Restricting affine Toda theory to the half-line

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We restrict affine Toda field theory to the half-line by imposing certain boundary conditions at $x = 0$. The resulting theory possesses the same spectrum of solitons and breathers as affine Toda theory on the whole line. The classical solutions describing the reflection of these particles off the boundary are obtained from those on the whole line by a kind of method of mirror images. Depending on the boundary condition chosen, the mirror must be placed either at, in front, or behind the boundary. We observe that incoming solitons are converted into outgoing antisolitons during reflection. Neumann boundary conditions allow additional solutions which are interpreted as boundary excitations (boundary breathers). For $a_n^{(1)}$ and $c_n^{(1)}$ Toda theories, on which we concentrate mostly, the boundary conditions which we study are among the integrable boundary conditions classified by Corrigan et.al. As applications of our work we study the vacuum solutions of real coupling Toda theory on the half-line and we perform semiclassical calculations which support recent conjectures for the $a_2^{(1)}$ soliton reflection matrices by Gandenberger.

Introduction and Overview

Affine Toda theories are integrable relativistic field theories in one space and one time dimension. They have been widely studied, both classically and at the quantum level, because of their remarkable properties and interesting algebraic structure. For a review see Cor94. The simplest affine Toda theory is the sine-Gordon model.

fig1.eps, width=6.5cm The interaction of a sine-Gordon soliton with an antisoliton. fig1

The affine Toda equations of motion possess soliton solutions Hol92,oli93b. These are kink configurations which interpolate between the different vacua of the Toda potential. The characteristic property of solitons is that they propagate without dispersion and that even after collision with other solitons they regain their original shape. As an example figure fig1 shows a solution of the sine-Gordon model which describes an antisoliton and a soliton moving in opposite directions. One sees how they move through each other, the only effect of their interaction being a time delay.

These solitons can be quantized and then their scattering is described by factorizable S-matrices obeying the axioms of crossing symmetry, unitarity, Yang-Baxter equation and the bootstrap principle Zam. These S-matrices can be obtained by exploiting the affine quantum group symmetry of affine Toda theory Ber.

Interestingly, certain of the classical multi-soliton solutions of affine Toda theory seem to make physical sense also if one views them only on the left half-plane. To illustrate this we have taken the solution of figure fig1 and have restricted it to $x < 0$ to arrive at figure fig2. Now it describes a reflection process off a
boundary at $x = 0$. The right-moving antisoliton hits the wall at $x = 0$ and is reflected back as a soliton with opposite velocity.

The solution of figure fig1 interpreted as describing the elastic reflection of an antisoliton off a boundary at $x = 0$. fig2

In this paper we will investigate this possibility of restricting affine Toda theory to the left half-line by putting a reflecting boundary at $x = 0$. In other words we will impose boundary conditions at $x = 0$ which among all multi-soliton solutions of Toda theory on the whole line select only those which, when viewed on the left half-line, describe the purely elastic reflection of solitons off the boundary.

The existence of these classical solutions describing solitons on the half line opens up the challenge of deriving the corresponding quantum theory. In addition to the S-matrix describing the scattering of the solitons among themselves one will also have to construct a reflection matrix describing the reflection of the solitons off the boundary. The complete set of axioms which such reflection matrices have to satisfy was given by Ghoshal and Zamolodchikov Gho extending the early work of Cherednik Che. Gandenberger Gan98 has found solutions to these axioms for the case of $a_2^{(1)}$ Toda theory.

In this paper we will prepare the ground for extending these investigations into the quantum theory of solitons on the half-line by performing classical and semiclassical calculations. By using the method of images we are able to complement the investigations which have already been performed on this subject. Corrigan et.al. Bow95 have classified boundary conditions which preserve integrability. We see that many of them descend from just the three boundary conditions which we study. The Durham group and in particular Bowcock Bow96 have studied the classical solutions of real coupling Toda theory on the half-line. There are conjectures for particle reflection matrices for real coupling Toda theory on the half-line Fri94b, Sas93,Kim96. In order to check these conjectures and to connect them with specific boundary conditions it is necessary to determine the classical vacuum solution Cor94a. Our analysis simplifies these calculations. Gandenberger Gan98 has determined quantum soliton reflection matrices by solving the boundary Yang-Baxter equation. We verify his results semiclassically by comparing with the time delays computed from our solutions. The idea of obtaining soliton solutions on the half-line from those on the whole line is of course not new. It has been applied to the sine-Gordon model in Sal94, to Toda lattice models in Fuj95 and to real coupling Toda theory Bow96.

There is one affine Toda theory $T(\hat{g})$ for every affine simple complex Lie algebra $\hat{g}$. It describes an $n$-component bosonic field, where $n$ is the rank of $\hat{g}$. Let $\iota_i$, $i = 1, \ldots, n$ be the simple roots of the finite-dimensional Lie algebra $g$ underlying $\hat{g}$ and let $\eta = -\sum_{i=1}^{n} \eta_i$ be the extra simple root that needs to be added to obtain the extended Dynkin diagram of $\hat{g}$. Define $\eta_0 = 1$. Then the corresponding affine Toda theory $T(\hat{g})$ is described by the equations of motion $\dddot{\phi} + m^2 \sum_{i=0}^{n} \eta_i e^{\phi} = 0$. phicom