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DARBOUX METHOD AND SEARCH OF INVARIANTS FOR THE LOTKA–VOLTERRA AND COMPLEX QUADRATIC SYSTEMS

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Abstract

The Darboux method introduces algebraic solutions quite useful to obtain invariant and first integrals of polynomial differential systems. Here we study the 2d Lotka–Volterra (LVS) and the complex quadratic system (QS) using straight lines for both and conics for the LVS. The conditions needed to obtain these invariants are given and a study of the phase space portrait is done.

1. Introduction

We consider the search of invariants for the 2–dimensional differential system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (1)$$

where P and Q are polynomials with coefficients in $\mathbf{F}$ where $\mathbf{F}$ is either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. We say that $m = \max \{\deg P, \deg Q\}$ is the *degree* of the polynomial differential system. This type of equations appear in the modelisation of natural phenomena described in different branches of the science such as biology, chemistry, astrophysics, fluid

mechanics, electronics, etc. Of particular interest are the systems such that $m = 2$. The polynomial differential systems of degree 2 will be called *quadratic systems* (QS). One particularly well known quadratic system is the *Lotka-Volterra system* (LVS) which has been used to model the time evolutions of conflicting species in biology and of chemical reactions [7, 1]. Among other applications, we find a QS in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics (with the assumption of quasi-neutrality to eliminate either the ion or electron equation) [11]. Moreover a reduced QS is obtained from a generalized Blasius equation for fluid flow around a wedge-shaped obstacle in boundary layer theory [5]. In the context of plasma physics, all the nonlinear terms represent binary interactions or model certain transport across the boundary of the system. There is a long history of research on finding sufficient conditions for which periodic solutions (center problem) exist for systems equivalent to the QS, and numerous results were obtained which we are not able to fully survey [10]. However, most of the previous works assumed that the origin is a linear center (i.e. having eigenvalues $\pm i$) which we do not assume here as starting point.

Although our main interest concerns the integrability, we will also consider cases for which one can exhibit a constant of the motion which is not a *first integral*. As it is well known in 2d a polynomial system is integrable if it possess a first integral. However, it can be interesting sometimes to know if it can have an *invariant*. Roughly speaking, with a first integral we can describe completely the phase portrait of the polynomial system, while with an invariant (a time-dependent first integral) we only can describe its asymptotic behaviour. In fact, an invariant is a function $I(x, y, t)$ such that when $(x, y) = (x(t), y(t))$ is a solution of (1), I is a constant. Consequently, I must satisfy the equation

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q = 0. \quad (2)$$

When I does not depend explicitly of t , the invariant is called a first integral.

In spite of the fact that they differ in the details of computation, all the methods to obtain invariants (Carleman linearisation [4, 1], Hamiltonian method [2, 8]) are based on an a priori hypothesis on the form of the invariant I . So, in the Carleman method this form is introduced in system (1) and the parameters entering in it are selected in order that (2) becomes an identity or, in the Hamiltonian method, through rescaling, the form of the equations leading to this invariant is determined and again we select the different parameters and unknowns which are at our disposal to identify these equations with the one we want to study.

Here we review a method proposed long time ago by Darboux [6]. A survey of many works triggered by the Darboux theorem together with a study of the integrability of real quadratic systems having an invariant conic is given in [3].

The Darboux method is based on the possibility of writing the invariant (or at least an integrating factor for system (1)) as the product of different functions $f_i(x, y)$ raised at a given power λ_i . It is on the form of these functions f_i that we introduce an ansatz and the subsequent need to identify the parameters. In that sense the Darboux method is not so different from the others cited above. Nevertheless the experience shows that we have somehow divided the difficulties of the unavoidable identification leading to algebraic but nonlinear equations with sometimes a number of equations greater than the number of unknowns with, consequently, conditions imposed on the parameters describing the dynamical system. From a physicist point of view the rule of the game is to find invariants with as few conditions as possible on the parameters of the given system.

The paper is organized as follows. In Section 2 we present the Darboux method, and apply it in Section 3 to the Lotka–Volterra system. In Section 4 we apply it to the quadratic systems after reduction to a canonical form already used by two of the authors in the Painlevé analysis of such systems [9]. In Section 5 we give our conclusion.

2. The Darboux method

Although the method can be extended to more than 2 dimensions we present it for the planar system (1) where P and Q are polynomials in x and y .

As has been said in the Introduction, the Darboux method is based on the existence of functions $f_i(x, y)$ which are in fact *algebraic solutions* of the differential system. Suppose that we can determine two polynomials $f_i(x, y)$ and $K_i(x, y)$ such that

$$\frac{\partial f_i}{\partial x}P + \frac{\partial f_i}{\partial y}Q = K_i f_i. \quad (3)$$

Then equation $f_i(x, y) = 0$ describes an algebraic curve which is formed by trajectories.

These f_i are going to be the “bricks” with which we will build the invariants. Suppose that we have obtained q functions f_i . The polynomial K_i is called the *cofactor* of f_i . We look for an invariant of the form

$$I = \prod_{i=1}^q f_i^{\mu_i}(x, y) \exp(st). \quad (4)$$

We obtain

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q = I \left[s + \sum_{i=1}^q \frac{\mu_i}{f_i} \left(\frac{\partial f_i}{\partial x}P + \frac{\partial f_i}{\partial y}Q \right) \right]. \quad (5)$$

Taking into account (3) and imposing that I is an invariant, we obtain

$$s + \sum_{i=1}^q \mu_i K_i = 0. \quad (6)$$

The equations for μ_i are now linear equations. How many f_i do we need? The equation (3) shows that if the system is of degree m , K_i is at most of degree $m-1$ independently of the degree of f_i . The left-hand-side of (6) is consequently a polynomial in x, y of degree at most $m-1$ with a total of $m(m+1)/2$ terms. Consequently (6) produces $m(m+1)/2$ equations where the unknowns are the μ_i . We see from (6) that if we allow s to be different from zero the system of equations for the μ_i is inhomogeneous, and we need $q = m(m+1)/2$. If we want a first integral then s must be zero and the system becomes homogeneous and we need a priori either a new f_i , or a new condition by imposing that the determinant of the system of equations for the μ_i is zero. We will call λ_i the solution of this system.

In fact it is not impossible to find cases where q is smaller than $m(m+1)/2$ with μ_i such that $\sum \mu_i K_i + s \equiv 0$. We see that the possibility of solving for μ_i in equation (6) depends on the conditions we put on the coefficients of (1). We should not forget that some conditions come from equation (6) and other ones come from equation (3).

Another possibility indicated also by Darboux is to obtain an integrating factor as follows. We seek an integrating factor of the form

$$R = \prod_{i=1}^q f_i^{\lambda_i}. \quad (7)$$

For R to be an integrating factor we must have

$$\operatorname{div}(RP, RQ) = \frac{\partial}{\partial x}(RP) + \frac{\partial}{\partial y}(RQ) = 0. \quad (8)$$

We have

$$\frac{\partial R}{\partial x} = R \sum_i \frac{\lambda_i}{f_i} \frac{\partial f_i}{\partial x}, \quad \frac{\partial R}{\partial y} = R \sum_i \frac{\lambda_i}{f_i} \frac{\partial f_i}{\partial y}, \quad (9)$$

and

$$\operatorname{div}(RP, RQ) = R \operatorname{div}(P, Q) + R \sum_{i=1}^q \frac{\lambda_i}{f_i} \left(P \frac{\partial f_i}{\partial x} + Q \frac{\partial f_i}{\partial y} \right). \quad (10)$$

Taking (3) into account we obtain

$$\operatorname{div}(RP, RQ) = R \left[\operatorname{div}(P, Q) + \sum_{i=1}^q \lambda_i K_i \right] = 0. \quad (11)$$

Therefore R will be an integrating factor if we can solve for λ_i the following system

$$\sum_i^q \lambda_i K_i + \operatorname{div}(P, Q) = 0, \quad (12)$$

which is of the same type as equation (6). They differ only through the inhomogeneous terms. (12) gives $m(m+1)/2$ equations and, a priori, we need the same number of pairs f_i, K_i . However, for some usually non generic cases a smaller number can be enough.

A comment is in order about the number of conditions and its variation with m and the order of the algebraic solutions tested. For the existence of algebraic solutions of high order we require larger number of conditions than for curves of smaller order. Since the number of algebraic solutions needed increases quickly with m , we see that the most interesting cases (i.e. with a not too high number of conditions) will be obtained using straight lines and conics as algebraic solutions for Lotka–Volterra and quadratic systems. This conclusion was already found using the Carleman or Hamiltonian method.

3. Lotka–Volterra system

The bidimensional LVS writes

$$\begin{aligned} \dot{x} &= x(a_1 + b_{11}x + b_{12}y), \\ \dot{y} &= y(a_2 + b_{21}x + b_{22}y), \end{aligned} \quad (13)$$

We begin with straight lines as algebraic solutions i.e.

$$f(x, y) = f_{00} + f_{10}x + f_{01}y, \quad K(x, y) = K_{00} + K_{10}x + K_{01}y. \quad (14)$$

Two first straight lines are obvious

$$f(x, y) = x, \quad K(x, y) = a_1 + b_{11}x + b_{12}y, \quad (15)$$

$$f(x, y) = y, \quad K(x, y) = a_2 + b_{21}x + b_{22}y. \quad (16)$$

We remark that these two straight lines do not impose any condition. The interest of using these straight lines was noted by Cairó and Feix who were seeking invariants of the form

$$x^\alpha y^\beta P(x, y) \exp(st),$$

$f(x, y)$	$K(x, y)$	Condition	n^0
x	$a_1 + b_{11}x + b_{12}y$	—	1
y	$a_2 + b_{21}x + b_{22}y$	—	2
$a_1 + b_{11}x$	$b_{11}x$	$b_{12} = 0$	3
$a_2 + b_{22}y$	$b_{22}y$	$b_{21} = 0$	4
$(b_{21} - b_{11})x + (b_{22} - b_{12})y$	$a_1 + b_{11}x + b_{22}y$	$a_1 = a_2$	5
$a_1a_2 + b_{11}x + b_{22}y$	$b_{11}x + b_{22}y$	$r_{12} = 0$	6

Table 1: Algebraic solutions and conditions for Theorem 1.

and have noted that the two terms $x^\alpha y^\beta$ were giving extra freedom. Now the fulfillment of (3) imposes either $f_{00} = 0$ or $K_{00} = 0$. We begin with the solution $K_{00} = 0$. We obtain

$$K_{10} = b_{11}, \quad K_{01} = b_{22}, \quad f(x, y) = f_{00} \left(1 + \frac{b_{11}}{a_1}x + \frac{b_{22}}{a_2}y \right), \quad (17)$$

but we need one condition namely

$$r_{12} = \frac{b_{11}}{a_1} (b_{12} - b_{22}) + \frac{b_{22}}{a_2} (b_{21} - b_{11}) = 0, \quad (18)$$

found in [1] to obtain the invariant of type III (in fact a first integral).

Note that the straight line $f(x, y) = 0$ with f given by (17) joins the two equilibrium points $-a_1/b_{11}, 0$ and $0, -a_2/b_{22}$ while the cofactor straight line $b_{11}x + b_{22}y = 0$ joins the origin to the equilibrium point outside the axes.

Another invariant straight line is obtained by taking $f_{00} = 0$. It is easily proved that we need one condition namely $a_1 = a_2 = a$, that $K_{10} = b_{11}, K_{01} = b_{22}$ and $K_{00} = a$. This condition is needed to obtain the invariant of type II.

Two other possible linear algebraic solutions are $f(x, y) = x + a_1/b_{11}$ with $b_{12} = 0$ as condition, and $f(x, y) = y + a_2/b_{22}$ with $b_{21} = 0$ as condition. Table 1 recapitulates the 6 possible invariant straight lines.

We see that to build an invariant with straight lines we must include first the straight lines 1 and 2 which do not bring any condition. If we add the invariant straight line 5 we recover the invariant II of Cairó and Feix, and if we add the straight line 6 we recover invariant III which is in fact a first integral. Straight lines 3 and 4 do not bring interesting

invariants since the associated invariant concerns the species fully decoupled from the other (we have either b_{12} or b_{21} equal to zero). Finally we try to build an invariant with no condition, and for that we select the two straight lines $f = x$ and $f = y$. If μ_1 and μ_2 are the respective exponents, we obtain, taking into account the expression of the cofactors

$$\begin{aligned}\mu_1 a_1 + \mu_2 a_2 + s &= 0, \\ b_{11} \mu_1 + b_{21} \mu_2 &= 0, \\ b_{12} \mu_1 + b_{22} \mu_2 &= 0.\end{aligned}\tag{19}$$

Since s is at our disposal we can ignore the first equation, but the two others being homogeneous will have a nontrivial solution only if $\Delta = b_{11}b_{22} - b_{12}b_{21} = 0$. This is the invariant of type I of [1]. The LVS need at least one condition to possess an invariant. So the following theorem can be established.

Theorem 1. *Let $f_1 = x = 0$ and $f_2 = y = 0$ be the two trivial algebraic solutions of a LVS. Then the following statement holds*

(a) *If $r_{12} = 0$ the expression $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor of the LVS with $\lambda_1 = [(a_2 - a_1)b_{22} + a_2 b_{12}]/(a_1 b_{22} - a_2 b_{12})$ and $\lambda_2 = (2a_1 b_{22} - a_2 b_{12})/(a_1 b_{22} - a_2 b_{12})$. If the condition is $\Delta = 0$, then the expression $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant of the LVS provided that μ_1 and μ_2 be a solution of the system (19).*

If a LVS has a third algebraic solution $f_3 = 0$ of degree 1, then modulus the symmetry $(x, y, b_{11}, b_{12}, a_1, b_{21}, b_{22}, a_2) \rightarrow (y, x, b_{22}, b_{21}, a_2, b_{12}, b_{11}, a_1)$ and provided that the LVS satisfies one of the conditions indicated in Table 1, there exist a Darboux invariant $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ and an integrating factor $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ having the following values for f_3 and the exponents λ_i, μ_i :

(b) *If $a_1 b_{11} b_{22} \neq 0$, and the condition 3 is satisfied, then*

$$\begin{aligned}f_3 &= b_{11}x + a_1, \\ \mu_1 &= -s/a_1, \quad \mu_2 = 0, \quad \mu_3 = s/a_1, \\ \lambda_1 &= (a_2 - a_1)b_{11}, \quad \lambda_2 = -2a_1 b_{11}, \quad \lambda_3 = a_1(b_{21} - b_{11}) - a_2 b_{11}.\end{aligned}$$

(c) *If $a_1(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, and the condition 5 is satisfied, then*

$$\begin{aligned}f_3 &= (b_{21} - b_{11})x + (b_{22} - b_{12})y, \\ \mu_1 &= s \frac{b_{22}}{a_1(b_{12} - b_{22})}, \quad \mu_2 = s \frac{b_{11}}{a_1(b_{21} - b_{11})}, \quad \mu_3 = s \frac{b_{11}b_{22} - b_{12}b_{21}}{a_1(b_{12} - b_{22})(b_{21} - b_{11})}, \\ \lambda_1 &= b_{12}(b_{21} - b_{11}), \quad \lambda_2 = b_{21}(b_{12} - b_{22}), \quad \lambda_3 = -[b_{11}(b_{12} - b_{22}) + b_{22}(b_{21} - b_{11})].\end{aligned}$$

(d) If $a_2^2 b_{11}^2 + a_1^2 b_{22}^2 \neq 0$, and the condition 6 is satisfied, then there exist a first integral $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3}$ with

$$f_3 = 1 + \frac{b_{11}}{a_1}x + \frac{b_{22}}{a_2}y,$$

$$\mu_1 = b_{22}(b_{21} - b_{11}), \quad \mu_2 = b_{11}(b_{12} - b_{22}), \quad \mu_3 = b_{11}b_{22} - b_{12}b_{21}.$$

Next we turn to quadratic algebraic solutions $f(x, y) = 0$, which we write

$$f(x, y) = f_{00} + f_{10}x + f_{01}y + f_{20}x^2 + f_{11}xy + f_{02}y^2.$$

The cofactor is of the form $K(x, y) = K_{00} + K_{10}x + K_{01}y$ but K_{00} must be zero if $f_{00} \neq 0$. The identification of the terms in x^3 and y^3 give

$$f_{20}K_{10} = 2f_{20}b_{11}, \quad f_{02}K_{01} = 2f_{02}b_{22},$$

and if $f_{20}f_{02} \neq 0$ (we will come back to this possibility later on), we get

$$K_{10} = 2b_{11}, \quad K_{01} = 2b_{22}. \quad (20)$$

Taking $f_{00} = 1$, the identification of terms in x and y gives

$$f_{10} = \frac{2b_{11}}{a_1}, \quad f_{01} = \frac{2b_{22}}{a_2}. \quad (21)$$

Identification of terms in x^2 and y^2 gives taking (20) and (21) into account

$$f_{20} = \frac{b_{11}^2}{a_1^2}, \quad f_{02} = \frac{b_{22}^2}{a_2^2}. \quad (22)$$

All quantities except f_{11} are known and we are left with 3 equations from the identification of terms in xy , x^2y and xy^2 . Consequently we will need two conditions. After some calculations we get

$$f_{11} = -2\frac{b_{11}b_{22}}{a_1a_2},$$

with the two conditions

$$b_{12} = b_{22}\left(2 + \frac{a_1}{a_2}\right), \quad b_{21} = b_{11}\left(2 + \frac{a_2}{a_1}\right). \quad (23)$$

The conic writes

$$f(x, y) = \left(1 + \frac{b_{11}}{a_1}x + \frac{b_{22}}{a_2}y\right)^2 - 4\frac{b_{11}b_{22}}{a_1a_2}xy,$$

x	y	x^3	y^3	
$f_{10} = 2b_{11}/a_1$	$f_{01} = K_{01}/a_2$	$K_{10} = 2b_{11}$	$f_{02} = 0$	
x^2	y^2	xy^2	x^2y	xy
$f_{20} = b_{11}^2/a_1^2$	$K_{01} = b_{22}$	$f_{11} = 0$	$b_{22} = 2b_{12}$	$b_{21} = b_{11}(2 + a_2/a_1)$

Table 2: Consequences of the identification of the different terms of (3)

and the associated cofactor writes

$$K(x, y) = 2b_{11}x + 2b_{22}y.$$

It must be realized that the presence of two conditions as given by (23) leaves a one parameter LVS equation. Indeed it is easily proved that (13) can be written as a 3 parameter system. Introducing $X = b_{11}x/a_1$, $Y = b_{22}y/a_2$, $C_1 = b_{12}a_2/b_{22}a_1$, $C_2 = b_{21}a_1/b_{11}a_2$, (13) writes

$$\frac{dX}{a_1 dt} = X(1 + X + C_1 Y), \quad \frac{dY}{a_2 dt} = Y(1 + C_2 X + Y), \quad (24)$$

where

$$C_1 = 1 + 2\frac{a_2}{a_1}, \quad C_2 = 1 + 2\frac{a_1}{a_2},$$

and, indeed, writing the system (24) in the form dY/dX , we are left with one relevant parameter, (a_1/a_2) .

We briefly discuss the other possibilities. The identification of the coefficients of the terms in x^3 and y^3 can also be obtained by taking f_{20} (for x^3) and f_{02} (for y^3) equal to zero. Identifying both leads to an uninteresting case: either $f_{11} = 0$ or $b_{12} = b_{21} = 0$. Breaking the symmetry we take $f_{02} = 0$. Table 2 gives the result of the identification of the different terms. We recover again two conditions

$$b_{22} = 2b_{12}, \quad b_{21} = b_{11} \left(2 + \frac{a_2}{a_1} \right).$$

One is identical to the previous case, the other is different. However, there is still another case for invariant conics of type (19). The identification of the y^2 term gives $f_{01}b_{22} = K_{01}f_{01}$ and assuming $f_{01} \neq 0$ we deduce $K_{01} = b_{22}$. Finally another solution is possible with $f_{01} = 0$. Since (4th column in Table 2) $f_{01} = K_{01}/a_2$, we must take $K_{01} = 0$ and the table is modified. The identification of the term in xy^2 gives

$$b_{12} + b_{22} = 0,$$

which is the first condition. The next term x^2y gives f_{11} and the term in xy gives the second condition

$$b_{21} = b_{11} \left(2 + \frac{a_2}{a_1} \right),$$

the same as in the previous two cases.

We must now consider a nonzero constant term in the cofactor with, consequently, no constant term in f with

$$\begin{aligned} K(x, y) &= K_{00} + K_{10}x + K_{01}y, \\ f(x, y) &= f_{10}x + f_{01}y + f_{20}x^2 + f_{11}xy + f_{02}y^2. \end{aligned}$$

The terms in x^3 and y^3 provide the usual results $K_{10} = 2b_{11}$, $K_{01} = 2b_{22}$ with f_{20} and f_{02} different from zero. Then the terms in x and y give $f_{10}K_{00} = f_{10}a_1$ and $f_{01}K_{00} = f_{01}a_2$ with a first condition $a_1 = a_2$ (assuming $f_{10} \neq 0$ and $f_{01} \neq 0$). However, we have found that this condition is sufficient to provide the invariant II. The existence of a conic as an invariant curve implies a second condition. Obtaining an invariant with this conic and the two straight lines $x = 0, y = 0$ does not present any interest since we can get an invariant with only the first condition. However, could it be that we recover a first integral? Pushing the computation we obtain for the conic the following relations (in addition to $a_1 = a_2 = a$)

$$\begin{aligned} f_{10} &= (a/b_{11})f_{20}, \quad f_{01} = (a/b_{22})f_{02}, \\ f_{20} &= (b_{21} - b_{11})/[2(b_{22} - b_{12})]f_{11}, \\ f_{02} &= (b_{12} - b_{22})/[2(b_{11} - b_{21})]f_{11}. \end{aligned}$$

The second condition comes from the identification of terms in x, y . We get in addition to $r_{12} = 0$, the relation

$$3b_{11}b_{22} + b_{12}b_{21} - 2b_{11}b_{12} - 2b_{22}b_{21} = 0, \tag{25}$$

which is the interesting one. Now the 3 cofactors are

$$a + b_{11}x + b_{12}y, \quad a + b_{21}x + b_{22}y, \quad a + 2b_{11}x + 2b_{22}y$$

and we obtain a first integral if the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ b_{11} & b_{21} & 2b_{11} \\ b_{12} & b_{22} & 2b_{22} \end{pmatrix}$$

statement	condition 1	condition 2
a	$b_{22} - 2b_{12} = 0$	$(2a_1 + a_2)b_{11} - a_1b_{21} = 0$
b	$b_{12} + b_{22} = 0$	$(2a_1 + a_2)b_{11} - a_1b_{21} = 0$
c	$(a_1 + 2a_2)b_{22} - b_{12}a_2 = 0$	$(2a_1 + a_2)b_{11} - a_1b_{21} = 0$
d	$a_1 = 2a_2$	$b_{21}(2b_{22} - b_{12}) - b_{11}(3b_{22} - 2b_{12}) = 0$
e	$a_1 = a_2$	$b_{21}(2b_{22} - b_{12}) - b_{11}(3b_{22} - 2b_{12}) = 0$

Table 3: Invariant conditions of Theorem 2.

cancels, but this gives again (25) which together with $a_1 = a_2$ provide the two conditions needed to obtain a first integral. If we assume that one of the coefficients, say f_{01} , is zero, then $K_{00} = a_1$ and the cancellation of the terms in x^2 and y^2 gives respectively

$$a_1 f_{20} = b_{11} f_{10} \quad \text{and} \quad a_1 = 2a_2. \quad (26)$$

Finally looking at the terms in xy^2 , x^2y and xy we obtain

$$f_{11} = 2f_{02}(b_{11} - b_{21})/(b_{12} - b_{22}), \quad (27)$$

$$f_{11} = 2f_{20}(b_{22} - b_{12})/(b_{21} - b_{11}), \quad (28)$$

$$f_{11} = (2b_{22} - b_{12})f_{10}/a_2. \quad (29)$$

Eqs. (28) and (29) and the first of (26) give the second condition, the first being $a_1 = 2a_2$,

$$b_{21}(2b_{22} - b_{12}) - b_{11}(3b_{22} - 2b_{12}) = 0.$$

The following theorem recapitulates the five cases obtained with a conic as algebraic solution in addition to the two straight lines $f_1 = x$, $f_2 = y$. We note that a trivial interchange between x and y , a_1 and a_2 , b_{11} and b_{22} , b_{12} and b_{21} , give three other invariants in cases (a), (b) and (d) while cases (c) and (e) are invariant under this interchange.

Figure 1

Theorem 2. Let $f_1 = x = 0$ and $f_2 = y = 0$. If a LVS has a third algebraic solution $f_3 = 0$ of degree 2, then modulus the symmetry $(x, y, b_{11}, b_{12}, a_1, b_{21}, b_{22}, a_2) \rightarrow (y, x, b_{22}, b_{21}, a_2, b_{12}, b_{11}, a_1)$, and under the conditions given in Table 3, a Darboux invariant $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ and an integrating factor $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ can be found having the following values for f_3 and the exponents μ_i, λ_i :

(a) If $a_1 a_2 b_{11} \neq 0$, then

$$\begin{aligned} f_3 &= \left(1 + \frac{b_{11}}{a_1} x\right)^2 + \frac{b_{22}}{a_2} y, \\ \mu_1 &= -s/a_1, \quad \mu_2 = 0, \quad \mu_3 = s/2a_1, \\ \lambda_1 &= -2, \quad \lambda_2 = (a_1 - a_2)/a_2, \quad \lambda_3 = -(2a_1 + a_2)/(2a_2). \end{aligned}$$

(b) If $a_1(a_1 + a_2)b_{11} \neq 0$, then

$$\begin{aligned} f_3 &= \left(1 + \frac{b_{11}}{a_1} x\right)^2 + 2 \frac{b_{11} b_{22}}{a_1(a_1 + a_2)} xy, \\ \mu_1 &= \mu_2 = -s/(a_1 + a_2), \quad \mu_3 = (3a_1 + a_2)s/2a_1(a_1 + a_2), \\ \lambda_1 &= -a_1/(a_1 + a_2), \quad \lambda_2 = -(2a_1 + a_2)/(a_1 + a_2), \quad \lambda_3 = (a_1 - a_2)/[2(a_1 + a_2)]. \end{aligned}$$

(c) If $a_1 a_2 b_{11} \neq 0$, then

$$\begin{aligned} f_3 &= \left(1 + \frac{b_{11}}{a_1} x + \frac{b_{22}}{a_2} y\right)^2 - 4 \frac{b_{11} b_{22}}{a_1 a_2} xy, \\ \mu_1 &= -s/(2a_1), \quad \mu_2 = -s/(2a_2), \quad \mu_3 = (a_1 + a_2)s/(2a_1 a_2), \\ \lambda_1 &= -1, \quad \lambda_2 = -1, \quad \lambda_3 = -1/2. \end{aligned}$$

(d) If $a_1 b_{11}(2b_{22} - b_{12}) \neq 0$, then

$$\begin{aligned} f_3 &= a_1 b_{11} x + [b_{11} x + (2b_{22} - b_{12}) y]^2, \\ \mu_1 &= 2b_{22}s/a_1(b_{12} - 2b_{22}), \quad \mu_2 = 2s/a_1, \quad \mu_3 = 2(b_{22} - b_{12})s/a_1(b_{12} - 2b_{22}), \\ \lambda_1 &= (b_{22} - b_{12})/(b_{12} - 2b_{22}), \quad \lambda_2 = -2, \quad \lambda_3 = b_{12}/[2(b_{12} - 2b_{22})]. \end{aligned}$$

(e) Finally, a first integral $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3}$ exists if $a_1 b_{11} b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$. Then

$$\begin{aligned} f_3 &= a_1 [b_{22}(b_{21} - b_{11})^2 x + b_{11}(b_{12} - b_{22})^2 y] + b_{11} b_{22} [(b_{21} - b_{11})x - (b_{12} - b_{22})y]^2, \\ \mu_1 &= b_{21} - 2b_{11}, \quad \mu_2 = b_{11}, \quad \mu_3 = b_{11} - b_{21}. \end{aligned}$$

The qualitative picture of the phase portrait at finite distance from the origin for the 4 first statements (quadratic invariants) is given in Figure 1.

4. The complex quadratic system

The LVS are a particular case of the *quadratic systems*. Its most general form is

$$\begin{aligned}\dot{x} &= \frac{dx}{dt} = k_1 + a_{11}x + a_{12}y + b_{11}x^2 + b_{12}xy + c_1y^2, \\ \dot{y} &= \frac{dy}{dt} = k_2 + a_{21}x + a_{22}y + b_{21}xy + b_{22}y^2 + c_2x^2.\end{aligned}\tag{30}$$

where the coefficients and the variables x, y are complex, but the independent variable t is real. Here we are going to consider a canonical version of (30), which is the result first, of a translation of the origin eliminating the constant terms k_1 and k_2

$$x = \bar{x} + m_1, \quad y = \bar{y} + m_2,\tag{31}$$

and second, of a linear transformation

$$x = \alpha_{11}\bar{x} + \alpha_{12}\bar{y}, \quad y = \alpha_{21}\bar{x} + \alpha_{22}\bar{y},\tag{32}$$

to eliminate the terms c_1y^2 and c_2x^2 in (30). Since the linear transformations (31) and (32) do not introduce higher order terms, we arrive at a new system with the same form as (30) (with of course $k_1 = k_2 = 0$) but with all the terms in over-bar variables. Of interest are

$$\begin{aligned}\bar{c}_1 &= -\alpha_{12}^3 c_2 + \alpha_{12}^2 \alpha_{22} (b_{11} - b_{21}) + \alpha_{12} \alpha_{22}^2 (b_{12} - b_{22}) + \alpha_{22}^3 c_1, \\ \bar{c}_2 &= \alpha_{11}^3 c_2 - \alpha_{11}^2 \alpha_{21} (b_{11} - b_{21}) - \alpha_{11} \alpha_{21}^2 (b_{12} - b_{22}) - \alpha_{21}^3 c_1.\end{aligned}$$

Introducing

$$l = \frac{\alpha_{12}}{\alpha_{22}}, \quad m = \frac{\alpha_{11}}{\alpha_{21}},\tag{33}$$

it is easily checked that the two equations $\bar{c}_1 = 0$ and $\bar{c}_2 = 0$ coincide and write

$$c_2 X^3 + (b_{21} - b_{11})X^2 + (b_{22} - b_{12})X - c_1 = 0.\tag{34}$$

In (34) we must take for l and m two different roots, otherwise $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 0$ and the linear transformation becomes degenerate. The transformation is non-degenerate if we take different roots among the three roots of (34) for l and m . Note at this point that when two roots are complex conjugate, the resulting complex quadratic system has complex coefficients, without modifying the generality of our subsequent study. Two coefficients (for example α_{11} and α_{22}) are still arbitrary and one could hope to be chosen in order to diagonalize the linear terms. If that were possible, we could reduce the quadratic system to the LVS. To see the impossibility we use (32) to diagonalize the linear terms. We obtain

a similar result, namely, that l and m , as given by (33), are the two different roots of the second degree equation

$$a_{21}X^2 + (a_{22} - a_{11})X - a_{12} = 0. \quad (35)$$

Of course the same l and m cannot be simultaneously the roots of (34) and (35). Excluding the case of a double root we can say that the quadratic system becomes, without loss of generality

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y + b_{11}x^2 + b_{12}xy, \\ \dot{y} &= a_{21}x + a_{22}y + b_{21}xy + b_{22}y^2, \end{aligned} \quad (36)$$

where we have, once again, dropped the over-bar on the variables. The only – but essential – difference with the LVS is the presence of the off-diagonal terms $a_{12}y$ and $a_{21}x$. It is clear that the above transformation, can in general, generate complex coefficients. For this reason system (36) is called *complex quadratic system*, which in short will be called here QS.

The search for invariant straight lines is quite similar to the one given above for the LVS. To begin with we look for a (complex) straight line going through the origin, i.e.

$$f(x, y) = f_{10}x + f_{01}y, \quad K(x, y) = K_{00} + K_{10}x + K_{01}y,$$

where we suppose first that f_{10} and f_{01} are different from zero. The terms in x^2 and y^2 in (3) gives $K_{10} = b_{11}$ and $K_{01} = b_{22}$. The terms in xy , x and y give respectively

$$f_{10}(b_{22} - b_{12}) + f_{01}(b_{11} - b_{21}) = 0, \quad (37)$$

$$f_{10}(a_{11} - K_{00}) + f_{01}a_{21} = 0, \quad f_{10}a_{12} + f_{01}(a_{22} - K_{00}) = 0. \quad (38)$$

If we suppose $b_{21} \neq b_{11}$ and $b_{12} \neq b_{22}$, we deduce f_{10}/f_{01} from (37) and introducing it in (38) we obtain K_{00} and a condition, the values of which are given below in (5) of Proposition 3. A particularly interesting case is obtained when $b_{12} = b_{22}$ and $b_{21} = b_{11}$. Then (37) is automatically fulfilled and the two equations (38) have non trivial solution only in the case of K_{00} being solution of the equation

$$K_{00}^2 - (a_{11} + a_{22})K_{00} + \Delta_a = 0, \quad (39)$$

where $\Delta_a = a_{11}a_{22} - a_{12}a_{21}$ and the two roots of K_{00} are the eigenvalues of the matrix (a)

$$(a) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (40)$$

The precise values are given below in (5') and (5'') of Proposition 3.

The other general case is obtained with

$$f(x, y) = f_{00} + f_{10}x + f_{01}y, \quad K(x, y) = K_{10}x + K_{01}y.$$

Again the terms in x^2 and y^2 give $K_{10} = b_{11}$ and $K_{01} = b_{22}$ (assuming $f_{00}, f_{10}, f_{01} \neq 0$). The terms in x and y give

$$f_{10}a_{11} + f_{01}a_{21} = f_{00}b_{11}, \quad f_{10}a_{12} + f_{01}a_{22} = f_{00}b_{22}. \quad (41)$$

The solution of (41) give f_{10} and f_{01} which introduced in (37) provides a condition (case (6) of Proposition 3).

As in the LVS we must also consider cases where one of f_{10} or f_{01} is equal to zero. In fact we recuperate the two axes and the two parallel to the axes given in Table 1 (under the numbers 1, 2, 3 and 4) with conditions differently written. Now these straight lines and especially 3 and 4, are more interesting. In the LVS two conditions were needed ($b_{12} = b_{21} = 0$), resulting in a decoupling of the two equations, while now, it is a relation between four coefficients (a_{12}, a_{11}, b_{12} and b_{11}) or (a_{21}, a_{22}, b_{21} and b_{22}) which are needed. Proposition 3 recapitulates the 8 different invariant straight lines, their cofactors and the conditions for their existence.

Proposition 3. *A QS has a linear algebraic solution $f = 0$ with cofactor K in the following cases*

- (1) *If $a_{12} = 0$ then $f = x$ and $K = a_{11} + b_{11}x + b_{12}y$.*
- (2) *If $a_{21} = 0$ then $f = y$ and $K = a_{22} + b_{21}x + b_{22}y$.*
- (3) *If $a_{11}b_{11} \neq 0$ and $a_{12} = a_{11}b_{12}/b_{11}$ then $f = a_{11} + b_{11}x$ and $K = b_{11}x + b_{12}y$.*
- (4) *If $a_{22}b_{22} \neq 0$ and $a_{21} = a_{22}b_{21}/b_{22}$ then $f = a_{22} + b_{22}y$ and $K = b_{21}x + b_{22}y$.*
- (5) *If $(b_{21} - b_{11})(b_{12} - b_{22}) \neq 0$ and $r_1 \equiv (b_{12} - b_{22})(b_{21} - b_{11})(a_{11} - a_{22}) + a_{12}(b_{21} - b_{11})^2 - a_{21}(b_{12} - b_{22})^2 = 0$ then $f = (b_{21} - b_{11})x + (b_{22} - b_{12})y$ and $K = a_{11} - a_{21}(b_{12} - b_{22})/(b_{21} - b_{11}) + b_{11}x + b_{22}y$.*
- (5') *If $b_{12} = b_{22}$ and $b_{21} = b_{11}$ then $f = a_{21}x + (K_{00}^{[1]} - a_{11})y$ and $K = K_{00}^{[1]} + b_{11}x + b_{22}y$, where $K_{00}^{[1]}$ is the first root of (39), which writes $(a_{11} + a_{22})/2 + [(a_{11} - a_{22})^2 + 4a_{12}a_{21}]^{1/2}/2$*
- (5'') *If $b_{12} = b_{22}$ and $b_{21} = b_{11}$ then $f = a_{12}y + (K_{00}^{[2]} - a_{22})x$ and $K = K_{00}^{[2]} + b_{11}x + b_{22}y$, where $K_{00}^{[2]}$ is the second root of (39), which writes $(a_{11} + a_{22})/2 - [(a_{11} - a_{22})^2 + 4a_{12}a_{21}]^{1/2}/2$*

(6) If $b_{11}b_{22}(b_{21} - b_{11})(b_{12} - b_{22})(a_{11}a_{22} - a_{12}a_{21}) \neq 0$ and $r_2 \equiv (a_{12}b_{11} - a_{11}b_{22})(b_{21} - b_{11}) + (a_{21}b_{22} - a_{22}b_{11})(b_{12} - b_{22}) = 0$ then $f = a_{11}a_{22} - a_{12}a_{21} + (a_{22}b_{11} - a_{21}b_{22})x + (a_{11}b_{22} - a_{12}b_{11})y$ and $K = b_{11}x + b_{22}y$.

A comment is in order about the two invariant straight lines (5') and (5''). They do not contain any b_{ij} and are the support of the two eigenvectors of the matrix (a) . Moreover it is easily shown that in this case ($b_{12} = b_{22}, b_{21} = b_{11}$) we have only three equilibrium points at finite distance from the origin: the origin, one on (5') and the last one on (5''). Moreover the invariant straight line (6) goes through these points.

Now with these eight "bricks" we are going to build the invariants, first integrals and integrating factors. Let us begin with the obtention of Darboux invariants. Getting q invariant straight lines labeled with i going from 1 to q , we obtain the identity

$$\sum_{i=1}^q \mu_i (K_{00i} + K_{10i}x + K_{01i}y) + s = 0,$$

i.e. two homogeneous linear equations in μ_i

$$\sum_{i=1}^q K_{10i}\mu_i = 0, \quad \sum_{i=1}^q K_{01i}\mu_i = 0, \quad (42)$$

and the inhomogeneous equation

$$\sum_{i=1}^q K_{00i}\mu_i + s = 0. \quad (43)$$

Since s is at our disposal, let us leave for a moment (43) which will simply fix s . For the two equations (42) to be fulfilled we need either 3 straight lines (with 3 μ_i) or 2 straight lines, but then the determinant of the system must be zero. Since one straight line implies one condition we need in principle 3 conditions. However, one can find exceptional cases with 2 straight lines having a determinant automatically equal to zero. For example the cofactors of straight lines (5) and (6) have the same K_{10i} and K_{01i} (respectively b_{11} and b_{22}) and we get an invariant with only 2 straight lines and 2 conditions.

Concerning the integrating factor, the things are somewhat similar except that now s is replaced by $\text{div}(P, Q)$, with $\text{div}(P, Q) = A_0 + A_1x + A_2y$. We have now 3 inhomogeneous equations with 3 conditions.

A priori the most expensive in conditions should be the first integral. Since we get (42) and (43) and this time with $s = 0$, we have a system of 3 homogeneous linear equations.

The identification of the determinant plus the obtention of the 3 straight lines gives 4 conditions, but again special cases can be found. For example straight lines (5'), (5'') and (6) altogether, ask for only 2 conditions ($b_{12} = b_{22}$ and $b_{21} = b_{11}$). Moreover they all possess the same $K_{10i} = b_{11}$ and $K_{01i} = b_{22}$ in their 3 cofactors and we get a first integral with only 2 conditions. Consequently each case must be separately examined.

In the rest of the paper we use the following notation: If a first integral, an integrating factor or a Darboux invariant of a complex QS has been obtained using for instance the algebraic solutions of statements (5) and (6) of Proposition 3, we only say that the integrable case is (56), etc. We present three types of results

(A) *Invariant (56).*

It is the invariant built with straight lines (5) and (6). The coexistence of straight lines (5) and (6) implies $r_1 = r_2 = 0$ and a first solution is

$$a_{12} = a_{22}(b_{22} - b_{12})/(b_{11} - b_{21}), \quad a_{21} = a_{11}(b_{11} - b_{21})/(b_{22} - b_{12}), \quad (44)$$

but it is easily checked that under conditions (44), the straight lines (5) and (6) are identical (the straight line (6) passing also through the origin) with, as a consequence, not enough straight lines to build an invariant. Fortunately another solution is possible with

$$b_{11}(b_{22} - b_{12}) + b_{22}(b_{11} - b_{21}) = 0, \quad (45)$$

$$(a_{11} - a_{22})b_{11}b_{22} - a_{12}b_{11}^2 + a_{21}b_{22}^2 = 0. \quad (46)$$

The cofactors being respectively $b_{11}x + b_{22}y + K_{00} = 0$ for (5) and $b_{11}x + b_{22}y = 0$ for (6), the equations write

$$b_{11}(\mu_1 + \mu_2) = 0, \quad b_{22}(\mu_1 + \mu_2) = 0, \quad K_{00}\mu_1 + s = 0.$$

Taking $\mu_2 = -\mu_1$, we get an invariant with only 2 conditions: (45) and (46). Now this result must be compared with the 2 conditions given for that system in [9]. For the passing of the Painlevé test as given in [9], (45) is the index condition (4.31) and (46) is the compatibility condition (4.33). Indeed with, for example, $\mu_1 = 1$ and $\mu_2 = -1$ we obtain an analytical invariant in agreement with the conclusion obtained in [9].

(B) *Invariant (5'5'') and (5'5''6).*

In the case where the invariant is built with (5') and (5'') or (5'), (5'') and (6) we have the two conditions $b_{21} = b_{11}$, $b_{12} = b_{22}$ and the corresponding cofactors are $b_{11}x + b_{22}y + K_{00}^{[1]}$,

$b_{11}x + b_{22}y + K_{00}^{[2]}$ and $b_{11}x + b_{22}y$, where $K_{00}^{[1]}$ and $K_{00}^{[2]}$ are the eigenvalues of matrix (a) given by (39). First we search an invariant built only with (5') and (5''). We get $\mu_1 + \mu_2 = 0, \mu_1 K_{00}^{[1]} + \mu_2 K_{00}^{[2]} + s = 0$ with, for example, $\mu_1 = 1, \mu_2 = -1$ and $s = K_{00}^{[2]} - K_{00}^{[1]}$. The invariant writes

$$I = \frac{a_{21}x + (K_{00}^{[1]} - a_{11})y}{(K_{00}^{[2]} - a_{22})x + a_{12}y} \exp\left(-\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right) t. \quad (47)$$

However, we can have a first integral in addition to (47) without introducing new conditions. For that we build with (5'), (5'') and (6) obtaining $\mu_1 + \mu_2 + \mu_3 = 0$ and $\mu_1 K_{00}^{[1]} + \mu_2 K_{00}^{[2]} + s = 0$. The interesting point is that we get twice the equation $\mu_1 + \mu_2 + \mu_3 = 0$. Now taking $\mu_1 = K_{00}^{[2]}$ and $\mu_2 = -K_{00}^{[1]}$ we obtain $s = 0$ (and consequently a first integral). Having a first integral and an invariant the problem is now fully algebraic and, consequently, this system can be completely solved and the integration is as complete as possible. Note, however that this case does not pass the Painlevé test a clear indication that passing the Painlevé test and getting explicit invariant, first integral or explicit solutions are two different approaches not always connected (although correlated in many cases).

(C) Other cases.

We exclude cases treated above built with both straight lines (5) and (6) and cases where $b_{12} = b_{22}, b_{21} = b_{11}$. Also we omit those invariants involving only one variable obtained when one equation is decoupled from the other one. As a result, 13 invariants are obtained: 8 build with 3 straight lines, 4 with 2 straight lines and 1 with 1 line. We denote the invariants with the names of the straight lines upon which they have been built. With 3 straight lines we have

(125), (236), (345), (146), which are the generalization of invariant II of the LVS. (125) is LVS ($a_{12} = 0, a_{21} = 0$) with the 3 straight lines passing through one of the equilibrium point.

(126), (235), (346), (145), which are the generalization of invariant III of the LVS. (126) is LVS with 3 straight lines forming a triangle.

With 2 straight lines, we have

(12), (14), (23), (34), which are the generalization of the invariant I of the LVS ((12) is LVS). These 4 cases require the third condition $\Delta = 0$.

A last case can be found with only one line. Obviously the cofactor must be a constant and the only interesting case is given by the straight line (5) with $b_{11} = b_{22} = 0$. Taking

$f = f_{10}x + f_{01}y$ and $K = K_{00}$ we obtain

$$f_{10}b_{12} + f_{01}b_{21} = 0, \quad (48)$$

$$f_{10}(a_{11} - K_{00}) + f_{01}a_{21}, \quad f_{10}a_{12} + f_{01}(a_{22} - K_{00}) = 0, \quad (49)$$

and the invariant is

$$I = (b_{21}x - b_{12}y) \exp[(a_{12}b_{21} - a_{22}b_{12})t/b_{12}], \quad (50)$$

with 3 conditions

$$b_{11} = b_{22} = 0, \quad a_{11} - a_{22} = a_{21} \frac{b_{12}}{b_{21}} - a_{12} \frac{b_{21}}{b_{12}}. \quad (51)$$

which is invariant 1 in [9].

It is worth to note that in all cases (including the last one) each invariant straight line goes through two equilibrium points. We have seen also that many cases are generalization of the types I, II and III of the LVS. One of the most interesting invariants ((56) since we need only 2 conditions) splits for the LVS in both invariants II and III. The introduction of the canonical form (36) through transformations (31) and (32) has greatly simplified the problem of expliciting invariants and conditions and has shown the very fundamental connection between LVS and QS. We can establish the following theorem.

Theorem 4. *A complex QS has a first integral, an integrating factor or a Darboux invariant formed by invariant straight lines in the following cases:*

(a) *If $b_{11}(2b_{22} - b_{12}) - b_{21}b_{22} = 0$, $r_2 = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, then the straight lines $f_1 = b_{11}x + b_{22}y = 0$ and $f_2 = a_{11}a_{22} - a_{12}a_{21} + (a_{22}b_{11} - a_{21}b_{22})x + (a_{11}b_{22} - a_{12}b_{11})y = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{22}y + a_{11} + a_{21}b_{22}/b_{11}$ and $K_2 = b_{11}x + b_{22}y$ respectively and $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant for $\mu_1 = -\mu_2$ and $\mu_2 = sb_{11}/(a_{21}b_{22} + a_{11}b_{11})$. This corresponds to the integrable case (56).*

(b) *If $b_{21} = b_{11}$, $b_{12} = b_{22}$, $a_{11}a_{22}b_{11}b_{22} \neq 0$, and let us call $K_{00}^{[1]}$, $K_{00}^{[2]}$ the eigenvalues of matrix (a), then the straight lines $f_1 = a_{21}x + (K_{00}^{[1]} - a_{11})y = 0$ and $f_2 = (K_{00}^{[2]} - a_{22})x + a_{12}y = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{22}y + K_{00}^{[1]}$ and $K_2 = b_{11}x + b_{22}y + K_{00}^{[2]}$ respectively such that $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_1 = (a_{11} + a_{22})/(2K_{00}^{[1]} - 2K_{00}^{[2]}) - 3/2$ and $\lambda_2 = -(a_{11} + a_{22})/(2K_{00}^{[1]} - 2K_{00}^{[2]}) - 3/2$. Moreover $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant for $\mu_1 = -s/(K_{00}^{[1]} - K_{00}^{[2]})$ and $\mu_2 = -\mu_1$. This corresponds to the integrable case (5'5").*

(c) If $b_{21} = b_{11}$, $b_{12} = b_{22}$, $a_{11}a_{22}b_{11}b_{22} \neq 0$, and let us call $K_{00}^{[1]}$, $K_{00}^{[2]}$ the eigenvalues of matrix (a) , then the straight lines $f_1 = a_{21}x + (K_{00}^{[1]} - a_{11})y = 0$, $f_2 = (K_{00}^{[2]} - a_{22})x + a_{12}y = 0$ and $f_3 = a_{11}a_{22} - a_{12}a_{21} + (a_{22}b_{11} - a_{21}b_{22})x + (a_{11}b_{22} - a_{12}b_{11})y = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{22}y + K_{00}^{[1]}$, $K_2 = b_{11}x + b_{22}y + K_{00}^{[2]}$ and $K_3 = b_{11}x + b_{22}y$ respectively and $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3}$ is a first integral for $\mu_1 = K_{00}^{[2]}$, $\mu_2 = -K_{00}^{[1]}$ and $\mu_3 = K_{00}^{[1]} - K_{00}^{[2]}$. This corresponds to the integrable case (5'5''6).

(d) If $a_{12} = 0$, $a_{21} = a_{22}b_{21}/b_{22}$, $r_2 = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, then the straight lines $f_1 = y = 0$, $f_2 = a_{11} + b_{11}x = 0$ and $f_3 = b_{11}x + (b_{22} - b_{12})y + a_{11}a_{22} = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{21}x + b_{22}y + a_{22}$, $K_2 = b_{11}x + b_{12}y$ and $K_3 = b_{11}x + b_{22}y$ respectively such that $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor for $\lambda_1 = -(a_{11} + a_{22})/a_{22}$, $\lambda_2 = b_{12}/(b_{22} - b_{12})$ and $\lambda_3 = a_{11}/a_{22} + b_{22}/(b_{12} - b_{22})$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant for $\mu_1 = -s/a_{11}$, $\mu_2 = sb_{11}/[a_{22}(b_{21} - b_{11})]$ and $\mu_3 = s/a_{22} + sb_{12}/[a_{11}(b_{12} - b_{22})]$. This corresponds to the integrable case (236).

(e) If $a_{12} = a_{11}b_{12}/b_{11}$, $a_{21} = a_{22}b_{21}/b_{22}$, $r_1 = 0$, and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$ then there exist the straight lines $f_1 = a_{11} + b_{11}x = 0$, $f_2 = a_{22} + b_{22}y = 0$ and $f_3 = a_{22}b_{11}x - a_{11}b_{22}y = 0$ which are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{12}y$, $K_2 = b_{21}x + b_{22}y$, $K_3 = b_{11}x + b_{22}y - (b_{12} - b_{22})a_{22}/b_{22}$ respectively, such that $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor for $\lambda_1 = a_{22}b_{11}/[a_{11}(b_{11} - b_{21})] - 1$, $\lambda_2 = b_{21}/(b_{11} - b_{21})$ and $\lambda_3 = b_{11}(a_{11} + a_{22})/[a_{11}(b_{21} - b_{11})]$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of the QS for $\mu_1 = -s/a_{22}$, $\mu_2 = sb_{22}/[a_{11}(b_{22} - b_{12})]$ and $\mu_3 = s/a_{22} + sb_{12}/[a_{11}(b_{12} - b_{22})]$. This corresponds to the integrable case (345).

(f) If $a_{12} = 0$, $a_{21} = a_{22}b_{21}/b_{22}$, $r_2 = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, then the straight lines $f_1 = x = 0$, $f_2 = a_{22} + b_{22}y = 0$ and $f_3 = (b_{11} - b_{21})x + b_{22}y + a_{11}a_{22} = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{12}y + a_{11}$, $K_2 = b_{21}x + b_{22}y + a_{22}$ and $K_3 = b_{11}x + b_{22}y + a_{11}$ respectively such that $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor for $\lambda_1 = -(a_{11} + a_{22})/a_{11}$, $\lambda_2 = b_{21}/(b_{11} - b_{21})$ and $\lambda_3 = b_{11}/(b_{21} - b_{11}) + a_{22}/a_{11}$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant for $\mu_1 = -s/a_{11}$, $\mu_2 = sb_{11}/[a_{22}(b_{21} - b_{11})]$ and $\mu_3 = sb_{21}/[a_{22}(b_{21} - b_{11})] + s/a_{11}$. This corresponds to the integrable case (146).

(g) If $a_{12} = a_{11}b_{12}/b_{11}$, $a_{21} = 0$, $r_1 = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, then the straight lines $f_1 = y = 0$, $f_2 = a_{11} + b_{11}x = 0$ and $f_3 = (b_{21} - b_{11})x - (b_{12} - b_{22})y = 0$ are algebraic solutions of the QS with cofactors $K_1 = a_{22} + b_{21}x + b_{22}y$, $K_2 = b_{11}x + b_{12}y$ and $K_3 = a_{11} + b_{11}x + b_{22}y$ respectively and $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3}$ is a first integral for $\mu_1 = b_{22}(a_{11} - a_{22})$, $\mu_2 = -a_{11}b_{12}$ and $\mu_3 = a_{22}b_{12}$. This corresponds to the integrable case (235).

(h) If $a_{11} + a_{22} = 0$, $a_{12} = a_{11}b_{12}/b_{11}$, $a_{21} = a_{22}b_{21}/b_{22}$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11})\Delta \neq 0$, then there exist the straight lines $f_1 = a_{11} + b_{11}x = 0$, $f_2 = a_{22} + b_{22}y = 0$ and $f_3 = a_{11}\Delta - b_{11}b_{22}(b_{21} - b_{11})x + b_{22}b_{11}(b_{12} - b_{22})y = 0$ which are algebraic solutions of the QS with the cofactors $K_1 = b_{11}x + b_{12}y$, $K_2 = b_{21}x + b_{22}y$, $K_3 = b_{11}x + b_{22}y$, such that $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3}$ is a first integral for $\mu_1 = b_{22}(b_{21} - b_{11})$, $\mu_2 = b_{11}(b_{12} - b_{22})$ and $\mu_3 = \Delta$. This corresponds to the integrable case (346).

(i) If $a_{12} = 0$, $a_{21} = a_{22}b_{21}/b_{22}$, $r_1 = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, then the straight lines $f_1 = x = 0$, $f_2 = a_{22} + b_{22}y = 0$ and $f_3 = (b_{21} - b_{11})x - (b_{12} - b_{22})y = 0$ are algebraic solutions of the QS with cofactors $K_1 = a_{11} + b_{11}x + b_{12}y$, $K_2 = b_{21}x + b_{22}y$ and $K_3 = a_{22} + b_{11}x + b_{22}y$ respectively and $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3}$ is a first integral for $\mu_1 = -a_{22}b_{21}$, $\mu_2 = b_{11}(a_{22} - a_{11})$ and $\mu_3 = a_{11}b_{21}$. This corresponds to the integrable case (145).

(j) If $a_{12} = 0$, $a_{21} = a_{22}b_{21}/b_{22}$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, then the straight lines $f_1 = x = 0$ and $f_2 = a_{22} + b_{22}y = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{12}y + a_{11}$ and $K_2 = b_{21}x + b_{22}y$ respectively such that with the additional condition $r_1 = 0$, $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_1 = -(a_{11} + a_{22})/a_{11}$ and $\lambda_2 = (a_{22}b_{12} - 2a_{11}b_{22})/(a_{11}b_{22})$. On the other hand if the additional condition is $\Delta = 0$ then $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant for $\mu_1 = -s/a_{11}$ and $\mu_2 = sb_{12}/(a_{11}b_{22})$. This corresponds to the integrable case (14).

(k) If $a_{12} = 0$, $a_{21} = a_{22}b_{21}/b_{22}$, $\Delta = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$ then the straight lines $f_1 = y = 0$ and $f_2 = a_{11} + b_{11}x = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{21}x + b_{22}y + a_{22}$ and $K_2 = b_{11}x + b_{12}y$ respectively such that with the additional condition $r_1 = 0$, $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_2 = -(a_{11} + a_{22})/a_{22}$ and $\lambda_1 = (a_{11}b_{21} - 2a_{22}b_{11})/(a_{22}b_{11})$. On the other hand if the additional condition is $\Delta = 0$ then $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant for $\mu_1 = -s/a_{22}$ and $\mu_2 = sb_{22}/(a_{22}b_{12})$. This corresponds to the integrable case (23).

(l) If $a_{12} = a_{11}b_{12}/b_{11}$, $a_{21} = a_{22}b_{21}/b_{22}$, $\Delta = 0$ and $a_{11}a_{22}b_{11}b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$ then the straight lines $f_1 = a_{11} + b_{11}x = 0$ and $f_2 = a_{22} + b_{22}y = 0$ are algebraic solutions of the QS with cofactors $K_1 = b_{11}x + b_{12}y$ and $K_2 = b_{21}x + b_{22}y$ respectively such that $f_1^{\mu_1} f_2^{\mu_2}$ is a first integral for $\mu_1 = b_{22}$ and $\mu_2 = -b_{12}$. This corresponds to the integrable case (34).

(m) If $b_{11} = b_{22} = 0$, $r_1 = 0$ and $b_{12}b_{21} \neq 0$, then the straight line $f = b_{21}x - b_{12}y = 0$, is an algebraic solution of the QS with cofactor $K = a_{11} - a_{21}b_{12}/b_{21}$ such that $f_1^{\mu} e^{st}$ is a Darboux invariant for $\mu = (a_{12}b_{21} - a_{22}b_{12})/b_{12}$. This corresponds to the integrable case (5).

A qualitative picture of the phase space portrait at a finite distance from the origin for each of the statements of Theorem 4 appears in Figure 2. The figures concerning the invariants (126) , (145) , (235) and (346) deserve a special comment. They suggest that the equilibrium point, which is not a summit of the triangle formed by the three straight lines is a center. Indeed the four cases are first integrals and if we show that this equilibrium point is a linear center, the possession of this first integral will confer a center status.

Figure 2.

To check this linear center property we will have to discuss the real or complex nature of the roots of the second degree equation obtained by linearisation of the system (36). While the relation between the coefficients of the QS, needed for the existence of the invariants or first integral remains equally valid for complex or real coefficients of (36), we will suppose real in this paragraph all the coefficients of (36).

Let x_0 and y_0 be the coordinates of an equilibrium point. The linearised equation for δx and δy with $x = x_0 + \delta x$ and $y = y_0 + \delta y$ writes

$$\begin{aligned}\delta\dot{x} &= (a_{11} + 2b_{11}x_0 + b_{12}y_0)\delta x + (a_{12} + b_{12}x_0)\delta y, \\ \delta\dot{y} &= (a_{21} + b_{21}y_0)\delta x + (a_{22} + 2b_{22}y_0 + b_{21}x_0)\delta y,\end{aligned}\tag{52}$$

which will describe a linear center if the two eigenvalues of the matrix are purely imaginary. As a consequence, two relations are necessary. The first concerns the trace of the matrix of the system. We must have

$$a_{11} + a_{22} + b_{12}y_0 + b_{21}x_0 + 2(b_{11}x_0 + b_{22}y_0) = 0,\tag{53}$$

relation which can be written, taking into account the fact that x_0, y_0 are the coordinates of an equilibrium point

$$a_{12}(y_0/x_0) + a_{21}(x_0/y_0) = b_{11}x_0 + b_{22}y_0.\tag{54}$$

The second relation is an inequality concerning the sign of the term under the square root to obtain a purely imaginary root. Taking into account (53), the eigenvalues are

$$\lambda_{1,2} = \sqrt{-D},\tag{55}$$

where

$$D = \begin{vmatrix} a_{11} + 2b_{11}x_0 + b_{12}y_0, & a_{12} + b_{12}x_0 \\ a_{21} + b_{21}y_0, & a_{22} + 2b_{22}y_0 + b_{21}x_0 \end{vmatrix}$$

and the inequality to obtain a linear center is $D > 0$. Now we show that the 4 cases mentioned above fulfill (54). We begin with **(346)** where the origin is the candidate to the status of a center. With $x_0 = y_0 = 0$, (53) becomes $a_{11} + a_{22} = 0$, which is precisely one of the conditions needed to obtain a first integral. If $a_{11}a_{22} - a_{12}a_{21} > 0$ the origin is consequently a center. For **(126)** we are in the LVS case ($a_{12} = a_{21} = 0$). Computing x_0 and y_0 and introducing it in (54) we check that this last relation is fulfilled. For values of the parameters such that $D > 0$ this equilibrium point will be a center. Finally we consider

(145) ((235) being symmetric). Taking into account the 3 conditions and doing a little algebra, we obtain for x_0 and y_0

$$x_0 = -a_{11}b_{22}/\Delta, \quad y_0 = a_{11}b_{21}/\Delta. \quad (56)$$

Introducing (56) into (53), we see that this last relation is fulfilled and we have just to check that $D > 0$.

Consequently the 4 equilibrium points of figures (126), (235), (346) and (145) fulfill the relations needed to have the status of a center and are, indeed, centers if $D > 0$.

5. Conclusion

Although some of the results presented here have been obtained before, we see that the Darboux method is more systematic and simplifies the obtention and resolution of the resulting algebraic - but non linear - equations. These equations are originated from the identification of the different terms after the introduction of ansatzes on the expression of the algebraic solutions (also called invariant curves). Each of these algebraic solutions brings its own conditions to exist. For bidimensional quadratic systems, 3 algebraic solutions are needed. The simplest are straight lines, each bringing one condition in the QS case. We need altogether 3 conditions except essentially in two cases. The LVS is the special quadratic system with $a_{12} = a_{21} = 0$ and, consequently only one more condition is needed. These algebraic solutions are invariant straight lines connecting two equilibrium points. For the LVS the two axes provide the two first algebraic solutions and if the third one is a conic we need two conditions. The invariant conics also connect two equilibrium points. Both an increase of the dimensionality and the degree of the system increase quickly the number of needed conditions.

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