QUANTUM MECHANICS OF CONFINEMENT AND CHIRAL SYMMETRY BREAKING IN TWO-DIMENSIONAL QCD

Alexei Nefediev
Institute of Theoretical and Experimental Physics, Moscow 117259, Russia

ABSTRACT

The system of light quark and heavy anti-quark source is studied in 1+1 QCD in the large \( N_C \) limit. Making use of the modified Fock–Schwinger gauge allows to consider simultaneously the spectroscopical problem of the \( q\bar{Q} \) bound states and the problem of the light quark Green function. The Dirac-type equation for the spectrum of the system is proved to be equivalent to the well-known 't Hooft one in the one body limit. The unitary transformation from the Dirac–Pauli representation to the Foldy–Wouthuysen one is carried out explicitly, and it is shown that the equation in the Foldy–Wouthuysen representation can be treated as a gap equation which defines the light quark self-energy in the modified Fock–Schwinger gauge. The Foldy–Wouthuysen angle is found to play the role of the Bogoliubov–Valatin one and to give the standard value of the chiral condensate. Connections of the given formalism to the standard four-dimensional QCD are outlined and discussed.

For the first time the two-dimensional model of QCD in the limit of infinite number of colours \( N_C \) was considered in 1974 by 't Hooft\(^1\) and the celebrated equation of the same name was derived in the light-cone gauge. Four years later, in 1978, this equation was re-derived in the axial gauge.\(^2\) So the model seems to have been studied well enough. Still it attracts considerable attention as a problem with features very much similar to those of standard four-dimensional QCD.

Besides the usual assumption \( N_C \to \infty \) limit that allows to sum up only planar diagrammes, we make use of the so-called modified Fock-Schwinger or Balitsky gauge\(^3\)

\[
A^a_1(x_0, x) = 0 \quad A^a_0(x_0, 0) = 0
\]

As soon as gauge (1) is a kind of radial one, the gluonic field can be expressed in terms of the field strength tensor that yields the gluon propagator in the form

\[
K_{\alpha \beta}^{ab}(x_0 - y_0, x, y) = \delta^{ab} \frac{g^2}{2} \delta(x_0 - y_0)(|x - y| - |x| - |y|) \equiv \delta^{ab} K(x, y),
\]

and other components equal to zero.
Note that \( K \) can be naturally broken into local \((K^{(1)} \sim |x-y|\delta(x_0-y_0))\) and non-local \((K^{(2)} \sim (|x|+|y|)\delta(x_0-y_0))\) parts.

Green function for the \( q - Q \) system under consideration has the form *

\[
S_{q\bar{Q}}(x, y) = \frac{1}{N_C} \int D\psi D\bar{\psi} DA_{\mu} \exp \left\{ -\frac{1}{4} \int d^2x F_{\mu\nu}^a \int d^2x \bar{\psi}(i\partial - m - \bar{A})\psi \right\} \times \bar{\psi}(x) S_{\bar{Q}}(x, y|A)\psi(y),
\]

where anti-quark Green function \( S_{\bar{Q}} \) is introduced. The main advantage of our peculiar gauge choice is the fact that the anti-quark is decoupled completely so that \( S_{\bar{Q}} \) can be substituted in a very simple form:

\[
S_{\bar{Q}}(x, y|A) = S_{\bar{Q}}(x - y); \quad S_{\bar{Q}} = -i \left( \frac{1+\gamma_0}{2} \theta(t)e^{-iMt} + \frac{1-\gamma_0}{2} \theta(-t)e^{iMt} \right) \delta(x).
\]

On integrating gluon degrees of freedom in (3), we arrive at the effective Lagrangian for the light quark which leads in turn to the Schwinger–Dyson equation

\[
(i\partial_x - m)S(x, y) + \frac{iN_C}{2} \int d^2z \gamma_0 S(x, z)\gamma_0 K(x, z)S(z, y) = \delta^{(2)}(x - y),
\]

where

\[
S(x, y) = \frac{1}{N_C} S_{\alpha}(x, y).
\]

It is very instructive to note here that the role played by Green function (6) is twofold. By construction \( S \) is the Green function of the light quark, but due to a very passive part of the static anti-quark it plays the role of the Green function of the whole \( q\bar{Q} \) system as well, so that the problem of the light quark Green function and the spectroscopical problem for the \( q\bar{Q} \) system can be considered simultaneously. We shall get back to this statement later on while discussing the chiral properties of the model.

Approach based on spectral decomposition of the Green function (6) turns out very useful \(^\dagger\), so one has

\[
S(q_{10}, q_1, q_{20}, q_2) = 2\pi \delta(q_{10} - q_{20}) \left( \sum_{\varepsilon_n > 0} \frac{\varphi_n^{(+)}(q_1)\varphi_n^{(+)}(q_2)}{q_{10} - \varepsilon_n} + \sum_{\varepsilon_n < 0} \frac{\varphi_n^{(-)}(q_1)\varphi_n^{(-)}(q_2)}{q_{10} + \varepsilon_n} \right).
\]

To proceed further we assume that the Foldy-Wouthuysen operator \( T(p) = e^{\frac{i}{2}\theta_F(p)\gamma_1} \) diagonalizing equation (5) exists and that angle \( \theta_F \) is the same for all \( n \).

With such an assumption applied Schwinger–Dyson equation (5) reduces to the Dirac-type equation in the Hamiltonian form \((f = \frac{2N_C}{4\pi}, \alpha = \gamma_0\gamma_1, \beta = \gamma_0, )\):

\[
(\alpha p + \beta m)\varphi_n^0(p) - \pi f \int dqdk(\beta\cos\theta_F(q) + \alpha\sin\theta_F(q))K(p - q, k - q)\varphi_n^0(k) = E_n\varphi_n^0,
\]

\(^*\)We adopt the following \( \gamma \)-matrix convention\(^2\): \( \gamma_0 = \sigma_3, \gamma_1 = i\sigma_2, \gamma_5 = \sigma_1 \)

\(^\dagger\)The approach based on diagrammatic technics leads to the same results\(^4\)
where $\varphi_n^0(p)$ being scalar wave function

$$
\varphi_n^{(+)}(p) = \varphi_n^0(p)T_n^+(1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi_n^{(-)}(p) = \varphi_n^0(p)T_n^-(0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

(9)

Let us consider only local (i.e. generated by $K^{(1)}$) part of interaction which reduces to a mass operator $\Sigma$ and can be naturally parametrized via two scalar functions $E(p)$ and $\theta(p)$ in the convenient form

$$
\Sigma(p) \equiv |E(p)\cos\theta(p) - m| + \gamma_1 |E(p)\sin\theta(p) - p|.
$$

(10)

Self-consistency condition for such a parametrization makes $E(p)$ and $\theta(p)$ satisfy a system of coupled equations:

$$
\begin{aligned}
E(p)\cos\theta(p) &= m + \frac{f}{2} \int \frac{dk}{(p-k)^2} \cos\theta(k) \\
E(p)\sin\theta(p) &= p + \frac{f}{2} \int \frac{dk}{(p-k)^2} \sin\theta(k),
\end{aligned}
$$

(11)

It is easy to verify that if we identify the Foldy-Wouthuysen angle $\theta_F$ with the Bogoliubov–Valatin one $\theta^\dagger$, then the non-local interaction diagonalizes as well, so that the Foldy-Wouthuysen representation of equation (8) takes the Schrödinger-type form

$$
\varepsilon_n \varphi_n^0(p) = E(p)\varphi_n^0(p) - f \int \frac{dk}{(p-k)^2} \cos\frac{\theta(p) - \theta(k)}{2} \varphi_n^0(k).
$$

(12)

Equation (12) is nothing but the one-body limit of the well-known 't Hooft equation.²

As mentioned above, Green function of the $q\bar{Q}$ system constructed from the solutions of equation (12) is the one of the light quark as well, so that the chiral condensate can be easily calculated

$$
<\bar{q}q> = -i \lim_{x \rightarrow y^+} Tr S(x, y) = -\frac{N_C}{\pi} \int_0^{\infty} dp \cos\theta(p).
$$

(13)

Condensate (13) does not vanish in the chiral limit $m \rightarrow \infty$ and coincides with the standard value³

$$
<\bar{q}q>_{m=0} = -0.29N_C\sqrt{2f}.
$$

(14)

A reasonable question may arise, whether it is worth reproducing old results with a new complicated method. Still the answer is positive, since the given method can be easily generalized to describe the four-dimensional QCD. The Schwinger–Dyson equation similar to (5) takes the following form in Euclidean space⁵

$$
(-i\partial_x - im)S(x, y) + \int d^4zK_{\mu\nu}(x, z)\gamma\mu S(x, z)\gamma\nu S(z, y) = \delta^{(4)}(x - y).
$$

(15)

$\frac{1}{2}\theta(p)$ obviously plays the role of the Bogoliubov–Valatin angle as it describes the rotation from the bare particle with the free dispersion law $\sqrt{p^2 + m^2}$ to the dressed “physical” particle with dispersion $E(p)$. 

³$\theta(p)$ obviously plays the role of the Bogoliubov–Valatin angle as it describes the rotation from the bare particle with the free dispersion law $\sqrt{p^2 + m^2}$ to the dressed “physical” particle with dispersion $E(p)$. 
The main object governing the quark dynamics is the correlation function $D$:

$$< F^a_{\mu\nu}(x) F^b_{\lambda\rho}(y) > = \frac{\delta^{ab}}{N_C^2 - 1} D(x - y) (\delta_{\mu\lambda} \delta_{\nu\rho} - \delta_{\mu\rho} \delta_{\nu\lambda})$$  (16)

with the kernel of equation (15) being proportional to $D$.

Function $D$ rapidly decreases at large Euclidean distances and this decrease is governed by the gluonic correlation length $T_g$. In the string picture $T_g$ defines the radius of the string formed between $q$ and $\bar{Q}$. The two limiting cases, $mT_g \gg 1$ and $mT_g \ll 1$, should be treated separately, as it is clearly seen from the non-relativistic expansion of the interaction in (15).

The case of large $mT_g$ gives quite a natural result for interaction

$$V(r) = \left(\frac{5}{6} + \frac{1}{6}\gamma_0\right) \sigma r + \text{corrections}, \quad V_{FW}(r) = \sigma r - \frac{\vec{\sigma} \vec{l}}{4m^2 r} + O\left(\frac{\sigma r}{mT_g}\right),$$  (17)

which is in agreement with the Eichten–Feinberg–Gromes results$^{9,10}$ whereas the opposite limit of small $mT_g$ leads to a self-inconsistency,$^{11}$ as the corrections $\sim O\left((mT_g)^{-2}\right)$ and diverge in the given limit.

A natural interpretation of such results is offered by the string picture of confinement. In the case of “thick” string ($T_g \gg \frac{1}{m}$) the quark interacts with the gluonic field rather than with a formed string, so the quark dynamics is local and potential.

The opposite limit of “thin” string ($T_g \ll \frac{1}{m}$) is just the case realized in the two-dimensional ’t Hooft model, where strings are infinitely thin (the system just lacks of extra transverse dimensions to allow the string to swell). The interaction is sufficiently non-local (see the integral term in the r.h.s. of equation (12)) and the same behaviour is expected in the string limit ($T_g \to 0$) of the four-dimensional QCD. Regge picture of the rotating string should follow from equation (5) in this case. Such a picture helps to resolve the well-known problem with local confinement: incorrect trajectory slope for scalar interaction vs Klein paradox for vector one.

References

1. G. ’t Hooft, Nucl. Phys. B72, 461 (1974);
2. I.Bars, M.B.Green, Phys.Rev. D17, 537 (1978);
5. Yu.A.Simonov, Phys.At.Nucl. 60, 2069 (1997);