The gauge symmetry inherent in the concept of manifold (General Gauge Symmetry) has been discussed. In particular, the spherical-symmetric solutions of the gauge invariant equations is considered. General form of the tensor fields those are invariant under time and space inversion has been given.

1. INTRODUCTION

According to the modern standpoint, space-time theory is the one that possesses a mathematical representation whose elements are a smooth four-dimensional manifold $M$ and geometrical object defined on $M$. There are two symmetry groups that are intimately connected with concept of manifold. One group is a group of transformations of the manifold $M$ itself, the manifold mapping group, and the other is a group of transformations acting in tangent vector spaces $T_p(M)$. The latter concept is clearly expounded in the treatise by Misner, Thorne, Wheeler [1]. The well-known manifold mapping group [2] is often called the group of general transformations of coordinates or the group of diffeomorphisms. The physical meaning of the manifold mapping group is that it is a group of symmetry of gravitational interactions in Einstein theory of gravity. A systematic and thorough consideration of the questions connected with space-time symmetry of General Relativity may be found in Ref. [2]. We emphasize only that the diffeomorphism group is evidently the widest group of space-time symmetry.

Let $S_{ij}$ be the components of a tensor field $S$ of type $(1,1)$ that satisfies the condition $\det(S_{ij}) \neq 0$. In this case, there exists a tensor field $S^{-1}$ with components $T^i_j$ such that $S^i_k T^k_j = \delta^i_j$. It is obvious that a tensor field $S$ can be regarded as a linear transformation

$$\bar{V}^i(x) = S^i_j(x)V^j(x) \quad \text{(1.1)}$$

in the space of vector fields; $S^{-1}$ will be the inverse transformation. It is easy to show that transformations (1.1) form a group that establishes an equivalence relation or a general gauge relativity in the tangent space of vector fields and in the spaces of other fields in question. In what follows this group of symmetry underlying the very notion of manifold will be called the general gauge group because it is evident, this is the natural and widest gauge group tightly connected with manifold.

Now we can formulate the gauge principle as follows. The tangent space $T_p(M)$ is identified here with the so-called gauge or internal space. But in the case in question there is a strict coupling between space-time and gauge space, while gauge models presuppose a strict local separation between them. Maybe, this feature is a clue to the whole question of fundamental processes. The group defined by equation (1.1) is a gauge-symmetry group. To complete the gauge principle, it is to be added with an important concept of a particle with spin.

Any pair $(p,V)$, where $p$ is a point of space-time $M$ and $V$ is a vector tangent to the manifold $M$ at the point $p$, will be called the particle with spin. Spin is associated with the vector $V$ and characterizes the polarization properties of the particle. In this article we will shortly represent the simplest gauge invariant equations for the gauge field defined by the gauge principle and consider the problem of spherical-symmetric gauge fields on the background Minkowski space-time. This is the natural first step on the way of search for a physical interpretation of new interactions.

2. BASIC FIELDS

To derive the simplest equations of motion of the spin vector we apply to symmetry considerations. In the general case the infinitesimal change of the vector field on curve $\gamma(t)$ is given by the expression [2]

$$\delta V^i = dV^i + \Gamma^i_{jk} \dot{x}^j V^k dt,$$
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where \( V^i \) are components of the vector \( V(t) \). We assume that at every moment of time \( t \) an infinitesimal change of a vector along the curve \( \gamma(t) \) is equal to an infinitesimal linear transformation of the vector induced by the general gauge group. If \( S^j_j = \delta^j_j + B^j_j \) is an infinitesimal gauge transformation, we have

\[
\delta V^i = B^i_j V^j dt,
\]

from which we obtain a system of ordinary linear homogeneous differential equations

\[
\frac{dV^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} V^k = B^i_j V^j, \tag{2.1}
\]

defining the law of change of the spin vector in the course of motion of a polarized particle along the curve \( \gamma(t) \).

Let us show that equations (2.1) allow us to establish the laws of change of fields \( \Gamma \) and \( B \) under gauge transformations in a natural way. Let vector fields \( \bar{V} \) and \( V \) be equivalent with respect to the general gauge group, then \( \bar{V}^i = S^i_j V^j \). If the components \( V^i \) obey equation (2.1), it is not difficult to verify that the components \( \bar{V}^i \) will be a solution of the equation

\[
\frac{d\bar{V}^i}{dt} + \bar{\Gamma}^i_{jk} \frac{dx^j}{dt} \bar{V}^k = \bar{B}^i_j \bar{V}^j,
\]

where

\[
\bar{\Gamma}^i_{jk} = S^i_l \Gamma^l_{jm} T^m_k + S^i_l \partial_j T^l_k, \quad \bar{B}^i_j = S^i_k B^k_l T^l_j,
\]

and \( T^i_j \) are components of the field \( S^{-1} \) inverse to \( S \). Thus, the transformation laws of fields \( \Gamma \) and \( B \) under gauge transformations are determined.

Let us mention some characteristic properties of the basis fields \( \Gamma \) and \( B \). For brevity, we will use the matrix notation

\[
B = (B^i_j), \quad \Gamma_j = (\Gamma^i_{jk}), \quad E = (\delta^i_j),
\]

in which the transformation law of the connection (or gauge potential) \( \Gamma \) is of the form

\[
\bar{\Gamma}_i = ST \Gamma_i S^{-1} + S \partial_i S^{-1} = \Gamma_i + S \nabla_i S^{-1}, \tag{2.2}
\]

where \( \nabla_i \) stands for the covariant derivative with respect to the connection \( \Gamma \)

\[
\nabla_i S = \partial_i S + \Gamma_i S - S \Gamma_i = \partial_i S + [\Gamma_i, S].
\]

The relation (2.2) clearly shows that the \( \Gamma \) is a gauge potential with respect to the general gauge symmetry defined above. As \( S \nabla_i S^{-1} \) is a tensor field of the type \( (1,2) \), then a gauge potential \( \bar{\Gamma} \) is the connection together with \( \Gamma \) with respect to the general coordinate transformations. Let

\[
(H^{k}_{ij}) = H_{ij} = \partial_i \Gamma_j - \partial_j \Gamma_i + [\Gamma_i, \Gamma_j] \tag{2.3}
\]

be components of the strength tensor of the gauge field \( \Gamma \), then from (2.2) and (2.3) we obtain

\[
\bar{H}_{ij} = S H_{ij} S^{-1}. \tag{2.4}
\]

Since a tensor field \( B \) transforms under gauge transformations as

\[
\bar{B} = S B S^{-1}, \tag{2.5}
\]

then the scalars \( B^i_i = \text{Tr} B \) and \( B^i_j B^j_i = \text{Tr}(B B) \) are evidently invariants of the general gauge group. It is known from the theory of linear operators that there also exist other invariants, but in what follows we will only use these simplest invariants. It is not difficult to see that in the framework of the general gauge principle the fields \( B \) and \( \Gamma \) are separated uniquely.
3. GAUGE-INVARIANT EQUATIONS

As it is noted above, the diffeomorphism group is responsible for gravitational interactions, and thus, the general gauge group is a symmetry group of new interactions. To simplify computations and to write equations in a symmetrical and manifestly gauge - invariant form, we introduce the notion of the gauge - covariant derivative. We will say that a tensor field $T$ of the type $(m,n)$ is of the gauge type $(p,q)$ if under the transformations of the gauge group there is the correspondence

$$T \Rightarrow \tilde{T} = S \cdots S T S^{-1} \cdots S^{-1},$$

where

$$0 \leq p \leq m \text{ and } 0 \leq q \leq n.$$

The gauge field strength $H_{ij}$ is a tensor field of type $(1,3)$ and according to (2.4) has the gauge type $(1,1)$. From (2.5) it follows that the field $B$ being a tensor field of the type $(1,1)$ has the gauge type $(1,1)$. As it follows from the consideration of the left - hand side of the Einstein equations, the Einstein potential $g_{ij}$ being a tensor field of the type $(0,2)$ is to be assigned the gauge type $(0,0)$.

Let $T$ be components of the tensor field (tensor density) of the gauge type $(1,1)$, then by definition

$$D_i T = \partial_i T + [\Gamma_i, T]$$

is the gauge - covariant derivative. For instance, for the gauge field strength tensor

$$D_i H_{jk} = \partial_i H_{jk} + [\Gamma_i, H_{jk}].$$

For the field $B$ the gauge- covariant derivative coincides with the standard covariant derivative:

$$D_i B = \partial_i B + [\Gamma_i, B] = \nabla_i B.$$

In general, the operator $D_i$ is not covariant since $D_i T$ are not always components of the tensor field together with $T$. However, the commutator $[D_i, D_j]$ is covariant, since

$$[D_i, D_j] T = [H_{ij}, T].$$

Hence we obtain the important and covariant relation for the strength tensor

$$[D_i, D_j] H_{kl} = [H_{ij}, H_{kl}]. \quad (3.1)$$

The basic property of the gauge - covariant derivative follows from its definition

$$\tilde{D}_i \tilde{T} = S(D_i T) S^{-1},$$

where $\tilde{T} = ST S^{-1}$ and $\tilde{D}_i$ is the gauge- covariant derivative with respect to the gauge potential

$$\tilde{\Gamma} = \Gamma + S \nabla S^{-1}.$$

So, the tensor fields $B$ and $D_i B$ are of the same gauge type.

As it is known, the determinant $|g_{ij}| \neq 0$, which actually allows to obtain, for the tensor field $g_{ij}$ the equations invariant under the transformations of manifold mapping group. By analogy, let us consider the case when the determinant $|B'_{ij}| \neq 0$. Under this condition the field $B$ has the inverse one for which the nonlinear gauge-invariant equations can be suggested. The simplest gauge-invariant Lagrangian has the form

$$L = -\frac{1}{2} \text{Tr}(D_i B D^i B^{-1}) + \lambda \text{Tr}B - \frac{1}{4} \text{Tr}(H_{ij} H^{ij}), \quad (3.2)$$

where

$$D^i = g^{ij} D_j, \quad H^{ij} = g^{ik} g^{jl} H_{kl},$$

and $\lambda$ is a constant. Taking into account that
\[ \delta B = -B(\delta B^{-1})B \]

and varying (3.2) with respect to \( B \) and \( \Gamma \) we obtain the following second order equations for the fields \( B \) and \( \Gamma \)

\[
\begin{align*}
D_i(\sqrt{|g|}B^{-1}D_j B) &= \lambda \sqrt{|g|}B, \\
D_i(\sqrt{|g|}H^{ij}) &= \sqrt{|g|}J^j,
\end{align*}
\]  

(3.3a)  

(3.3b)

where \( |g| \) is the absolute value of the determinant of the matrix \( (g_{ij}) \) and

\[ J^i = [B^{-1}, D^i B]. \]

The source tensor current \( J \) has to satisfy the equation

\[ D_i(\sqrt{|g|}J^i) = 0 \]

as in accordance with (3.1),

\[ D_i D_j(\sqrt{|g|}H^{ij}) = 0. \]

Since this is really so, the system of equations (3.3) is consistent. Varying the Lagrangian (3.2) with respect to \( g_{ij} \) we obtain the gauge-invariant metric tensor of energy - momentum

\[ T_{ij} = \text{Tr}(D_i BD_j B^{-1}) + \text{Tr}(H_{ik} H^k_j) + g_{ij}L \]

which satisfies the equation

\[ T^{ij;i} = 0. \]  

(3.4)

The semicolon denotes the covariant derivative with respect to the Levi-Civita connection belonging to the field \( g_{ij} \)

\[ \{^i_{jk}\} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}). \]

While deriving (3.4), besides the field equations, one should use the standard relations for the Christoffel symbols \( \{^i_{jk}\} \) given in [3] and the identity

\[ D_i H_{jk} + D_j H_{ki} + D_k H_{ij} = 0 \]

which can easily be obtained from the relation

\[ [D_i, D_j]B = [H_{ij}, B]. \]

From (3.4) and the gauge invariance of the metric tensor of energy-momentum it follows that the complete system of equations derived from the Lagrangian \( L_F = L_g + L \), where \( L_g \) is the Einstein-Hilbert Lagrangian, will be consistent.

### 4. SPHERICAL-SYMMETRIC GAUGE POTENTIALS

In this section we consider Minkowski spacetime with the metric to be related to spherical coordinates that is most convenient under the consideration of \( SO(3) \) symmetry. First of all we would like to find gauge potentials those are invariant under the displacement along the time axis \( t \to t + a \) and inversion of time \( t \to -t \). From the first condition

\[ \Gamma^i_{jk}(x^0, x^1, x^2, x^3) = \Gamma^i_{jk}(x^0 + a, x^1, x^2, x^3) \]

follows that all \( \Gamma^i \)'s are independent of \( t \), while the second condition gives

\[ \Gamma^0_{\alpha \nu} = 0, \quad \Gamma^0_{\beta \beta} = 0, \quad \Gamma^0_{\mu \lambda} = \Gamma^0_{\nu \mu} = \Gamma^0_{\lambda \nu} = 0, \quad \Gamma^1_{\mu \rho} = \Gamma^2_{\mu \rho} = \Gamma^3_{\mu \rho} = 0, \quad \mu, \nu = 1, 2, 3. \]  

(4.1)

Now we shall look for \( SO(3) \) invariant gauge potentials. Generators of \( SO(3) \) group in spherical coordinates have the form [4]
Let us now write the equations

\[ X_1 = \sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi}, \]

(4.2a)

\[ X_2 = -\cos \varphi \frac{\partial}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial}{\partial \varphi}, \]

(4.2b)

\[ X_3 = -\frac{\partial}{\partial \varphi}. \]

(4.2c)

The problem is to find solutions of the equations \( L_X \Gamma = 0 \) when Lie derivative are taken along the vectors fields of \( SO(3) \) group. The equation \( L_X \Gamma = 0 \) can be written in the following matrix form

\[ L_X(i) \Gamma_j = V_{(i)}^j \partial \Gamma_j + [\Gamma_j, A_{(i)}] + \Gamma \ell A_{(i)}^j + \partial_j A_{(i)} = 0, \]

(4.3)

Here, \( V_{(i)}^j \) are defined from \( X_{(i)} = V_{(i)}^j \partial_j \) as

\[ V_{(1)}^j = (0, 0, \sin \varphi, \cot \theta \cos \varphi), \]

\[ V_{(2)}^j = (0, 0, -\cos \varphi, \cot \theta \sin \varphi), \]

\[ V_{(3)}^j = (0, 0, 0, -1). \]

The matrices \( A_{(i)} \) here take the forms

\[ A_{(1)} = A_{(1)}^i j = \partial_j V_{(1)}^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos \varphi / \sin^2 \theta & -\cot \theta \sin \varphi \\ 0 & 0 & 0 & \cos \varphi \end{pmatrix}, \]

\[ A_{(2)} = A_{(2)}^i j = \partial_j V_{(2)}^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin \varphi \\ 0 & 0 & -\sin \varphi / \sin^2 \theta & -\cot \theta \cos \varphi \end{pmatrix}, \]

\[ A_{(3)} = A_{(3)}^i j = \partial_j V_{(3)}^i = 0. \]

Let us now write the equations \( L_X(i) \Gamma_j = 0 \) explicitly. The equations \( L_X(i) \Gamma_j = 0 \) now have the form

\[ \sin \varphi \frac{\partial \Gamma_0}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial \Gamma_0}{\partial \varphi} + [\Gamma_0, A_{(1)}] = 0 \]

(4.4a)

\[ \sin \varphi \frac{\partial \Gamma_1}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial \Gamma_1}{\partial \varphi} + [\Gamma_1, A_{(1)}] = 0 \]

(4.4b)

\[ \sin \varphi \frac{\partial \Gamma_2}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial \Gamma_2}{\partial \varphi} + [\Gamma_2, A_{(1)}] - \frac{\cos \varphi}{\sin^2 \theta} \Gamma_3 + \frac{\partial A_{(1)}}{\partial \varphi} = 0, \]

(4.4c)

\[ \sin \varphi \frac{\partial \Gamma_3}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial \Gamma_3}{\partial \varphi} + [\Gamma_3, A_{(1)}] + \cos \theta \sin \varphi \Gamma_3 + \frac{\partial A_{(1)}}{\partial \varphi} = 0, \]

(4.4d)

while the equations \( L_X(i) \Gamma_j = 0 \) read

\[ -\cos \varphi \frac{\partial \Gamma_0}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial \Gamma_0}{\partial \varphi} + [\Gamma_0, A_{(2)}] = 0, \]

(4.5a)

\[ -\cos \varphi \frac{\partial \Gamma_1}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial \Gamma_1}{\partial \varphi} + [\Gamma_1, A_{(2)}] = 0, \]

(4.5b)

\[ -\cos \varphi \frac{\partial \Gamma_2}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial \Gamma_2}{\partial \varphi} + [\Gamma_2, A_{(2)}] - \frac{\sin \varphi}{\sin^2 \theta} \Gamma_3 + \frac{\partial A_{(2)}}{\partial \varphi} = 0, \]

(4.5c)

\[ -\cos \varphi \frac{\partial \Gamma_3}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial \Gamma_3}{\partial \varphi} + [\Gamma_3, A_{(2)}] + \sin \varphi \Gamma_2 + \cot \theta \cos \varphi \Gamma_3 + \frac{\partial A_{(2)}}{\partial \varphi} = 0. \]

(4.5d)

Finally for \( L_X(i) \Gamma_j = 0 \) we obtain
\[ \frac{\partial \Gamma_j}{\partial \varphi} = 0. \]  

(4.6)

Here presuppose that \( \Gamma_j \) are taken in the form

\[
\Gamma_j = \begin{pmatrix}
\Gamma^0_{j0} & \Gamma^0_{j1} & \Gamma^0_{j2} & \Gamma^0_{j3} \\
\Gamma^1_{j0} & \Gamma^1_{j1} & \Gamma^1_{j2} & \Gamma^1_{j3} \\
\Gamma^2_{j0} & \Gamma^2_{j1} & \Gamma^2_{j2} & \Gamma^2_{j3} \\
\Gamma^3_{j0} & \Gamma^3_{j1} & \Gamma^3_{j2} & \Gamma^3_{j3}
\end{pmatrix}.
\]

(4.7)

From the equation (4.6) it follows that the \( \Gamma_j \)'s are independent of \( \varphi \). Taking into account that the \( \Gamma_j \)'s are independent of \( t \) and \( \varphi \) we finally combine the foregoing equations (4.4) and (4.5) in the form

\[
[\Gamma_0, C] = 0, \quad \frac{\partial \Gamma_0}{\partial \vartheta} + [\Gamma_0, D] = 0,
\]

(4.8)

\[
[\Gamma_1, C] = 0, \quad \frac{\partial \Gamma_1}{\partial \vartheta} + [\Gamma_1, D] = 0,
\]

(4.9)

\[
[\Gamma_2, C] - \frac{1}{\sin^2 \vartheta} \Gamma_3 + \frac{\partial C}{\partial \vartheta} = 0, \quad \frac{\partial \Gamma_2}{\partial \vartheta} + [\Gamma_2, D] + \frac{\partial D}{\partial \vartheta} = 0,
\]

(4.10)

\[
[\Gamma_3, C] + \Gamma_2 + D = 0, \quad \frac{\partial \Gamma_3}{\partial \vartheta} + [\Gamma_3, D] - \cotan \vartheta \Gamma_3 - C = 0,
\]

(4.11)

where we define

\[
C = \cos \varphi A(1) + \sin \varphi A(2) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1/\sin^2 \vartheta & 0
\end{pmatrix},
\]

\[
D = \sin \varphi A(1) - \cos \varphi A(2) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\cotan \vartheta & 0
\end{pmatrix}.
\]

Solving the equations (4.8 – 4.11), for \( \Gamma_j \)'s we find

\[
\Gamma_0 = \begin{pmatrix}
a & \alpha & 0 & 0 \\
\beta & d & 0 & 0 \\
0 & 0 & e & -f/\sin \vartheta \\
0 & 0 & f/\sin \vartheta & e
\end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix}
\gamma & c & 0 & 0 \\
d & \delta & 0 & 0 \\
0 & 0 & \mu & -\nu \sin \vartheta \\
0 & 0 & \nu/\sin \vartheta & \mu
\end{pmatrix},
\]

\[
\Gamma_2 = \begin{pmatrix}
0 & 0 & p \sin \vartheta & 0 \\
0 & 0 & \sigma \tau \sin \vartheta & 0 \\
u/\sin \vartheta & \varepsilon/\sin \vartheta & 0 & \cotan \vartheta
\end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix}
0 & 0 & -q \sin \vartheta & p \sin^2 \vartheta \\
0 & 0 & -\tau \sin \vartheta & \sigma \sin^2 \vartheta \\
u/\sin \vartheta & \varepsilon/\sin \vartheta & 0 & \cotan \vartheta
\end{pmatrix}.
\]

(4.12)

Since the \( \Gamma_j \)'s are invariant under time inversion, then taking into account (4.1) we rewrite (4.12) as follows

\[
\Gamma_0 = \begin{pmatrix}
0 & \alpha & 0 & 0 \\
\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix}
\gamma & 0 & 0 & 0 \\
0 & \delta & 0 & 0 \\
0 & 0 & \mu & -\nu \sin \vartheta \\
0 & 0 & \nu/\sin \vartheta & \mu
\end{pmatrix}.
\]
Once the Riemann tensor is defined, we immediately undertake to write the equations for the functions under consideration. Here the functions $\alpha, \beta, \gamma, \delta, \mu, \nu, \sigma, \tau, \varepsilon, \lambda$ are the arbitrary functions of $r$ only. Now the problem is to write these functions explicitly. In doing so, we first define the Riemann tensors. From (2.3) we know that

$$R_{ij} = \partial_i \Gamma_j - \partial_j \Gamma_i + [\Gamma_i, \Gamma_j] = -R_{ij}. \quad (4.14)$$

A straightforward calculation gives the following non-trivial Riemann tensors

$$R_{10} = \begin{pmatrix} \phi & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{20} = \begin{pmatrix} 0 & -\alpha \sigma & -\alpha \tau \sin \theta \\ 0 & 0 & 0 \\ \lambda \beta & 0 & 0 \\ \varepsilon \beta / \sin \theta & 0 & 0 \end{pmatrix},$$

$$R_{30} = \begin{pmatrix} 0 & 0 & -\alpha \tau \sin \theta - \alpha \sin^2 \theta \\ 0 & 0 & 0 \\ -\varepsilon \beta \sin \theta & 0 & 0 \\ \lambda \beta & 0 & 0 \end{pmatrix}, \quad R_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & \varepsilon / \sin \theta & 0 & 0 \end{pmatrix},$$

$$R_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tau \sin \theta & -\alpha \sin \theta \\ 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 \sin \theta & 0 \\ 0 & 0 & 0 & -B \sin \theta \end{pmatrix},$$

where we define

$$\phi := \alpha' - \alpha (\delta - \gamma), \quad \beta := \beta' - \beta (\delta - \gamma),$$

$$\sigma := \sigma' + \sigma (\delta - \mu) - \tau \nu, \quad \tau := \tau' + \tau (\delta - \mu) + \sigma \nu,$$

$$\lambda := \lambda' - \lambda (\delta - \mu) - \varepsilon \nu, \quad \varepsilon := \varepsilon' - \varepsilon (\delta - \mu) + \lambda \nu,$$

$$A := \varepsilon \sigma - \tau \lambda, \quad B := \varepsilon \tau + \sigma \lambda + 1.$$

Once the Riemann tensor is defined, we immediately undertake to write the equations for the functions under consideration. To this end we invoke the equation

$$\frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} R^{ij}) + [\Gamma_i, R^{ij}] = 0. \quad (4.16)$$

Here $R^{ij} = R_{pq} g^{ip} g^{jq}$, $|g| = r^2 \sin \theta$ and $g_{ij} = \text{diag}(1, -1, -r^2, -r^2 \sin \theta)$. Inserting (4.13) and (4.15) into (4.16) we obtain

$$\alpha' + \frac{2}{r} \alpha + \frac{2}{r^2} (B - 1) \alpha = 0, \quad (4.17a)$$

$$\beta' + \frac{2}{r} \beta + \frac{2}{r^2} (B - 1) \beta = 0, \quad (4.17b)$$

$$\alpha \beta - \beta \alpha = 0, \quad (4.17c)$$

$$\sigma \lambda - \lambda \sigma + \tau \varepsilon - \varepsilon \tau = 0, \quad (4.17d)$$

$$\lambda \tau - \tau \lambda + \sigma \varepsilon - \varepsilon \sigma = 0, \quad (4.17e)$$

$$\sigma' - (\mu - \delta) \sigma - \alpha \beta \sigma - \tau \nu + \frac{1}{r^2} (B \sigma + 3 A \tau) = 0, \quad (4.17f)$$

$$\vartheta' - (\mu - \delta) \vartheta - \alpha \beta \tau + \sigma \nu + \frac{1}{r^2} (B \tau - 3 A \sigma) = 0, \quad (4.17g)$$

$$\lambda' + (\mu - \gamma) \lambda - \alpha \beta \lambda - \varepsilon \nu + \frac{1}{r^2} (B \lambda - 3 A \varepsilon) = 0, \quad (4.17h)$$

$$\varepsilon' + (\mu - \gamma) \varepsilon - \alpha \beta \varepsilon + \lambda \nu + \frac{1}{r^2} (B \varepsilon + 3 A \lambda) = 0. \quad (4.17i)$$
Note that we have ten functions. The system \((4.17)\) contains six equations and three constraints for \(\alpha, \beta, \sigma, \tau, \varepsilon, \lambda\). The functions \(\gamma, \delta, \mu, \nu\) enter in these expressions, but there is no equation to determine them.

Let us demand the \(\Gamma_j\)'s be invariant under space inversion. Under this condition the functions \(\nu, \tau, \varepsilon\) become trivial. \(\gamma, \delta, \mu, \nu\) enter in these expressions, but there is no equation to determine them.

We also demand the \(\Gamma_1\) to be trivial, i.e., \(\gamma, \delta, \mu, \nu\) to be trivial. Under these conditions from \((4.17)\) we find
\[
\begin{align*}
\alpha'' + \frac{2}{r} \alpha' + \frac{2}{r^2} \sigma \lambda \alpha &= 0, \\
\beta'' + \frac{2}{r} \beta' + \frac{2}{r^2} \sigma \lambda \beta &= 0, \\
\alpha \beta' - \beta \alpha' &= 0, \\
\sigma'' - \alpha \beta \sigma + \frac{1}{r^2} (\sigma \lambda + 1) \sigma &= 0, \\
\lambda'' - \alpha \beta \lambda + \frac{1}{r^2} (\sigma \lambda + 1) \lambda &= 0, \\
\lambda \sigma' - \sigma \lambda' &= 0.
\end{align*}
\]

From \((4.18)\) follow \(\beta = c_0 \alpha\) and \(\lambda = d_0 \sigma\), where \(c_0\) and \(d_0\) are some arbitrary constants. Comparing the \(\Gamma_j\)’s with those for Minkowski ones in spherical coordinates, i.e.,
\[
\tilde{\Gamma}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\Gamma}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1/r \end{pmatrix},
\]
\[
\tilde{\Gamma}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & \cot \theta \end{pmatrix}, \quad \tilde{\Gamma}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \\ 0 & 0 & 0 & -\sin \theta \cos \theta \\ 0 & 1/r & \cot \theta & 0 \end{pmatrix},
\]

we conclude that the functions \(\alpha, \beta, \sigma, \lambda\) and the constant \(d_0\) have the following dimensions: \([\alpha] = L^{-1}\), \([\beta] = L^{-1}\), \([\sigma] = L^{-1}\), \([\lambda] = L\), \([d_0] = L^2\). The constant \(c_0\) is dimensionless. Since the constant \(d_0\) is not dimensionless, let us consider the case when \(d_0 = 0\). It means, the function \(\lambda\) is also zero. Under this condition from \((4.18)\) we find \(\beta = c_0 \alpha = c_0 q_0 / r\), \(\lambda = d_0 \sigma = 0\) and \(\sigma = p_0 / r\) with \(p_0\) being the arbitrary constant and the constant \(q_0\) obeying the relation \(c_0 q_0^2 = 3\).

Now, since the \(\Gamma_j\)’s are invariant under gauge transformation, together with \(L_X \Gamma_j = 0\), we demand the following equality to be fulfilled
\[
L_X \tilde{\Gamma}_j = 0,
\]
where \(\tilde{\Gamma}_j\) is defined by \((2.2)\), i.e., \(\tilde{\Gamma}_j = S \Gamma_j S^{-1} + S \partial_i S^{-1}\). The relation \((4.20)\) can be converted to
\[
\partial_j T + [\Gamma_j, T] = \nabla_j T = 0, \quad T = S L_X S^{-1},
\]
From \((4.21)\) follows
\[
[\nabla_i, \nabla_j] T = 0
\]
which can be rewritten as
\[
[R_{ij}, T] = 0.
\]
Putting \((4.15)\) into \((4.22)\) we find \(T = a I\), where \(a\) is some arbitrary constant and \(I\) is the unit matrix. Further it can be shown that \(a = 0\). Now, from the definition \(T = S L_X S^{-1}\) we obtain the gauge \(S\) to be
\[
S = \begin{pmatrix} s & t & 0 & 0 \\ u & v & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix},
\]

with \( s, t, u, v, z \) being some arbitrary constant. It follows from the fact that \( \bar{\Gamma}_1 = S\Gamma_1 S^{-1} + S\partial_s S^{-1} \), with \( \Gamma_1 \) and \( \bar{\Gamma}_1 \) being zero according to our assumption.

Thus we found spherically symmetric \( \Gamma_j \) those are invariant under time and space inversion and gauge transformation of type (4.23):

\[
\begin{align*}
\Gamma_0 &= \begin{pmatrix}
0 & q_0/r & 0 & 0 \\
0 & r & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, & \Gamma_1 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\Gamma_2 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & p_0/r & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \cot \theta & 0
\end{pmatrix}, & \Gamma_3 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & p_0 \sin^2 \theta/r & 0 \\
0 & 0 & \sin \theta \cos \theta & 0 \\
0 & 0 & \cot \theta & 0
\end{pmatrix}.
\end{align*}
\]

(4.24)

Let us now go back to the system of equations (4.17). If instead of spatial inversion we demand the metric connection

\[
g_{ij,k} = \Gamma^\ell_{ki} g_{\ell j} + \Gamma^\ell_{kj} g_{i \ell},
\]

(4.25)

where \( \Gamma^\ell_{ki} \) are the components of \( \Gamma_i \)'s defined by (4.13) and \( g_{ij} = \text{diag}(1, -1, r^2, r^2 \sin^2 \vartheta) \), to be fulfilled, we find

\[
\alpha = \beta, \quad \gamma = 0, \quad \delta = 0, \quad \mu = 1/r \\
\sigma = -\lambda r^2, \quad \tau = -\varepsilon r^2.
\]

We further demand \( \nu \) to be trivial which means we assume \( \Gamma_1 \) to be coincided with that of Minkowski one, i.e., with \( \bar{\Gamma}_1 \) in (4.19). In this case from (4.17) we obtain

\[
\frac{\partial^2 \alpha}{\partial r^2} + \frac{2}{r} \frac{\partial \alpha}{\partial r} - 2(1 + A^2) \varepsilon^2 \alpha = 0, \quad \lambda = A \varepsilon,
\]

(4.26a)

\[
\frac{\partial^2 \varepsilon}{\partial r^2} + \frac{2}{r} \frac{\partial \varepsilon}{\partial r} - \alpha^2 \varepsilon + \frac{1}{r^2} \varepsilon - (1 + A^2) \varepsilon^3 = 0,
\]

(4.26b)

\[
\frac{\partial \omega}{\partial \eta} = \omega - \eta + \eta^3
\]

(4.27)

where we put \( \varepsilon = \eta(\rho)/r\sqrt{1 + A^2} \) and \( \omega = \partial \eta / \partial \rho \) with \( \rho = \ln r \). The second-type abel equation is well studied and the existence theorem has been worked out by several authors. Unfortunately, the exact solution to the equation (4.27) is yet to be found.

5. CONCLUSION