu}(\tilde{A})] dx \right) \delta(2\tilde{A}_\mu^a + 2M^{-1}_{\mu \nu} \partial_\mu G^a_{\nu \nu}), \]

(4)

where the factor 2 in the delta function was introduced for convenience. If now in this expression the square root of the determinant and the Dirac delta function are written as exponentials, after a partial integration we get

\[ Z = \sqrt{\pi} \int D G \bar{D} A D \Omega A, \exp \left( \int \{ G^a_{\mu \nu}[-\alpha G^a_{\mu \nu} + F^a_{\mu \nu}(\tilde{A}) - 2D^{(\tilde{A})}A^a_\nu] - M^{-1}_{\mu \nu} \Omega^a_\mu \Omega^b_\nu \} dx \right) \],

(5)

which after some manipulations turns out to be

\[ Z = \pi \int D G D \tilde{A} \exp \left( \int G^a_{\mu \nu}[-\alpha G^a_{\mu \nu} + F^a_{\mu \nu}(\tilde{A})] dx \right), \]

(6)

where \( \tilde{A} = \tilde{A} - \Lambda \). This result shows the way back to the model we started from, as well as the covariance of the partition function (3) [13].

Further, the procedure stated above can be followed to find (3) for the non-abelian case with a \( CP \)-violating \( \theta \)-term, on the manifold \( X \). The action is given by

\[ I_{YM,\theta} = \frac{1}{8\pi} \int_X d^4x \sqrt{g} \left( \frac{4\pi}{g_{YM}^2} \text{tr}[F_{\mu \nu}F^{\mu \nu}] + \frac{i\theta}{4\pi} \varepsilon_{\mu \nu \rho \sigma} \text{tr}[F^{\mu \nu}F^{\rho \sigma}] \right), \]

(7)

where \( \theta \) is the non-Abelian theta-vacuum and \( g_{YM} \) is the Yang-Mills coupling constant. Equivalently,

\[ I_{YM,\theta} = \frac{i}{8\pi} \int_X d^4x \sqrt{g} \left( \bar{\tau} \text{tr}^{+}[F_{\mu \nu} + F^{\mu \nu}] - \tau \text{tr}[-F_{\mu \nu} - F^{\mu \nu}] \right), \]

(8)

here \( \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \) and \( \bar{\tau} \) its complex conjugate, and \( \pm F_{mn} \) are the self-dual and anti-self-dual field strengths respectively and which are given by \( \pm F^a_{\mu \nu} = \partial_\mu \pm A^a_{\mu} - \partial_\nu \pm A^a_{\nu} + f^a_{bc} \pm A^b_{\mu} \pm A^c_{\nu} \).

Thus, both terms in (8) have the same form as the usual Yang-Mills action and the \( S \)-dual Lagrangian to (7) can be obtained as the sum of the corresponding dual and anti-dual terms.

Thus, one finds the dual Lagrangian of (8) [12–14] to be of the form

\[ \bar{I}_{YM,\theta} = \frac{i}{8\pi} \int_X d^4x \sqrt{g} \left( -\frac{1}{\tau} + G^a_{\mu \nu} + G^{\mu \nu a} + \frac{1}{\bar{\tau}} - G^a_{\mu \nu} - G^{\mu \nu a} + 2(\pm M)^{-1}_{\mu \nu} \partial_\mu \partial_\nu + G^{\mu \nu}_{a\sigma} \partial_\sigma + G^a_{\nu \sigma} \right), \]

(9)
where $\pm G$ are, as mentioned, arbitrary two-forms on $X$ and $\pm M$ are, as previously, the adjoints of $\pm G$, correspondingly.

### III. GAUGE THEORIES OF GRAVITY

Let us review the MM proposal [23,24]. The starting point in the construction of this theory is to consider an SO(3,2) gauge theory with a Lie algebra-valued gauge potential $A^A_{\mu}$, where the indices $\mu = 0, 1, 2, 3$ are space-time indices and the indices $A, B = 0, 1, 2, 3, 4$ are internal indices.

For the gauge potential $A^A_{\mu}$, we may introduce the corresponding field strength

$$F^{AB}_{\mu\nu} = \partial_{\mu} A^A_{\nu} - \partial_{\nu} A^A_{\mu} + \frac{1}{2} f^{AB}_{CDEF} A^C_{\mu} A^D_{\nu},$$

(10)

where $f^{AB}_{CDEF}$ are the structure constants of SO(3,2). The MM approach chooses in a rather ad hoc way a vanishing “torsion”

$$F^{a4}_{\mu\nu} \equiv 0,$$

(11)

and as a consequence the gauge group breaks from SO(3,2) to SO(3,1)$^2$.

Then, the MM action is given by [23]

$$I_{MM} = \int_X d^4x \varepsilon^\mu_{\alpha\beta} \varepsilon_{abcd} F^{ab}_{\mu\alpha} F^{cd}_{\nu\beta},$$

(12)

where $a, b, \ldots$ etc. $= 0, 1, 2, 3$, $\varepsilon^\mu_{\alpha\beta}$ and $\varepsilon_{abcd}$ are the Levi-Civita symbols in four dimensions defined by $\varepsilon^{0123} = +1$.

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$^2$Recently Wilczek has argued a most natural way of avoiding this choice. Adding consistent potential terms to the MM Lagrangian, this breaking of the group can be seen as a spontaneously symmetry breaking. Thus the Riemann-Einstein structure follows naturally from the gauge theory and a volume form [29].
From Eq. (10), it turns out that

\[ F_{\mu}^{ab} = H_{\mu}^{ab} + K_{\mu}^{ab}, \tag{13} \]

where

\[ H_{\mu}^{ab} = \partial_\mu A_{\nu}^{ab} - \partial_\nu A_{\mu}^{ab} + \frac{1}{2} f_{\nu}^{ab} A_c^{\mu} A_f^{\rho}, \tag{14} \]

with \( f_{\nu}^{ab} \) the structure constants of \( \text{SO}(3,1) \) and

\[ K_{\mu}^{ab} = -\lambda^2 (A_{\nu}^{a4} A_{\rho}^{b4} - A_{\nu}^{a4} A_{\rho}^{b4}). \tag{15} \]

The proposed action of Ref. [25] is just the (anti)self-dual part of the MM action (12)

\[ I_{\pm}^{\text{MM}} = \int_X d^4 x \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} \pm F_{\mu \nu}^{ab} \pm F_{\mu \nu}^{cd}, \tag{16} \]

where \( \pm A_{\mu}^{ab} = \frac{1}{2} (A_{\mu}^{ab} - \frac{i}{2} \varepsilon^{ab}_{\rho \sigma} A_{\mu}^{\rho \sigma}) \) are the (anti)self dual parts of \( A_{\mu}^{ab} \) and \( \pm K_{\mu}^{ab} = \pm (K_{\mu}^{ab} - \frac{1}{2} i \varepsilon_{\rho \sigma}^{ab} K_{\mu \rho \sigma}^{cd}). \) Then

\[ \pm F_{\mu \nu}^{ab} = \pm H_{\mu \nu}^{ab} + \pm K_{\mu \nu}^{ab}, \tag{17} \]

where

\[ \pm H_{\mu \nu}^{ab} = \partial_\mu \pm A_{\nu}^{ab} - \partial_\nu \pm A_{\mu}^{ab} + \frac{1}{2} f_{\nu}^{ab} \pm A_c^{\mu} A_f^{\rho}. \tag{18} \]

It was found in [25] that the action (16) leads to a gauge theory based in the action

\[ I_{\pm}^{\text{MM}} = \frac{1}{2} \int_X d^4 x \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} H_{\mu \nu}^{ab} H_{\alpha \beta}^{cd} \pm i \int_X d^4 x \varepsilon^{\mu \nu \alpha \beta} \eta_{ab} \eta_{cd} H_{\mu \nu}^{ab} H_{\alpha \beta}^{cd} \]

\[ -2\lambda^2 \int_X d^4 x \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} \pm K_{\mu \nu}^{ab} H_{\alpha \beta}^{cd} + 2\lambda^2 \int_X d^4 x \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} A_{\mu}^{a4} A_{\nu}^{b4} A_{\alpha}^{c4} A_{\beta}^{d4}. \tag{19} \]

If now we identify the gauge fields \( A_{\mu}^{ab} \) with the Ricci rotation coefficients and \( A_{\mu}^{a4} \) with the space-time vierbein, then the first two terms are of topological nature and are proportional to the Euler characteristic and the signature of \( X \), respectively. The last two terms represent dynamical gravity with cosmological term in the form of Plebański-Ashtekar [26].
IV. GRAVITATIONAL S-DUALITY

In the section II we have seen that for Yang-Mills theories a kind of “dual theories” with inverted couplings can be found [12–14]. One starts with the Yang-Mills action plus a CP violating $\theta$-term. It is shown that these two terms are equivalent to a linear combination of the self-dual and anti-self-dual Yang-Mills actions.

One can search whether the construction of a linear combination of the corresponding self-dual and anti-self-dual parts of the MM action can be also reduced to the standard MM action plus a kind of gravitational $\Theta$-term [30] and, moreover, if by this means one can find the “dual-theory” associated with the MM theory.

Let us consider the action

$$S = \int_X d^4x \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \left( + \tau^+ F^{ab}_{\mu\nu} + F^{cd}_{\alpha\beta} - \tau^- F^{ab}_{\mu\nu} - F^{cd}_{\alpha\beta} \right),$$

where

$$\pm F^{ab}_{\mu\nu} = \frac{1}{2} \left( F^{ab}_{\mu\nu} \pm \tilde{F}^{cd}_{\mu\nu} \right),$$

with $\tilde{F}^{ab}_{\mu\nu} = -\frac{1}{2} i \varepsilon^{ab}_{cd} F^{cd}_{\mu\nu}$. It can be shown [31], that this action can be rewritten as

$$S = \frac{1}{2} \int d^4x \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \left[ (+\tau^- \tau) F^{ab}_{\mu\nu} F^{cd}_{\alpha\beta} + (+\tau^+ \tau^-) F^{ab}_{\mu\nu} \tilde{F}^{cd}_{\alpha\beta} \right].$$

MM showed in their original paper [23] that the first term in this action reduces to the Euler topological term plus the Einstein-Hilbert action with a cosmological term. Similarly, as can be seen from Eq. (19), the second term results to be equal to $i P$ where $P$ is the Pontrjagin topological term [25]. Thus, this term is a genuine $\Theta$ term with $\Theta$ given by the sum $^+\tau^- + ^-\tau$.

Let us write the action (20) as follows

$$S = S_+ + S_- = \int_X \mathcal{L}_+ + \int_X \mathcal{L}_-$$

where $\mathcal{L}_\pm = \pm \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} (\pm \tau^\pm F^{ab}_{\mu\nu} F^{cd}_{\alpha\beta})$. This additivity property of the lagrangian is similar to the one which in the Yang-Mills case leads to the factorization rule of the partition
function. However, in this case the vanishing torsion condition spoils this factorization rule, because it cannot be separated in self-dual and anti-self-dual parts.

Our second task is to find the “dual theory”, in a similar sense as in Yang-Mills theories [13,14]. For that purpose we consider the parent action

\[ I = \int_X L_+ + \int_X L_- \]

\[ = \int_X d^4x \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \left( c_1 G^{\alpha\beta}_{\mu\nu} + c_2 \mathcal{R}^{\alpha\beta}_{\mu\nu} + c_3 + F^{ab}_{\mu\nu} G^{cd}_{\alpha\beta} \right) + \int_X d^4x \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \left( c_2 - G^{\alpha\beta}_{\mu\nu} - c_3 - F^{ab}_{\mu\nu} - G^{cd}_{\alpha\beta} \right). \]  

(24)

First of all we will show how to recover the original action (20) from the parent action (24). For simplicity we focus in the partition function for the ‘self-dual’ part of Lagrangian (24) (the anti-self-dual part \( L_- \) follows the same procedure). Integrating out the additional degrees of freedom \( +G \), we get the “effective Lagrangian” \( L^*_+ \), i.e.

\[ Z_+ (+\tau) = \int D^+ G \exp \left( -\int_X L_+ \right) \]  

(25)

where \( Z_+ (+\tau) \) is given by

\[ Z_+ (+\tau) = \exp \left( -\int_X L^*_+ \right). \]  

(26)

Performing the Feynman functional integral for the self-dual and anti-self-dual sectors we get finally

\[ L^* = L^*_+ + L_- \]  

(27)

\[ L^* = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \left[ -\frac{c_3^2}{4c_1^4} + F^{ab}_{\mu\nu} + F^{cd}_{\alpha\beta} \left( -\frac{c_4^2}{4c_1^4} - F^{ab}_{\mu\nu} - F^{cd}_{\alpha\beta} \right) \right]. \]  

(28)

Which is precisely the Lagrangian (20) from which we started from, if in (28) we make the identifications

\[ c_1 = -\frac{1}{4^+\tau}, \quad c_2 = -\frac{1}{4^-\tau}, \quad c_3 = c_4 = 1. \]  

(29)
In order to get the “dual theory” we follow reference [13]. Therefore, one should start from the partition function

\[
\tilde{Z}(\tau) = \int \mathcal{D}^+ G \mathcal{D}^- G \mathcal{D} A_{\mu}^{ab} \mathcal{D} A_{\mu}^{a4} \exp \left( - \int X L \right).
\]  

(30)

Given the definition of \(\pm F\) in (21) and the fact that \(\pm G\) are (anti) self-dual fields, one can rewrite the second and fourth terms in (24) in the following manner

\[
4i\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^{ab} \left( c_3^+ G_{\alpha\beta ab} - c_4^- G_{\alpha\beta ab} \right).
\]  

(31)

If we take into account the second term of the expression (13) for \(F_{\mu\nu}^{ab}\), and which is quadratic in \(A_{\mu}^{a4}\) (15), then the integral over the components \(A_{\mu}^{a4}\) is Gaussian, giving after integration \((\det G)^{-1/2}\), where \(G\) is a matrix given by

\[
G_{\mu\nu}^{ab} = 8i\lambda^2 \varepsilon^{\mu\nu\alpha\beta} \left( c_3^+ G_{\alpha\beta ab} - c_4^- G_{\alpha\beta ab} \right).
\]  

(32)

Thus, the partition function (30) can be written as

\[
Z(\tau) = \int \mathcal{D}^+ G \mathcal{D}^- G \mathcal{D} A_{\mu}^{ab} (\det G)^{-1/2} \exp \left( -\Pi \right)
\]  

(33)

where

\[
\Pi = \int X IL = 2i \int d^4 x \varepsilon^{\mu\nu\alpha\beta} \left( c_1^+ G_{\mu\nu} + c_2^- G_{\mu\nu} - c_2^- G_{\mu\nu}^+ G_{\alpha\beta ab} + 2H_{\mu\nu}^{ab} (c_3^+ G_{\alpha\beta ab} - c_4^- G_{\alpha\beta ab}) \right),
\]  

(34)

with \(H_{\mu\nu}^{ab} = \partial_{\mu} A_{\nu}^{ab} - \partial_{\nu} A_{\mu}^{ab} + \frac{1}{2} f_{cde}^{ab} A_{\mu}^{cd} A_{\nu}^{ef}\) is the SO(3,1) field strength.

We now decompose \(H_{\mu\nu}^{ab}\) in its self-dual and anti-self-dual components as follows

\[
H_{\mu\nu}^{ab} = H_{\mu\nu}^{+} - H_{\mu\nu}^{-}
\]  

(35)

and integrate out separately in their (anti)self-dual connections \(\pm A_{\mu}^{ab}\).

The integration with respect to \(\pm A_{\mu}^{ab}\) (the anti-self-dual part \(\Pi L_-\) follows the same procedure) is given by
\[
\exp\left(-\int_X \bar{L}^+_+\right) = \int \mathcal{D}^+ A^a_b \exp\left(-\int_X L_+\right). \tag{36}
\]

The part of the Lagrangian (34) relevant to this integration is
\[
L_+ = \ldots + 8i c_3 \varepsilon^{\mu \nu \alpha \beta} A^a_b \partial_\nu + G_{\alpha \beta a b} + 2i c_3 \varepsilon^{\mu \nu \alpha \beta} c_3 f^a_{\, c d e f} + A^a_{\nu} + A^a_{\mu} + \ldots . \tag{37}
\]

Hence we finally obtain
\[
\tilde{Z}(\tau) = \int \mathcal{D}^+ G \mathcal{D}^+ \mathcal{D}^+ (\det G)^{-1/2} \det (+ M)^{-1/2} \det (- M)^{-1/2} \exp\left(-\int_X \tilde{L}\right), \tag{38}
\]

with
\[
\tilde{L} = \varepsilon^{\mu \nu \alpha \beta} \left[ -\frac{1}{4 \tau} + G_{\mu \nu}^a + G_{\alpha \beta a b} + \frac{1}{4 \tau} - G_{\mu \nu}^a - G_{\alpha \beta a b} - 2 \partial_\nu G_{\alpha \beta a b}^+(+ M)^{-1} \varepsilon^{\lambda \rho \sigma \tau} \partial_\theta + G_{\alpha \beta c d}^a \\
- 2 \partial_\nu G_{\alpha \beta a b}(- M)^{-1} \varepsilon^{\lambda \rho \sigma \tau} \partial_\theta - G_{\rho \sigma c d}^a \right], \tag{39}
\]

and
\[
\pm M^{\mu \nu \rho \sigma}_{a b} = \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma}_{a b} \left( -\delta^c_d G_{a}^b + \delta^c_d G_{b}^a + \delta^d_c G_{b}^a - \delta^d_c G_{a}^b \right). \tag{40}
\]

The condition of (anti) self-dual factorization for the above Lagrangian is not longer valid due to the \((\det G)^{-1/2}\) factor in (38).

As mentioned, MM were able to reduce their theory to standard general relativity with cosmological constant plus the Euler topological term by identifying \(A^a_b\) with the spin connection \(\omega^a_{\, b}\) and \(A^{a4}_a\) with the tetrad \(e^a_{\, \mu}\). Obviously then \(H^a_{\mu \nu} \equiv R^a_{\mu \nu}\).

The dynamics of the \(\pm G\) results rather complicated, the right action should de defined taking into account the determinants appearing in the partition function (38). Actually the presence of the degrees of freedom of the gauge fields \(A^{a4}_a\) manifest itself in the condition
\[
\varepsilon_{\mu \nu \rho \sigma} G_{\alpha}^{\mu \nu \rho \sigma} = 0, \tag{41}
\]

It is interesting to remark that the fields \(H^a_{\mu \nu} \equiv R^a_{\mu \nu}\), which depend only on \(A^a_b \equiv \omega^a_{\, b}\), the Ricci rotation coefficients, can also be considered in an action like (20) or (22) and
then the action would be pure topological because the first term in (22) would result in the Euler characteristic and the second one in the signature. As has been shown in [32], in this last case, it is possible to define a dual theory similar to (38) but without the \((\text{det} G)^{-1/2}\) factor. Also a kind of \(BF\) topological gravitational theory [33] and its dual theory have been analyzed. This case would correspond to take \(K_{ab}^{\mu\nu}\) in all the MM formalism, in Sec. III, as an independent field not having any relation with \(A_{\mu}^a\).

V. FINAL REMARKS AND FUTURE RESEARCH

In this paper we have defined a gravitational analog of \(S\)-duality, following similar procedures to those well known in non-abelian non-supersymmetric Yang-Mills theories [13,14].

We have shown how to construct the Lagrangians for ‘dual’ gauge theories of gravitation, in particular for the case of the MM theory. We could have also studied other approaches as the one of Pagel, based on the \(O(5)\) group [24]. We have also argued that the cases of pure topological gravity and \(BF\) topological gravity [33] can be understood as a kind of non-dynamical limits of the case presented here.

We are aware that this analogy was carried out only at the level of the structure group of the frame bundle over \(X\), and not over all the genuine symmetries which arise in Einstein gravity theories, such as diffeomorphisms group \(\text{Diff}(X)\) of \(X\). Related to this, it is interesting to note that in the ‘dual’ Lagrangian in Eq. (39) only partial derivatives of the \(G's\) fields appear, instead of covariant derivatives. However, it can be shown that following a procedure similar to that presented in the introduction for Yang-Mills fields, one can get the covariance of the partition functions. As it has been pointed out by Atiyah [34], in field theory the \(S\)-duality symmetry is a duality between the fundamental homotopy group of the circle and the space of group characters of representation theory of the circle. That means
a sort of ‘identification’ between algebraic properties of a topological space. It would be very interesting to investigate what is the mathematical interpretation of the ‘gravitational $S$-duality’.

We feel that in the framework of the already shown “gravitational duality” interesting questions arise:

- The structure that we have shown is as already argued, similar to that in Yang-Mills theories (see for example [35]). This points out to the possible existence of “gravitational monopoles” and/or solitons.

- It is of interest to consider supergravity versions of the work presented here. In general supersymmetry improves the mathematical consistence, and improves also the duality properties of Yang-Mills theories [7–9].

The self-dual extension of the MM $\mathcal{N} = 1$ supergravity has been already obtained [36]. So the supersymmetric equivalent to Eq. (20) can be straightforwardly constructed and then its “$S$-dual” can be searched for. For $\mathcal{N} = 2$ (or larger) supersymmetries we can try to follow a similar procedure that should allow the construction of an action with squared [37] self-dual field strengths and one would expect this action to be a generalization of the Ashtekar formulation of $\mathcal{N} = 2$ supergravity theories [38].

- It will be also of interest to investigate the relation of the “gravitational duality” of this paper with the gravitational duality proposed by Hull by means of the new gravitational branes arising in type II superstrings and M theory [18].

The previous points are under current investigation and will be reported elsewhere.

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