Interaction of D-string with F-string: 
A Path-Integral Formalism 

Supriya Kar† and Yoichi Kazama‡ 

Institute of Physics, University of Tokyo 
Komaba, Meguro-ku, Tokyo 153, Japan 

Abstract 

A path integral formalism is developed to study the interaction of an arbitrary curved Dirichlet (D-) string with elementary excitations of the fundamental (F-) string in bosonic string theory. Up to the next to leading order in the derivative expansion, we construct the properly renormalized vertex operator, which generalizes the one previously obtained for a D-particle moving along a curved trajectory. Using this vertex, an attempt is further made to quantize the D-string coordinates and to compute the quantum amplitude for scattering between elementary excitations of the D- and F-strings. By studying the dependence on the Liouville mode for the D-string, it is found that the vertex in our approximation consists of an infinite tower of local vertex operators which are conformally invariant on their respective mass-shell. This analysis indicates that, unlike the D-particle case, an off-shell extension of the interaction vertex would be necessary to compute the full amplitude and that the realization of symmetry can be quite non-trivial when the dual extended objects are simultaneously present. Possible future directions are suggested. 

†supriya@hep1.c.u-tokyo.ac.jp; Address after October 1, Dept. of Theoretical Physics, Chalmers Institute of Technology & Goteborg University, Sweden. 
‡kazama@hep1.c.u-tokyo.ac.jp
1 Introduction

The idea of D-branes [1, 2] has led to so many new results in the past few years that it is now an indispensable part of our thinking in string-related areas [3, 4]. Nevertheless, when it comes to quantum dynamics of D-branes, our knowledge is still far from complete: A considerable number of calculations have been performed to understand the interaction between a D-brane and string states or between D-branes [5]–[17] (and additional references in [4]), but in most of these works D-branes are treated as infinitely heavy backgrounds. The non-linear dynamics of D-brane(s) in interaction with massless background fields is neatly coded in the Dirac-Born-Infeld action [18, 19, 20] but its quantization is in general difficult if not intractable. At low energies, this can be approximated by the celebrated super Yang-Mills theory (SYM) on the worldvolume[21], but again, in general, analysis of the quantum dynamics is not an easy task.

The exceptions are the cases of D-particles and D-instantons. For the former, the SYM theory becomes a matrix quantum mechanics and moreover it was brilliantly reinterpreted as a promising candidate for a microscopic description of M-theory in the light-cone frame[22]. Its quantum properties have been under vigorous investigations and many results have already been obtained [23]. Likewise, the SYM theory for D-instantons was recognized as providing a non-perturbative definition of the type IIB superstring[24] and is being actively pursued. Progress on the quantum dynamics of D-particles has also been made from the string theory point of view. One of us (Y.K.), in collaboration with S.Hirano, developed a path-integral formalism to quantize a D-particle in interaction with closed string states, thereby incorporating the recoil effects [25]. He further showed [26], with three complementary methods, how one can quantize the system of two D-particles in string theory and obtained the fully quantum amplitude that generalizes the result obtained in [5].

In this paper, we will focus on the dynamics of D-string in interaction with F-string (i.e. the usual string). The long range motivation behind this work is the desire to understand the S-duality of the IIB theory [27], one of the key symmetries of the string theory. Although a number of evidences exist, they are either classical or indirect. A more direct fully quantum mechanical demonstration would require that we be able to treat the D-string and the F-string on equal footing. In particular, we should be able to quantize the excitations of D-string. This in turn requires, as a first step, a proper treatment of an arbitrary curved D-string interacting with F-string.
For this purpose, we shall extend the path-integral formalism developed in [25] and derive the vertex operator describing the interaction of a curved D-string with excitations of F-string in bosonic string theory. The necessary calculations are much more involved compared with the D-particle case, but, employing the derivative expansion which was successful in the D-particle case, we will be able to obtain a relatively compact result, very similar in form to the one for D-particle. In this process, a short-distance divergence, which depends on the extrinsic curvature of the D-string, appears. It is gratifying that this divergence can be neatly absorbed by a renormalization of the D-string coordinates. This is an important check of our formalism.

Using the renormalized vertex operator so obtained, we will make an attempt to compute the amplitude for the scattering between elementary excitations of D- and F-strings. This requires further path-integration over the D-string coordinates including the additional vertex operators for the D-string excitations and with the proper weight, i.e. the exponential of the D-string action. In the case of D-particles, the corresponding program was successfully accomplished [25], despite the highly non-linear nature of the interaction vertex: The effect of the quantization could be summarized in a simple rule that the (proper-)time derivative \( \dot{f}^\mu \) of the D-particle coordinate be replaced by the average \( \frac{1}{2}(p^\mu + p'^\mu) \) of the incoming and the outgoing momenta of the D-particle, which had been first conjectured in [28]. In the present case of D-string, the situation turned out to be much more involved because of the stringent requirement of the conformal invariance which is absent in the D-particle case. The analysis of the Liouville mode of the D-string in the conformal gauge indicates that our approximation only picks up the on-shell intermediate states and the full conformal invariance is not achieved unless an appropriate off-shell extension of the vertex will be made. We may draw from this an important lesson that the realization of symmetry, such as the conformal invariance in the present case, can be highly non-trivial when the dual extended objects are simultaneously present.

The organization of the rest of the article is as follows: In Sec.2, we describe the general setup of the problem, including the explanation of the geodesic normal coordinate expansion, the orthonormal moving frame to be used, and the constraints at the boundary of the open string world-sheet. The actual computation begins in Sec.3, where the path-integral over the open string coordinates is performed. The remaining integrations over the fields at the boundary will be treated in Sec.4. In Sec.5, the renormalization of the vertex operator is conducted and the its \( SL(2,R) \) transformation property is explained.
We then describe, in Sec. 6, an attempt to compute the amplitude for the scattering of excitations of D- and F- strings. After a brief résumé of the quantization procedure, we examine the question of conformal invariance of the vertex operator, analyze the nature of the problem, and suggest future directions.

2 Path integral formalism

2.1 The general setup

We begin with the characterization of an arbitrary curved D-string. Let $f^\mu(t, \sigma)$ be the coordinates of a curved D-string embedded in a flat space-time, with its world-sheet described by $(t, \sigma)$. In this article, we will take the topology of the worldsheet of the open string attached to the D-string to be a disk $\Sigma$ and parametrize its boundary $\partial \Sigma$ by the polar angle $\theta$, $(0 \leq \theta \leq 2\pi)$. Since the ends of the open string may lie anywhere on the world-sheet of the D-string, the Lorentz covariant condition at the boundary which characterizes the D-string is given by [18]

$$X^\mu(\theta) = f^\mu \left( t(\theta), \sigma(\theta) \right),$$

where $X^\mu$ denote the open string coordinates, and $t(\theta)$ and $\sigma(\theta)$ are arbitrary functions describing where on the world-sheet the ends of the open string land.

The object of our interest is the vertex operator (i.e. a functional of the D-string coordinates $f^\mu(\theta, \sigma)$) for a D-string interacting with the states of closed F-string. In the path integral formalism, it is obtained by integration of an appropriate weight over $X^\mu(z)$, $t(\theta)$ and $\sigma(\theta)$ with the constraint (1) imposed, together with the insertions of the vertex operators for the closed F-string states. Also, as usual, we will include in our formalism the coupling of the ends of the open string to the abelian gauge field $A_\mu(X)$. Then the precise form of the vertex operator to be studied is

$$\mathcal{V} \left( f^\mu, \{ k_i \} \right) = \frac{1}{g_s} \int D^2 z \ Dt(\theta) \ D\sigma(\theta) \ \delta \left( X^\mu(\theta) - f^\mu \left( t(\theta), \sigma(\theta) \right) \right)$$

$$\cdot \exp \left( - S[X, A] \right) \prod_i g_s \ V_i(k_i),$$

where $g_s$ is the string coupling constant and $V_i(k_i)$ are the vertex operators for closed string states carrying momenta $k_i$. For definiteness, we will consider the tachyon emission vertices $V_i(k_i) = \int d^2 z_i \exp \left( ik_i \cdot X(z_i) \right)$. $S[X, A]$ is the open string action coupled to the
gauge field and is of the familiar form[19, 20]

\[ S[X, A] = \frac{1}{4\pi\alpha'} \int_E d^2z \, \partial_a X^\mu \partial_\bar{a} X_{\mu} + i \int_{\partial E} d\theta \, A_\mu(X) \, \partial_\theta X^\mu , \]  

where \( \bar{a} = 0, 1 \) labels the coordinates on the F-string world-sheet.

### 2.2 Geodesic normal coordinate expansion

Let us first consider the integrals over \( t(\theta) \) and \( \sigma(\theta) \), describing the fluctuations of the ends of the F-string. Although these fluctuations play an important role in producing the effective action of the D-string, they cannot be dealt with exactly. Hence, as in the case of a D-particle [25], we will employ the geodesic normal coordinate expansion [29] and organize their effects order by order in the derivatives of the D-string world-sheet coordinates \( f^\mu(t, \sigma) \). Clearly such an expansion preserves the general coordinate invariance on the world-sheet of the D-string.

Adapting the method of [29] to our case, the expansion of \( f^\mu(t(\theta), \sigma(\theta)) \) around \( f^\mu(t, \sigma) \) is worked out as

\[
 f^\mu(t(\theta), \sigma(\theta)) = f^\mu(t, \sigma) + \partial_a f^\mu(t, \sigma) \zeta^a(\theta) + \frac{1}{2} K^\mu_{ab} \zeta^a(\theta) \zeta^b(\theta) \\
+ \frac{1}{3!} K^\mu_{abc} \zeta^a(\theta) \zeta^b(\theta) \zeta^c(\theta) + O(\zeta^4) ,
\]

where \( \zeta^a(\theta) \) are the normal coordinates with \( a = (t, \sigma) \) and \( K^\mu_{ab}(t, \sigma) \) and \( K^\mu_{abc}(t, \sigma) \) are the extrinsic curvatures of the D-string sub-manifold. They can be expressed as

\[
 K^\mu_{ab} = P^{\mu\nu} \partial_a f^\nu, \\
 K^\mu_{abc} = \partial_a \partial_b f^\mu - \partial_a \Gamma_{bc}^d f^d + 2 \Gamma^d_{ab} \Gamma_{dc}^\mu ,
\]

where \( P^{\mu\nu} \) is the projection operator normal to the world-sheet and \( \Gamma^a_{bc} \) is the Christoffel connection. More explicitly, \( P^{\mu\nu} \) is given in terms of the projector \( h^{\mu\nu} \) tangential to the world-sheet as

\[
 \eta^{\mu\nu} = h^{\mu\nu} + P^{\mu\nu} ,
\]

where \( h^{\mu\nu} \) is constructed out of the inverse \( h^{ab} \) of the metric \( h_{ab} \) induced on the world-sheet of the D-string:

\[
 h^{\mu\nu} = \partial_a f^\mu h^{ab} \partial_b f^\nu , \\
 h_{ab} = \partial_a f^\mu \partial_b f^\mu .
\]
The intuitive picture of this expansion is that to the leading order the entire boundary of the disk is attached to a point \((t, \sigma)\) on the D-string world-sheet and the effects of the fluctuations from this configuration is taken into account by the integration over the normal coordinates \(\zeta^a(\theta)\).

Now to facilitate the computation, we split the string coordinate \(X^\mu(z, \bar{z})\) into a constant and non-constant modes:

\[
X^\mu(z, \bar{z}) = x^\mu + \xi^\mu(z, \bar{z}).
\]

Then the \(\delta\)-function expressing the constraint (1) on the boundary decomposes into two parts:

\[
\delta\left( X^\mu(\theta) - f^\mu(t(\theta), \sigma(\theta)) \right) = \delta\left( x^\mu - f^\mu(t, \sigma) \right) \\
\cdot \delta\left( \xi^\mu(\theta) - \partial_a f^\mu(t, \sigma) \zeta^a(\theta) - \ldots \right).
\]

The first \(\delta\)-function makes the integration over \(x^\mu\) trivial and produces, from the product of tachyon vertices, a factor

\[
V_0 = \exp\left( i \sum k^\mu f_\mu(t, \sigma) \right),
\]

where \(k^\mu \equiv \sum_i k_i^\mu\) is the total momentum of the tachyons. The effect of the second \(\delta\)-function will be discussed later in Sec.3.1.

### 2.3 Orthonormal frame

Just as in the D-particle case[25], integration over the non-constant mode \(\xi^\mu(z, \bar{z})\) will be simplified by the use of an appropriate orthonormal moving frame. Since, to the zeroth order in the normal coordinate expansion, the entire boundary of the disk is mapped on to a point \((t, \sigma)\) on the D-string world-sheet, the most natural choice is a frame where two of the basis vectors, \(\hat{e}_a^\mu, a = 0, 1\), lie on the tangential plane at that point, with the rest, \(\hat{e}_\alpha^\mu, \alpha = 2, 3, \ldots, 25\), being orthogonal to them. In this way, the boundary conditions for the open string will be simply Neumann along \(\hat{e}_a^\mu\) and Dirichlet along \(\hat{e}_\alpha^\mu\), to this order of approximation.

Specifically, we construct the orthonormal vectors \(\hat{e}_A^\mu\) for \(A = (a, \alpha)\) in the following way:
\[
\begin{align*}
\mathring{e}_\mu^0 &= \frac{\dot{f}_\mu}{\sqrt{-h_{00}}}, \\
\mathring{e}_\mu^1 &= \sqrt{-h_{00}} \left( f^{\mu}_\nu + \mathring{e}_0^\nu \mathring{e}_0^\nu f'_\nu \right), \\
\mathring{e}_A^B &= \eta_{AB}, \\
\eta^{\mu\nu} &= \mathring{e}_A^\mu \mathring{e}_B^\nu \eta^{AB}, \\
h^{\mu\nu} &= \sum_a \mathring{e}_a^\mu \mathring{e}_a^\nu = -\mathring{e}_0^\mu \mathring{e}_0^\nu + \mathring{e}_1^\mu \mathring{e}_1^\nu, \\
P^{\mu\nu} &= \sum_a \mathring{e}_a^\mu \mathring{e}_a^\nu.
\end{align*}
\]

Here, \( h \) stands for \( \det h_{ab} \), a dot and a prime on \( f^\mu \) correspond to time (\( t \)) and spatial (\( \sigma \)) derivatives respectively, and we have displayed as well the expressions for the projectors in terms of the basis vectors. We will actually use more compact notations given by

\[
\begin{align*}
\mathring{e}_a^\mu &\equiv N_a \partial_a f^\mu(t, \sigma), \\
\mathring{e}_A^\mu &\equiv N_A e^\mu_A,
\end{align*}
\]

where \( a = (0, 1), N_0 = \frac{1}{\sqrt{-h_{00}}}, N_1 = \sqrt{h_{00}/h} \) and \( N_A \equiv (N_a, 1, \ldots, 1) \). (Apart from being orthogonal to \( \mathring{e}_a^\mu \), explicit forms for \( \mathring{e}_a^\mu \) will not be needed.) Now in this frame, the non-constant mode \( \xi^\mu(z, \bar{z}) \) can be expanded as

\[
\xi^\mu(z, \bar{z}) = \sum_A \mathring{e}_A^\mu \rho^A(z, \bar{z}),
\]

where \( \rho_a(z, \bar{z}) \) and \( \rho_a(z, \bar{z}) \) correspond to the fluctuations in the tangential and the transverse directions respectively.

### 2.4 “Constraints” and “conditions” at the boundary

Before starting the path integration, we must clarify the conditions imposed at the boundary and how we treat them. There are two types of such conditions governing the fluctuations of the string coordinates \( \rho_A(\theta) \) and the geodesic coordinates \( \zeta^a(\theta) \).

The first set of conditions, which we call the “boundary constraints”, come from the \( \delta \)-function in Eq.(11). Expressed as relations between \( \rho_A(\theta) \) and \( \zeta^a(\theta) \), they take the form

\[
\rho_a(\theta) = N_a^{-1} \eta_{ab} \left( \zeta^b(\theta) - \frac{1}{3!} \left[ K^\lambda_{im} K^\lambda_{nlp} h^{bp} + \Gamma^b_{lp} \Gamma^p_{mn} + \partial_l \partial_m f^\lambda \partial_q f^\lambda \partial_n h^{bq} \right] \right)
\]
\[ + 2\partial_t \partial_m f^\lambda \partial_q f_\lambda \Gamma_{np}^q h^{bp} - \partial_r f^\lambda \partial_q f_\lambda \Gamma_{np}^r \Gamma_{bn}^m h^{bp} ] \zeta^i(\theta) \zeta^m(\theta) \zeta^n(\theta) \right) + \mathcal{O}(\zeta^4) \] (16)

and

\[ \rho_\alpha(\theta) = \hat{\epsilon}_\alpha \left( \frac{1}{2} \partial_a \partial_b f_\lambda \zeta^a(\theta) \zeta^b(\theta) + \frac{1}{3!} \left[ \partial_a \partial_b \partial_c f_\lambda - 3\Gamma_{be}^d \partial_d \partial_a f_\lambda \right] \cdot \zeta^a(\theta) \zeta^b(\theta) \zeta^c(\theta) \right) + \mathcal{O}(\zeta^4) . \] (17)

These are much more complicated than their counterparts in the D-particle case[25] due to the nature of the general two dimensional induced metric on the D-string sub-manifold. We will regard them as constraining the integrations over \( \zeta^a(\theta) \), which will be performed after integrating over \( \rho_A(\theta) \).

The second type of conditions are the usual boundary conditions for \( \rho_A(\theta) \) that arise from the consistency of the variation of the action. To find them we need to expand the gauge field \( A_\mu(X) \) in Eq.(3) around the constant mode \( x^\mu \). To avoid unnecessary complications, we will deal with the case of constant field strength [20], for which the boundary interaction becomes quadratic in \( \xi^\mu \):

\[ i \int_{\partial \Sigma} d\theta A_\mu(x + \xi) \partial_b X^\mu = i \frac{1}{2} F_{\mu\nu} \int_{\partial \Sigma} d\theta \xi^\mu \partial_b \xi^\nu . \] (18)

Then the action (3) in the orthonormal frame becomes

\[ S[X, A] = \frac{1}{4\pi\alpha'} \left[ - \int_{\Sigma} d^2 z \rho_A \rho^A + \int_{\partial \Sigma} d\theta \left( \rho_A \partial_n \rho^A + i N_A N_B \tilde{F}_{AB} \rho^A \partial_\theta \rho^B \right) \right] , \] (19)

where we introduced the notations \( \tilde{F}_{AB} \equiv 2\pi\alpha' F_{AB} \), \( \partial_A \equiv e^a_A \partial_a \) and \( A_B \equiv e^a_B A_\mu \). From this action, it is straightforward to obtain the following set of consistent boundary conditions:

\[ \partial_n \rho_\alpha(\theta) + i N_A N_B \tilde{F}_{ab} \partial_\theta \rho^b(\theta) = 0 , \]
\[ \rho_\alpha(\theta) = 0 . \] (20)

Note that, as expected, the D-string only sees \( \tilde{F}_{ab} \), the components of the gauge field along its world-sheet.
3 Path integral over the string coordinates $\rho^A(z, \bar{z})$

We are now ready to perform the integration over the non-zero modes $\rho_A(z, \bar{z})$, which exist on the boundary as well as in the bulk. Rather than treating the boundary components separately, we shall regard them as bulk quantities with appropriate $\delta$-functions, so that the integration can be performed in a unified manner.

3.1 Reformulation of the boundary interactions

Let us describe this procedure more explicitly. First consider the non-zero mode part of the $\delta$-function constraints (11). It can be rewritten as an integral over a Lagrange multiplier $\nu^\mu(\theta)$ in the form

$$
\delta\left(\xi^\mu(\theta) - \partial_a f^\mu \zeta^a(\theta) - \frac{1}{2} K^\mu_{ab} \zeta^a(\theta) \zeta^b(\theta) - \frac{1}{3!} K^\mu_{abc} \zeta^a(\theta) \zeta^b(\theta) \zeta^c(\theta) \ldots\right) = \int D\nu^\mu(\theta) \exp\left( i \int d\theta \nu^\mu(\theta) \rho_{a}(\theta) \right) \cdot \exp\left( -i \int d\theta \nu_{a}(\theta) \left[ \partial_a f^\mu \zeta^a(\theta) + \frac{1}{2} K^\mu_{ab} \zeta^a(\theta) \zeta^b(\theta) + \frac{1}{3!} K^\mu_{abc} \zeta^a(\theta) \zeta^b(\theta) \zeta^c(\theta) + \ldots \right] \right),
$$

where $\nu^A(\theta) \equiv \nu_{\mu} \hat{e}^{\mu A}$ are the components in the orthonormal frame. Together with the use of the boundary condition (20) the above expression can be written explicitly as

$$
\int D\nu^\alpha(\theta) D\nu^\alpha(\theta) \exp\left( i \int d\theta \nu^\alpha(\theta) \rho_{a}(\theta) \right) \cdot \exp\left( -i \int d\theta \eta_{ab} \nu_{b}(\theta) \right)
\cdot \left[ \zeta^b(\theta) - \frac{1}{3!} \left( K^\lambda_{im} K_{\lambda np} h^{bp} + \Gamma^b_{lp} \Gamma^p_{mn} + \partial_l \partial_m f^\lambda \partial_q f_\lambda \partial_n h^{bp} + 2 \partial_l \partial_m f^\lambda \partial_q f_\lambda \Gamma^p_{nm} h^{bp} - \partial_r f^\lambda \partial_q f_\lambda \Gamma^q_{nm} \Gamma^p_{rl} h^{bp} \right) \zeta^l(\theta) \zeta^m(\theta) \zeta^n(\theta) \right] \cdot \exp\left( -i \int d\theta \nu^\alpha(\theta) \hat{e}^\lambda_{\alpha} \left[ \frac{1}{2} \partial_a \partial_b f_\lambda \zeta^a(\theta) \zeta^b(\theta) - \frac{1}{3!} \left( \partial_a \partial_b \partial_c f_\lambda - 3 \Gamma^d_{bc} \partial_d \partial_a f_\lambda \right) \zeta^a(\theta) \zeta^b(\theta) \zeta^c(\theta) \right] + O(\xi^4) \right).
$$

The boundary value of $\rho_A(z, \bar{z})$ appears only in the first exponent, which can be re-expressed as a bulk integral $i \int_{\Sigma} d^2z \delta_{\Delta A} \delta\left( |z| - 1 \right) \nu^A(z, \bar{z}) \rho_A(z, \bar{z})$.

Next consider the boundary interaction with the gauge field appearing in (19). Since the transverse coordinates $\rho^\alpha(\theta)$ drop out due to the boundary conditions (20), we can write it in the bulk form as
\[ i \, N_A N_B \, \tilde{F}_{AB}(x) \int_{\partial \Sigma} d\theta \, \rho^A(\theta) \partial_b \rho^B(\theta) = - N_a N_b \, \tilde{F}_{ab}(x) \int_{\Sigma} d^2 z \, \delta(|z| - 1) \rho^a(z, \bar{z}) \partial_z \rho^b(z, \bar{z}) \, . \quad (23) \]

Now from Eqs.(19),(22) and (23), the path integral over \( \rho_A(z, \bar{z}) \) with tachyon vertex insertions can be written as

\[
I_{\rho} = \int D\rho_A \exp \left( - \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 z \left[ \partial_b \rho^A \partial^b \rho_A - N_A N_B \tilde{F}_{AB}(x) \delta(|z| - 1) \rho^A \partial_z \rho^B \right] \right) \cdot \exp \left( i \int_{\Sigma} d^2 z \, J_A \rho^A \right) , \quad (24)
\]

where

\[
J_A = \sum_i k_{iA} \delta^{(2)}(z - z_i) + \delta_A a \, \delta(|z| - 1) \nu_a(\theta) \quad (25)
\]

and \( k_{iA} = k_{i\mu} \hat{e}_A^\mu \) are the momenta of the tachyons expressed in the orthonormal frame.

### 3.2 Evaluation of \( \rho^A \)-integrals

Clearly the integrals over the transverse coordinates \( \rho^a \), satisfying the Dirichlet boundary condition, can be performed rather trivially with the use of the well-known Dirichlet function \( D(z, z') \) on the unit disk. We write the result as

\[
I_{\rho}^t = \int D\rho_a \exp \left( \frac{\alpha'}{2} \int d^2 z \, d^2 z' \, J_a(z) G_{aa}(z, z') J_a(z') \delta_{\alpha\alpha} \right) . \quad (26)
\]

where

\[
G_{aa}(z, z') = D(z, z') = \ln |z - z'| - \ln |1 - z \bar{z}'| . \quad (27)
\]

The remaining integral over the longitudinal components \( \rho^a(z, z') \) is of the form

\[
I_{\rho}^l = \int D\rho_a \exp \left( \frac{1}{4\pi\alpha'} \int d^2 z \, \rho^a(z) \, \Delta_{ab} \rho^b(z) + i \int d^2 z \, J_a(z) \rho^a(z) \right) , \quad (28)
\]

where \( \Delta_{ab} \) is a non-trivial operator given by

\[
\Delta_{ab} = \eta_{ab} \partial^2 + N_A N_B \tilde{F}_{ab} \delta\left(|z| - 1\right) \partial_z . \quad (29)
\]
The path integral (28) is easily done, with the result

\[ I^I_\rho = (-\det \Delta_{ab})^{-1/2} \int \exp \left( \frac{\alpha'}{2} \int d^2z \, d^2z' \, J_a(z) G_{ab}(z, z') J_b(z') \eta_{ab} \right), \]  

(30)

where \( \tilde{F}_{ab} = \epsilon_{ab} \tilde{f} \) with \( \epsilon_{01} = 1 = -\epsilon_{10} \). The propagator matrix \( G_{ab}(z, z') \) in Eq.(30) for \( a = (0, 1) \) is the Neumann function on the unit disk and in the bulk it satisfies

\[ \Delta_{ab} G_{ab}(z, z') = \partial^2 G_{ab}(z, z') = 2\pi \eta_{ab} \delta^{(2)}(z, z') . \]  

(31)

Also, due to the boundary condition (20), \( G_{ab} \) satisfies

\[ \partial_n G_{ab}(z, z') + iN_a N_b \tilde{F}_{ab} \partial_\theta G_{ab}(z, z') = 0 , \]  

(32)

on the boundary \( \partial \Sigma \). Explicitly, they are given in matrix form as

\[ G_{ab}(z, z') = \delta_{ab} \ln |z - z'| + \frac{1}{2} \left( \frac{\sqrt{-h} - \tilde{F}}{\sqrt{-h} + \tilde{F}} \right)_{ab} \ln \left( 1 - \frac{1}{zz'} \right) \]

\[ + \frac{1}{2} \left( \frac{\sqrt{-h} + \tilde{F}}{\sqrt{-h} - \tilde{F}} \right)_{ab} \ln \left( 1 - \frac{1}{z'z} \right) . \]  

(33)

It can be checked that \( G_{00}(z, z') = G_{11}(z, z') \equiv G(z, z') \) and can be expressed as

\[ G(z, z') = \ln |z - z'| + \left( \frac{h + \tilde{f}^2}{h - \tilde{f}^2} \right) \ln \left| 1 - \frac{1}{zz'} \right| . \]  

(34)

For \( F_{ab} = 0 \), the off-diagonal part of the Neumann function vanishes and the diagonal element (34) corresponds exactly to that of a D-particle [25] in the mutually orthogonal directions.

On the boundary \( \partial \Sigma \), the diagonal part of the Neumann function \( G(\theta, \theta') \) diverges as \( \theta' \to \theta \) and needs to be regularized. We adopt the usual method [19] with a cut off \( \epsilon \) and write

\[ G(\theta, \theta') = -2h(h - \tilde{f}^2)^{-1} \sum_{n=1}^{\infty} \frac{e^{-en}}{n} \cos n(\theta - \theta') . \]  

(35)

Then its inverse \( G^{-1}(\theta, \theta') \) satisfies
\[
\frac{1}{2} \int d\theta \, d\theta' \, G^{-1}(\theta, \theta') \, G(\theta, \theta') = -1
\]
\[
\frac{1}{2\pi^2} \, G(\theta, \theta') \, \partial_\theta \partial_{\theta'} \, G(\theta, \theta') = \tilde{\delta}(\theta - \theta') .
\] (36)

where \( \tilde{\delta}(\theta - \theta') = \delta(\theta - \theta') - (1/2\pi) \). With this regularization, \( G(\theta, \theta) \) is given by

\[
G(\theta, \theta) = 2 \, h \left( h - \bar{f}^2 \right)^{-1} \ln \epsilon .
\] (37)

This divergence will be seen to be absorbed by the renormalization of the D-string coordinates.

Finally, we need to evaluate the Jacobian factor in Eq.(30). Using the Fourier mode expansion on the boundary circle and the usual \( \zeta \)-function regularization, we find

\[
\left( -\det \triangle_{ab} \right)^{-\frac{1}{2}} = \prod_{n=1}^{\infty} \left( \frac{h + \bar{f}^2}{h} \right)^{-1} = \exp \left[ -\zeta(0) \, \ln \left( \frac{h + \bar{f}^2}{h} \right) \right] = \left( \frac{h + \bar{f}^2}{h} \right)^{-\frac{1}{2}} .
\] (38)

This completes the path integral over the string coordinates \( \rho_A(z, z') \). Putting everything together, the result takes the form

\[
I_\rho = \sqrt{\frac{h + \bar{f}^2}{h}} \cdot \exp \left( \frac{\alpha'}{2} \sum_{ij} k_{ia} k_{jb} \, G_{ab}(z_i, z_j) \, \eta_{ab} \right) \cdot \exp \left( \alpha' \int d\theta \sum_i k_{ia} \, G_{ab}(z_i, \theta) \, \nu_b(\theta) \, \eta_{ab} \right) \cdot \exp \left( \alpha' \int d\theta' \, \nu_a(\theta) \, G_{ab}(\theta, \theta') \, \nu_b(\theta') \eta_{ab} \right) \cdot \exp \left( \frac{\alpha'}{2} \sum_{ij} k_{ia} k_{j\beta} \, G_{\alpha\beta}(z_i, z_j) \, \delta_{\alpha\beta} \right) .
\] (39)

4 Path integral over the boundary fields

What remains to be performed is the integral over the boundary fields, namely over the Lagrange multiplier fields \( \nu_a(\theta) \) and the normal coordinates \( \zeta_a(\theta) \).
4.1 Integration over the Lagrange multiplier $\nu_a(\theta)$

Assembling the relevant terms from Eqs.(22) and (39), the path integral over the Lagrange multipliers $\nu_a(\theta)$ takes the form

$$I_\nu = \int \mathcal{D}\nu_a(\theta) \cdot \exp \left( \frac{\alpha'}{2} \int d\theta\ d\theta' \nu_a(\theta) G_{ab}(\theta, \theta') \nu_b(\theta') \eta_{ab} \right) \cdot \exp \left( i \int d\theta \ j_a(\theta) \nu_b(\theta) \eta_{ab} \right),$$

where the source term is given by

$$\frac{j_a(\theta)}{\sqrt{\alpha'}} = \eta_{ab} \left( -i\sqrt{\alpha'} \sum_i k_{ia} G_{ab}(z_i, \theta) + \frac{N_a^{-1}}{\sqrt{\alpha'}} \left[ \zeta^b(\theta) \right. \right.$$

$$\left. \right. - \frac{1}{3!} \left( K^\lambda_m K^\lambda_{np} h_{bp} + \Gamma^b_{lp} \Gamma^p_{mn} + \partial_l \partial_m f^\lambda \partial_q f_\lambda \partial_n h_{bp} + 2 \partial_l \partial_m f^\lambda \partial_q f_\lambda \Gamma^q_{np} h_{bp} \right) + \partial_r f^\lambda \partial_q f_\lambda \Gamma^q_{np} \Gamma^r_{lm} h_{bp} \right) \right) + \mathcal{O}(\zeta^4).$$

The integration is straightforward, with the result

$$I_\nu \equiv \exp \left( \frac{1}{2} \int d\theta\ d\theta' \ j_a(\theta) G_{ab}^{-1}(\theta, \theta') \ j_b(\theta') \eta_{ab} \right),$$

where $\tilde{j}_a(\theta) = j_a(\theta)/\sqrt{\alpha'}$.

4.2 Integration over the normal coordinates $\zeta_a(\theta)$

Now we come to the final stage of the calculation, namely to the integration over the normal coordinates $\zeta_a(\theta)$. To simplify the calculation, it is convenient to make a rescaling

$$\zeta_a(\theta) = \sqrt{\alpha'} N_a \tilde{\zeta}_a(\theta).$$

This induces a change in the functional measure, which can be computed just like for $(-\det \Delta_{ab})^{-1/2}$, with the result

$$\mathcal{D}\zeta_a(\theta) = \frac{1}{\alpha'} \sqrt{-h} \mathcal{D}\tilde{\zeta}_a(\theta).$$

Then, assembling all the relevant terms from (40), (39) and (22), the integral over $\zeta(\theta)$, to order $\mathcal{O}(\zeta^4)$, becomes

$$I_\zeta = \int \mathcal{D}\tilde{\zeta}_a \exp \left( \frac{1}{2} \int d\theta\ d\theta' \tilde{\zeta}_a(\theta) G_{ab}^{-1}(\theta, \theta') \tilde{\zeta}_b(\theta') \eta_{ab} \right)$$
In the framework of the derivative expansion, then the integral becomes

\[ \text{det} \equiv \exp \left( -\frac{i\alpha'}{2} N_a N_b \int d\theta \hat{\nu}^\lambda(\theta) \partial_a \partial_b f_\lambda \tilde{\zeta}^a(\theta) \tilde{\zeta}^b(\theta) \eta_{ab} \right) \]

\[ \exp \left( -\frac{i}{2} \int d\theta d\theta' j_{\tilde{c}}^a(\theta') \tilde{\zeta}^b(\theta') \eta_{ab} \right) \]

\[ \exp \left( -\frac{\alpha'}{2} \int d\theta d\theta' \tilde{k}_a(\theta) G_{ab}^{-1}(\theta,\theta') \tilde{k}_b(\theta') \eta_{ab} \right) \]

\[ \exp \left( -\frac{\alpha'}{3!} N_a^{-1} N_b N_m N_m [K_{lm}^a K_{\lambda mn} h_{bp} + \Gamma_{lp}^b \Gamma_{mn}^p + \partial_l \partial_m f^\lambda \partial_q f_\lambda \partial_n h_{bp} \right. \]

\[ \left. + 2 \partial_l \partial_m f^\lambda \partial_q f_\lambda \Gamma_{np}^q h_{bp} + \partial_r f^\lambda \partial_q f_\lambda \Gamma_{np}^q \Gamma_{lm}^r h_{bp} \right] \]

\[ \cdot \int d\theta d\theta' \tilde{\zeta}^l(\theta) \tilde{\zeta}^m(\theta) \tilde{\zeta}^n(\theta) G_{ab}^{-1}(\theta,\theta') \tilde{\zeta}^b(\theta') \eta_{ab} \right), \quad (45) \]

where

\[ j_{\tilde{c}}^a(\theta') \equiv \sqrt{\alpha'} \tilde{k}_b G_{ab}^{-1}(\theta,\theta') \delta_{ab}. \quad (46) \]

To simplify the exponent, which contains a term linear in \( \tilde{\zeta} \), let us define \( D_{ab}(\theta,\theta') \) and \( \tilde{\zeta}^a(\theta) \) by

\[ D_{ab}(\theta,\theta') = G_{ab}^{-1}(\theta,\theta') - i\alpha' N_a N_b \hat{\nu}^\lambda(\theta) \partial_a \partial_b f_\lambda, \quad (47) \]

\[ \tilde{\zeta}^a(\theta) = \tilde{\zeta}^a(\theta) - i D_{ab}^{-1}(\theta,\theta') j_{\tilde{c}}^b(\theta'). \quad (48) \]

Then the integral becomes

\[ I_{\tilde{c}} = \int \mathcal{D}\tilde{\zeta}^a(\theta) \exp \left( \frac{1}{2} \int d\theta d\theta' \tilde{\zeta}^a(\theta) D_{ab}(\theta,\theta') \tilde{\zeta}^b(\theta') \eta_{ab} \right) \]

\[ \exp \left( -\frac{\alpha'}{3!} N_a^{-1} N_b N_m N_m [K_{lm}^a K_{\lambda mn} h_{bp} + \Gamma_{lp}^b \Gamma_{mn}^p + \partial_l \partial_m f^\lambda \partial_q f_\lambda \partial_n h_{bp} \right. \]

\[ \left. + 2 \partial_l \partial_m f^\lambda \partial_q f_\lambda \Gamma_{np}^q h_{bp} + \partial_r f^\lambda \partial_q f_\lambda \Gamma_{np}^q \Gamma_{lm}^r h_{bp} \right] \]

\[ \cdot \int d\theta d\theta' \tilde{\zeta}^l(\theta) \tilde{\zeta}^m(\theta) \tilde{\zeta}^n(\theta) G_{ab}^{-1}(\theta,\theta') \tilde{\zeta}^b(\theta') \eta_{ab} \right). \quad (49) \]

In the framework of the derivative expansion, \( \mathcal{O}(\tilde{\zeta}^4) \) terms in the exponent with an extra \( \alpha' \) will be treated perturbatively. The basic Gaussian integral gives the determinant

\[ \left( -\text{det} D_{aa} \right)^{-\frac{1}{2}} = \exp \left( -\frac{i\alpha'}{2} N_a^2 \partial_a \partial_a f_\lambda \int d\theta G_{aa}(\theta,\theta) \hat{\nu}^\lambda(\theta) \right) \quad (50) \]

\( G_{aa}(\theta,\theta) \), containing \( \ln \epsilon \) divergence, is actually independent of \( \theta \) and the integral fortunately vanishes due to the absence of the zero mode in \( \hat{\nu}^\lambda(\theta) \), i.e. by \( \int d\theta \hat{\nu}^\lambda(\theta) = 0 \). Thus we get \( \left( -\text{det} D_{aa} \right)^{-\frac{1}{2}} = 1. \)
Evaluation of the effect of the quartic interaction is straightforward using the propagator \( \langle \tilde{\zeta}^a(\theta) \tilde{\zeta}^b(\theta') \rangle = \eta^{ab} G_{ab}(\theta, \theta') \), which at the coincident point has the form

\[
\langle \tilde{\zeta}^a(\theta) \tilde{\zeta}^b(\theta') \rangle = 2 \eta^{ab} h \left( h - \bar{f}^2 \right)^{-1} \ln \epsilon .
\]

After some algebra, we get a remarkably simple result:

\[
\alpha' \ln h \left( h - \bar{f}^2 \right)^{-1} N_a^2 K_{aa}^\lambda K_{\lambda ab} h^{ab} .
\]

Putting everything together, the path integral over the boundary fields finally yields

\[
I_{\tilde{\zeta}} \equiv \frac{1}{g_s \alpha'} \sqrt{-h} \left( 1 - \alpha' h \left( h - \bar{f}^2 \right)^{-1} N_a^2 \eta_{aa} K_{aa}^\lambda K_{\lambda ab} h^{ab} \ln \epsilon \right) .
\]

5 Renormalized vertex operator and its property

5.1 Renormalization

Substituting the above result (52) into (39), we arrive at the disk amplitude to the next to leading order in the derivative expansion:

\[
\frac{1}{g_s \alpha'} \sqrt{-h} \left( 1 - \alpha' h \left( h - \bar{f}^2 \right)^{-1} N_a^2 \eta_{aa} K_{aa}^\lambda K_{\lambda ab} h^{ab} \ln \epsilon \right) .
\]

The leading term reproduces precisely the Dirac-Born-Infeld action \([30]\) for a D-string with tension \(1/(g_s \alpha')\). In our formalism, it is generated from a combination of the bulk integral and the boundary integral.

The subleading term is the correction due to the boundary interaction. It contains the extrinsic curvature of the curved D-string world-sheet and is a non-trivial generalization of the D-particle case. In fact one can check that by substituting \(N_1 = 0\), the above amplitude reduces to that obtained in \([25]\) for a D-particle.

Now we must ask if this divergent correction can be properly absorbed by a renormalization of the D-string coordinate \(f(t, \sigma)\), just as in the D-particle case \([25]\). The answer is in the affirmative. Let \(f^\mu_R(t, \sigma)\) be the renormalized coordinate which is related to the bare coordinate by

\[
f^\mu = f^\mu_R + \sum_a \delta_a f^\mu_R ,
\]

where

\[
\delta_a f^\mu_R = -\alpha' \left( \frac{h_R + \bar{F}}{h_R - \bar{F}} \right) \eta_{aa} N_a^2 K_{aa}^\mu \ln \epsilon .
\]
Then, after a long but straightforward calculation, we find, up to $O(\alpha')$
\begin{equation}
\sqrt{- (h + F)} = \sqrt{- (h_R + \tilde{F}) \left(1 + \alpha' h_R \left(h_R - \tilde{F}^2\right)^{-1} N_a^2 \eta_{baa} K^\lambda_{a} K_{\lambda ab} h_R^{ab} \ln \epsilon\right)} \tag{56}
\end{equation}
Putting this into (53), we see that the renormalized expression for the amplitude takes precisely the form of the DBI action
\begin{equation}
\frac{1}{g_s \alpha'} \sqrt{- (h_R + \tilde{F})} \tag{57}
\end{equation}
In the original derivation [18], the action of this form was obtained by quite a different logic. In that treatment, the action was determined so that the equation of motion derived from it gives a flat D-brane, namely the vanishing of the extrinsic curvature. In the present treatment, we are dealing with a curved D-string and our result shows that the DBI action is valid including this case if one properly renormalizes its coordinates.

### 5.2 Vertex operator for scattering with tachyons

We are now in a position to write down the detailed form of the vertex operator, defined in (2), describing the scattering of tachyons from an arbitrary curved D-string. From (39) and (57) it takes the form
\begin{equation}
\mathcal{V}_T (f^\mu, \{k_i\}) = \frac{1}{g_s \alpha'} \int dt d\sigma \sqrt{- (h_R + \tilde{F})} \cdot \exp \left( i k^\mu f^\mu \right)
\cdot \prod_i (g_s d^2 z_i) \cdot \exp \left( \frac{\alpha'}{2} \sum_{ij} k_i^\mu k_j^\nu \left[ h_{\mu \nu} G(z_i, z_j) + P_{\mu \nu} D(z_i, z_j) \right] \right), \tag{58}
\end{equation}
where $k^\mu = \sum_i k_i^\mu$ is the total momentum of tachyons and the prime on the summation implies that the singular part of the Green’s function for $i = j$ is omitted.

In the above expression, there still remain divergences due to the appearance of the bare coordinates $f^\mu(t, \sigma)$ and its derivatives in the exponent. They are of the following form: The one in the factor $\exp(ik^\mu f^\mu)$ is proportional to $k_\mu \delta_a f_R^\mu = k_\mu K^\mu_{aa}$, while $h_{\mu \nu}$ in the second exponential contains the piece proportional to $k^\mu \partial_\mu f_R^\mu$. As we shall explain below, they actually vanish due to the “current conservation” that follows from the requirement of $SL(2,R)$ invariance of the vertex operator.

For simplicity, let us consider the two tachyon case for illustration. The $SL(2,R)$ transformation can be written as
\begin{equation}
z \rightarrow \tilde{z} = \frac{\alpha z + \beta}{\beta z + \tilde{\alpha}}, \quad |\alpha^2| - |\beta^2| = 1 \tag{59}
\end{equation}
The measure transforms as \( d^2 \tilde{z} = d^2 z |\tilde{\beta} z + \tilde{\alpha}|^{-4} \), while the exponent becomes

\[
\frac{\alpha'}{2} \left( 2k_1^\mu k_2^\nu h_{\mu\nu} \Delta G(z_1, z_2) + k_1^\mu k_1^\nu h_{\mu\nu} \Delta G(z_1, z_1) + k_2^\mu k_2^\nu h_{\mu\nu} \Delta G(z_2, z_2) + 2k_1 \cdot k_2 D(z_1, z_2) - (k_1^2 \ln |1 - |z_1|^2| + k_2^2 \ln |1 - |z_2|^2|) \right)
\]  

(60)

where \( \Delta G \equiv G - D \). It is easy to see that the part containing \( h_{\mu\nu} \) is invariant if and only if the current conservation condition \( k^\mu \partial_\mu f = 0 \) is met. For the rest, \( D(z_1, z_2) \) is invariant by itself, while the last two terms cancel the contribution from the measures if and only if the tachyon on-shell conditions \( k_1^2 = k_2^2 = 4/\alpha' \) are satisfied. This result persists for the general multi-particle case and hence the \( SL(2,R) \)-invariance condition reads

\[
k_\mu \hat{e}_a^\mu \equiv k_\mu \partial_a f = 0
\]

and

\[
\alpha' \sum_a k_{ia}^2 + \alpha' \sum_a k_{ia}^2 - 4 = \alpha' k_i^2 - 4 = 0.
\]

(61)

It is easy to see that with the current conservation condition the divergent pieces vanish and we have a well-defined vertex operator in terms of the renormalized fields. This structure is again quite analogous to the D-particle case discussed in [25].

6 Attempt at quantized scattering amplitude

Having obtained the vertex operator, let us make an attempt to obtain a quantized scattering amplitude. Contrary to the D-particle case, a D-string is an extended object containing infinite number of excitations. Hence the natural amplitude is the one for the scattering of the elementary excitations of D-string with those of F-string. As the effect of the latter is already encoded in the vertex operator itself, what we need to do is to insert the vertex operators for the D-string excitations and then perform the path integral over the D-string coordinates, \( i.e. \) quantize the D-string.

6.1 Brief summary of covariant quantization of D-string

Let us begin by making a brief summary of the covariant quantization of D-string, which has been developed over the past year[31, 32, 33, 34]. We follow the approach of [34], which is most suitable for our purpose.
One begins with the DBI action for a D-string given by

\[ S = -\tilde{T} \int d^2 \sigma \sqrt{-h_F}, \]  

where

\[ \tilde{T} \equiv \frac{T}{g_s}, \quad T = \frac{1}{2\pi \alpha'}, \]  
\[ h_F \equiv \det(h + \bar{F})_{ab} = h + \bar{f}^2, \]  
\[ h_{ab} = \partial_a f^\mu \partial_b f_\mu, \quad \bar{F}_{ab} = \epsilon_{ab} \bar{f}, \]  
\[ \bar{f} = \epsilon^{ab} \partial_a A_b = \dot{A}_1 - A'_0. \]

The definitions of the momentum and the electric field are

\[ p_\mu = \frac{\partial L}{\partial \dot{f}_\mu} = \frac{\tilde{T}}{\sqrt{-h_F}}(\dot{f}_\mu h_{11} - f_\mu h_{01}), \]  
\[ E \equiv E^1 = \frac{\partial L}{\partial \dot{A}_1} = \frac{\tilde{T} \bar{f}}{\sqrt{-h_F}}. \]

From these follow the primary constraints

\[ L_\pm = \frac{1}{2}(p \pm T_E f')^2 = 0, \]  
\[ E^0 = 0, \]

where \( T_E \equiv \sqrt{\tilde{T}^2 + E^2} \). Further, from the consistency of \( E^0 = 0 \) with the time-development, one gets, in the usual manner, the Gauss law constraint \( \partial_1 E = 0 \). Under the Poisson bracket, \( L_\pm \) can be shown to form the left and right Virasoro algebras even in the presence of the electric field. One can then construct the BRS charges and perform the gauge-fixing in the standard manner.

The convenient gauge choice is the conformal gauge defined by

\[ h_{ab} = \left( \begin{array}{cc} -\epsilon^- \bar{\phi} & 0 \\ 0 & \epsilon^+ \bar{\phi} \end{array} \right), \]

where \( \bar{\phi} \) is the D-Liouville field. In this gauge, the action takes the form [34]

\[ S = \int d^2 \sigma \left( p \cdot \dot{f} - \frac{1}{2T_E}(p^2 + T_E^2 f'^2) + E \partial_0 A_1 \right). \]

Upon integration over \( A_1 \), we find that \( E \) is a constant. Finally, further integration over the momenta yields the usual Polyakov-type action

\[ S = \frac{T_E}{2} \int d^2 \sigma \partial_a f^\mu \partial^a f_\mu. \]

Since \( E \) is constant, so is the D-string tension \( T_E \) and hence the quantization of this action can be performed just like for F-string.
6.2 Scattering amplitude and the problem of conformal invariance

For simplicity, let us take the excitations of the D-string to be tachyons (to be called D-tachyons, to be distinguished from the tachyons of F-string.) Since the action is of the usual Polyakov form, the vertex operator for a D-tachyon carrying momentum $p$ should be, in conformal gauge,

$$g_D \int d^2 \sigma \sqrt{-h} e^{ipf} = g_D \int d^2 \sigma e^{\tilde{\phi}(\sigma)} e^{ipf},$$

where $g_D$ is the D-string coupling constant, which in our present formalism is a free parameter. Also, in this gauge $\sqrt{-h_F}$ and $h_{\mu\nu}$ take the following form:

$$\sqrt{-h_F} = \frac{T}{T_E} e^{\tilde{\phi}}, \quad h_{\mu\nu} = e^{-\tilde{\phi}} \partial_{\mu} f_{\nu} \partial_{\nu} f_{\mu}.$$

The amplitude for the scattering of $N$ tachyons with $M$ D-tachyons then becomes

$$A(k, p) = \int Df^\mu \exp \left( -\frac{1}{4\pi \alpha'} \int d^2 \sigma \partial_{\mu} f^\nu \partial_{\nu} f_{\mu} \right) \frac{T}{g_s T_E} \int d^2 u e^{\tilde{\phi}(u)} e^{ik \cdot f(u)} \cdot \prod_{i=1}^N (g_s d^2 z_i) \exp \left( \frac{\alpha'}{2} e^{-\tilde{\phi}(u)} \sum_{i,j} k^\mu_i k^\nu_j \partial_{\mu} f_{\mu}(u) \partial_{\nu} f_{\nu}(u) \Delta G(z_i, z_j) \right) \cdot \exp \left( \frac{\alpha'}{2} \sum_{i,j} k_i \cdot k_j D(z_i, z_j) \right) \cdot \prod_{I=1}^M (g_D d^2 w_I) e^{ip_I f(w_I) e^{\tilde{\phi}(w_I)}},$$

where, as before, $\Delta G \equiv G - D$. The Neumann function $G(z, z')$, defined in (34), appears to depend on the D-string coordinate through $h$. However, from the definition of the electric field, which is constant, one easily finds $h \pm f^2 = h \left( 1 \mp E^2 T_E \right)$ = constant and thus $G$ actually does not depend on $f^\nu(t, \sigma)$.

Now we come to face the question of conformal invariance. There are several places where the D-Liouville field $\tilde{\phi}$ appears and for the theory to be conformally invariant they

\[\text{a}^\text{It should be determinable in a more complete theory of D-string.}\]
must be eliminated. To study this problem, it is instructive to recall how the Liouville fields disappear in the usual string theory [35].

Consider first the tachyon vertex insertions. After performing the integral over the string coordinates, one gets the structure

$$\prod_i e^{\phi(z_i)} e^{(\alpha'/2) \sum_i p_i \cdot p_j G(z_i, z_i)},$$  \hspace{1cm} (77)

where $G(z', z) = \frac{1}{2} \ln |z' - z|^2$ is the closed string Green’s function. At the coincident point, this diverges and we must regularize. Define $G_{\text{reg}}(z, z) \equiv \frac{1}{2} \ln |\Delta z|^2$, where $|\Delta z|$ is the cutoff distance. What we have to fix actually is the invariant distance $\epsilon = e^{\phi(z)} |\Delta z|^2$.

Thus,

$$G_{\text{reg}}(z, z) = \frac{1}{2} \ln(e^{-\phi(z)} \epsilon) = \frac{1}{2} \ln \epsilon - \frac{1}{2} \phi(z).$$  \hspace{1cm} (78)

Then the dependence on the Liouville field at $z_i$ is

$$e^{\phi(z_i)} \cdot e^{-(\alpha'/4)p^2_i \phi(z_i)}.$$  \hspace{1cm} (79)

Therefore the Liouville dependence disappears for on-shell tachyon, i.e. for $p^2 = 4/\alpha'$. (The $\ln \epsilon$ piece disappears due to the momentum conservation, $\sum_i p_i \cdot p_j = (\sum_i p_i)^2 = 0$.)

Consider next the case of the graviton-dilaton vertex. The vertex is of the form

$$\int d^2 z \sqrt{-h} e^{\mu \nu} \partial \alpha \partial \beta \partial \gamma \partial \delta e^{ik \cdot X} = \int d^2 ze^{\phi} \left\{ e^{-\phi} e^{\mu \nu} \partial \alpha \partial \beta \partial \gamma \partial \delta e^{ik \cdot X} \right\} = \int d^2 ze^{\phi} \partial \alpha \partial \beta \partial \gamma \partial \delta e^{ik \cdot X}.$$  \hspace{1cm} (80)

So it does not have the Liouville factor. This in turn means that the Liouville dependence from the Green’s function must vanish by itself. This requires precisely the on-shell condition for the graviton-dilaton, i.e. $k^2 = 0$. Similar mechanism works for higher rank tensor fields $^b$.

Now let us go back to our amplitude (76). If we expand the exponential in the third line and combine each of the term with the factor $\int d^2 u e^{\phi(u)} e^{ik \cdot f}$ in the second line, we see that our vertex operator describing the interaction of D- and F- strings consists of an infinite sum of vertex operators for tensor fields of various ranks. They come precisely

\footnote{Such a mechanism has been implicitly applied when we performed the integration over the F-string. Omission of the contribution from the singular part of the Green’s function at the coincident point is the consequence of this mechanism.}
with the expected D-Liouville factors discussed above for the ordinary string. Each vertex operator is thus conformally invariant for the respective on-shell value of $k^2$, namely $k^2 = -4n/\tilde{\alpha}'$, $n = -1, 0, 1, \ldots$, where $\tilde{\alpha}' = 1/2\pi T_E$ is the slope parameter for D-string. This means that our vertex operator is dominated by the effect of an infinite collection of on-shell resonances. This unfortunately does not respect the conformal invariance as a whole.

This must have occurred due to our approximation. In our scheme, we have started with the contribution of the configuration where the ends of the open string are attached to a point on the D-string worldsheet and then tried to take into account the fluctuation from this limit in the derivative expansion. The first correction so obtained is the factor $\sqrt{-h + \bar{F}}$ (and its renormalization). Therefore, up to this level, the interaction is essentially point-like and hence only the effects of the on-shell intermediate states are picked up.

Thus we have learned that, unlike in the case of a D-particle interacting with a string, the requirement of symmetry governing the system of two different types of extended objects is much more stringent and is non-trivial to implement. Our work, which attempted to go beyond the existing knowledge by quantizing the D-string coordinates, revealed this feature in an explicit manner.

How would one overcome this difficulty? There may be several directions for progress:

- One way is to try to investigate the effect of higher order corrections in the present scheme: One should then be able to see how the off-shell intermediate states would begin to contribute and may get a hint for constructing satisfactory off-shell vertex operator.

- Alternatively, one may look for a scheme which would automatically guarantee the conformal invariance. Imposition of BRST invariance in the operator approach may be among such possibilities. In this picture, the amplitude that we attempted to compute is represented by the expression of the form

$$D\langle 0|V_D(p_1) \cdots V_D(p_M) \mathcal{V} V_F(k_1) \cdots V_F(k_N)|0\rangle_F,$$

where $D\langle 0|$ and $V_D(p_i)$ are, respectively, the vacuum state and the vertex operator in the D-string sector and similarly the ones with the subscript $F$ denote the corresponding quantities in the F-string sector. In the middle is the vertex operator $\mathcal{V}$ which converts between D- and F- sectors, just like the fermion emission vertex of
the RNS formalism of superstring theory. For conformal (or BRST) invariance, $\mathcal{V}$ must satisfy $Q_D \mathcal{V} = \mathcal{V} Q_F$, where $Q_D$ and $Q_F$ are, respectively, the BRST charge in the D- and F-sector. This equation should restrict the form of $\mathcal{V}$ considerably and may even be powerful enough to determine $\mathcal{V}$.

The studies suggested above are, however, beyond the scope of the present investigation and are left for the future.

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