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ASYMPTOTIC LIPSCHITZ MAPS, COMBABLE GROUPS AND HIGHER SIGNATURES

T. KATO

sw 9832

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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T.Kato

Introduction

Study of geometric infinite groups is developping actively. From the point of view of metric structure, many important classes of discrete groups have appeared, word hyperbolic groups, automatic groups, combable groups, etc. In this paper we study combable groups and their variant, in relation with Novikov's conjecture on homotopy invariance of higher signatures. In recent days, it turns out that in order to develop Novikov's conjecture, it is very effective to study geometric infinite groups. Higher signatures are characteristic numbers of manifolds which are deeply related to their fundamental groups. At this time there are many partial solutions to the problem by restricting fundamental groups to various classes of discrete groups. One of particular works is by [CGM], where Novikov conjecture is verified for many geometric classes of discrete groups, including word hyperbolic groups. It was discovered there that proper Lipschitz maps have a deep relation to the Novikov conjecture. In particular they constructed sufficiently many proper Lipschitz maps from contractible spaces to Euclidean spaces.

From the point of view of asymptotic property of Lipschitz maps, in this paper we shall introduce asymptotic proper Lipschitz maps. It turns out that we can naturally construct asymptotic proper contracting maps from combable groups to themselves. In order to apply these maps to the Novikov conjecture, we will use E theory as a basic tool. In [CH], A.Connes and N.Higson introduced a sophisticated theory, which they call E theory. It is a bivariant functor from C^* algebras to abelian groups. By restricting to continuous functions on spaces, $C(X)$, it is isomorphic to Kasparov's KK theory. It turns out that asymptotically proper Lipschitz maps naturally fit into equivariant E theory. In fact using asymptotic Lipschitz structure of maps, one can produce sufficiently many elements in $E_\Gamma(\mathbb{C}, \mathbb{C})$.

Our main theorem is the following:

Theorem *Let Γ be a torsion free proper combing. Then the analytic assembly map is rationally injective. In particular the higher signatures of Γ are oriented homotopy invariants.*

The analytical assembly map is expressed:

$$A : E(B\Gamma, *) \mapsto E(*, C^*\Gamma)$$

in terms of E theory. It is enough to hold rational injectivity of A for the Novikov conjecture. To show this theorem, we have the following:

Proposition *Let Γ be a torsion free proper combing. Then there exists a family of approximations of $B\Gamma$:*

$$P_0/\Gamma \subset \cdots \subset P_i/\Gamma \subset \cdots \subset B\Gamma$$

and non degenerate pairings over $\mathbb{Q}$:

$$E(\bar{P}_i/\Gamma, *) \times E(\tilde{P}_i/\Gamma, *) \mapsto \mathbb{Z}$$

such that by restricting the above, we have the following commutative diagram:

$$E(\bar{P}_i/\Gamma, *) \times E(\tilde{P}_{i+1}/\Gamma, *) \mapsto \mathbb{Z}$$

$$\downarrow A \times \sigma_{p_0} \qquad \qquad \parallel$$

$$E(C, C^*(\Gamma)) \times E(C^*(\Gamma), C) \mapsto \mathbb{Z}$$

where the lower composition is E theory intersection product.

The essential step is to construct a natural morphism $\sigma_{p_0} : E(\tilde{P}_{i+1}/\Gamma, *) \mapsto E(C^*(\Gamma), C)$, using a family of asymptotically proper Lipschitz maps from Rips complexes to Euclidean spaces. To construct these maps, we shall make an asymptotically proper contracting maps from groups to themselves. At this point we will use combability essentially.

Combable groups have the strong property that each P_i is contractible in $P_{i'}$ for some $i \leq i'$. In part II, we study locally combable groups, which are subclass of asynchronously combable groups. In this class, we are no longer able to use the above property. In order to overcome this difficulty, we will use the approximation spaces of $B\Gamma$ at the same time. At this point, some infinite dimensional geometry enters.

In order to treat this class, we need some algebraic topological condition on groups. This is related to the so-called Mittag-Leffler condition on E or K theory of $B\Gamma$ (4.1). Under this condition, we will show the following:

Theorem *Let Γ be a torsion free, locally quasi geodesic combing. Then the analytic assembly map is rationally injective. In particular the higher signatures of Γ are oriented homotopy invariants.*

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I Properly combable groups and higher signatures

Let X, Y be metric spaces and try to find Lipschitz maps $f : X \mapsto Y$. When these spaces are non compact, we require them to be proper. Suppose given a proper map $g : X \mapsto Y$, and try to deform it by a proper homotopy so that it becomes a proper Lipschitz map. We start from constructing proper Lipschitz maps, and next find a proper homotopy between them. There are obvious obstruction to find such homotopy, $\pi_i(Y)$. Thus if Y is contractible, this procedure becomes reasonable. We have a relative situation to this; suppose we have a proper inclusion of metric spaces $Y \subset Z$ which are Lipschitz equivalent. Let g be as above and try to find a proper Lipschitz map $f : X \mapsto Z$ which is proper homotopic to g . In this case also, the above procedure works when $\text{im}\pi_i(Y) \rightarrow \pi_1(Z) = 0$. If X satisfies certain kind of convexity, then it turns out that this process works well enough, by constructing self, or asymptotic self contracting maps ([CGM], [Ka]).

Non-proper Lipschitz maps can be constructed from *radial projections*. Let $L \subset X$ be a compact subset. A radial projection to t ball is a non proper map $F_t : X \setminus L \mapsto \partial B_t(x_0)$, where $x_0 \in X$ is a fixed point and $B_t(x_0) \supset L$ is t ball. This map is trivially contracting on any δ separated subset for sufficiently large δ . A metric space X_0 is δ separated if for any different $x, y \in X_0$, we have $d(x, y) \geq \delta$. By restricting to some δ separated subset $X_0 \subset X$, we get a $2t\delta^{-1}$

Lipschitz map, $F_t|_{X_0 \setminus L}$. In particular if δ is sufficiently large so that $2t\delta^{-1} < \epsilon$, then this becomes a contracting map with sufficiently small contracting constant.

Example Suppose there is a continuous family of maps $g : X \times [0, \infty) \mapsto X$ such that $g(x)$ is a geodesics with $g(x)(0) = x_0$, $g(x)(t) = x$ for large t . Then $F_t(x) = g|_{X \setminus B_t(x_0)}(x)(t)$ is a radial projection. We have such g when X is an Hadamard manifold, simply connected, complete Riemannian manifold of non positive curvature.

Throughout this paper, when we construct maps, our steps will proceed as follows; first we construct contracting maps from some δ separated subset to itself, and next extend this to maps from X , holding small contracting constants, and then to make them continuous with respect to the parameter t . This step will produce a family of non proper contracting maps. In order to make it proper, we need to deform it on the domain near infinity. Then we will get a family of proper maps which is contracting on some compact subset for each t , and these subsets become larger and larger as t increases. This motivates the following:

Definition A family of maps $F : [1, \infty) \times X \mapsto Y$ is called asymptotically proper Lipschitz if there exist a constant C and an increasing proper function $\mu : [0, \infty) \mapsto [0, \infty)$ such that:

$$(1) \quad F_t|_{B_t(x_0)} \text{ is } C \text{ Lipschitz}, \quad (2) \quad F_t(X \setminus B_t(x_0)) \subset Y \setminus B_{\mu(t)}(y_0)$$

where $x_0 \in X$ and $y_0 \in Y$ are fixed points. If $0 < C < 1$, we say that F_t is asymptotically proper contracting.

In practice, we will construct asymptotically proper self contracting maps G on X , with decreasing contracting constants, $C = C(t) \mapsto 0$, $t \mapsto \infty$. By a trivial construction, we may have a proper map $F : X \mapsto Y$, with increasing Lipschitz constants on $B_t(x_0)$. Our construction allows $C(t)$ to be arbitrarily small, thus the composition $F \circ G$ becomes the desired asymptotically proper Lipschitz map. One of the essential step to construct G is to deform radial projection, in order to make them proper. If we have assigned proper paths from any point in X to a fixed point, then we can perform it along these paths. In our case, this role will be played by paths of combing of groups. In partII, we will deal with groups which seems to be difficult to make deformation. For this case, we use radial projection directly. After mapped to Y by F , we expand its image. We will use the property of radial projection, that the image of G is small ϵ away from the fixed point.

1 Asymptotic morphisms

1.1 Asymptotic morphisms: In order to make the above maps live in algebraic topology, we will use E theory. In this section we summary basic definitions on E theory.

Let P be a complete Riemannian manifold of dimension N , and $Cliff(TP)$ be the complex Clifford algebra of TP . If TP is trivial, then there is a natural Dirac operator on $\eta(P) = L^2(P, Cliff(R^N))$. More precisely let $\{e_i\}$ be an orthonormal basis on R^N , and $\hat{e}_i$ be the operator on $Cliff(R^N)$ by the right multiplication: $\hat{e}_i \xi = (-1)^{\deg \xi} \xi e_i$. Then we define $D = \sum \hat{e}_i \frac{\partial}{\partial x_i}$. Let us take $f \in S = C_0(R)$. Then using functional calculus, we obtain a compact operator $f(D) \in K(\eta(P))$.

Let $SK(\eta(P))$ be a C^* algebra $C_0(R) \hat{\otimes} K(\eta(P))$, where $S = C_0(R)$ is graded by even, or odd functions, and $Cl_0(P) = C_0(P, Cliff(R^N))$. Let us define a map:

$$\varphi : SCl_0(P) \mapsto K(\eta(P))$$

by $f \hat{\otimes} h \mapsto f(D)M(h)$, where $M(h) \in End(\eta(P))$ is the left multiplication by h . Let us introduce a parameter $t \in [1, \infty)$ by $\varphi_t : SCl_0(P) \mapsto K(\eta(P))$ by $f \hat{\otimes} h \mapsto f_t(D)M(h)$, $f_t(x) = f(t^{-1}x)$. φ is not a $*$ homomorphism, however, φ_t approaches to such morphisms as $t \rightarrow \infty$. The latter is called an asymptotic morphism.

For convenience, we recall the definition of asymptotic morphisms and bivariant E theory([CH]). Let A and B be C^* algebras. An *asymptotic morphism* from A to B is a norm continuous family of bounded functions:

$$\varphi_t : A \mapsto B \quad t \in [1, \infty)$$

such that the followings all converge to 0 in operator norm as $t \rightarrow \infty$: For $a_1, a_2 \in A$, $\lambda \in C$,

$$\begin{aligned} (1) \quad & \varphi_t(a_1)\varphi_t(a_2) - \varphi_t(a_1a_2), \quad (2) \quad \varphi_t(a_1) + \varphi_t(a_2) - \varphi_t(a_1 + a_2), \\ (3) \quad & \lambda\varphi_t(a_1) - \varphi_t(\lambda a_1), \quad (4) \quad \varphi_t(a_1^*) - \varphi_t(a_1)^*. \end{aligned}$$

Let Γ be a group, and suppose Γ acts on A and B by automorphisms. Then a Γ asymptotic morphism is an asymptotic morphism $\varphi_t : A \mapsto B \quad t \in [1, \infty)$ with the additional condition:

$$(5) \quad \varphi_t(ga) - g\varphi_t(a) \mapsto 0 \quad t \rightarrow \infty.$$

Two (Γ) asymptotic morphisms $\varphi_t, \varphi'_t : A \rightarrow B$ are equivalent if for all $a \in A$,

$$\varphi_t(a) - \varphi'_t(a) \mapsto 0 \quad t \rightarrow \infty.$$

φ_t and φ'_t are (Γ) asymptotic homotopy equivalent if there is a (Γ) asymptotic morphism $\psi_t : A \mapsto C([0, 1], B)$ such that $\text{ev}_0 \circ \psi_t = \varphi$ and $\text{ev}_1 \circ \psi_t = \varphi'$, where $\text{ev}_s : C([0, 1], B) \mapsto B$ is the evaluation at $s \in [0, 1]$. Clearly equivalent asymptotic morphisms are homotopic, and later we will say only equivalent, which means both equivalent and homotopy equivalent.

$E(A, B)$ is the set of equivalent classes of asymptotic morphisms from $S \hat{\otimes} A \hat{\otimes} K$ to $S \hat{\otimes} B \hat{\otimes} K$, where K is a C^* algebra of compact operators on a separable, Hilbert space H . It admits a groups structure by:

$$[\varphi_t] + [\phi_t] = \begin{bmatrix} \varphi_t & 0 \\ 0 & \phi_t \end{bmatrix}$$

identifying $H \cong H \oplus H$. For any $f \in S$, let us define $\tilde{f} \in S$ by $\tilde{f}(x) = f(-x)$. Then the inverse $-[\varphi_t]$ is given by $-[\varphi_t](f \hat{\otimes} a \hat{\otimes} k) = [\varphi_t](\tilde{f} \hat{\otimes} a \hat{\otimes} k)$. For $\Gamma - C^*$ algebras A and B , we have a similar definition $E_\Gamma(A, B)$.

Let $\varphi_t : A \mapsto B$ and $\phi_t : B \mapsto C$ be (Γ) asymptotic morphisms. Then there is a natural composition $[\phi_t] \circ [\varphi_t] : A \mapsto C$ by the following :

Lemma (CH) *There exists a proper function $\delta : [1, \infty) \mapsto [1, \infty)$ $\delta(t) \gg t$ such that $\phi_{\delta(t)} \circ \varphi_t : A \mapsto C$ defines a (Γ) asymptotic morphism. Its equivalent class is independent of choice of δ .*

In particular the above composition induces the corresponding bilinear map:

$$E_\Gamma(A, B) \times E_\Gamma(B, C) \mapsto E_\Gamma(A, C).$$

Let P be a complete Riemannian manifold with proper, and free action of Γ . For our purpose, it is enough to assume that $T(P/\Gamma)$ is trivial. Let:

$$C : R^N \hookrightarrow \text{Cliff}(R^N), \quad X : R \mapsto R$$

be essential self adjoint, unbounded multipliers of $Cl_0(R^N)$ and S by embedding and the identity respectively, with domain the compactly supported functions (and by left multiplication). Let $\alpha : P \mapsto R^N$ be a continuous map. Then by pulling back C , we have an essentially self adjoint unbounded multiplier of $Cl_0(P)$, $\alpha^*(C)$. Let us define a $*$ homomorphism $\beta : S \mapsto S \hat{\otimes} M(Cl_0(P))$ by $\beta(f) = f(X \hat{\otimes} 1 + 1 \hat{\otimes} \alpha^*(C))$. If α is proper, then it defines:

$$\beta : S \mapsto SCl_0(P) = S \hat{\otimes} Cl_0(P).$$

1.2 Lipschitz effect: In order to make β a Γ asymptotic morphism, we need more geometric structure on α . Let X and Y be metric spaces. A map $f : X \mapsto Y$ is said to be *strictly Lipschitz* (s-Lipschitz) if for any $\delta \geq 0$, and any $p, p' \in X$, $d(p, p') \leq \delta$, there exists $C = C(\delta)$ such that for a fixed point $y_0 \in Y$,

$$d_Y(y_0, f(p))d_Y(f(p), f(p')) \leq C.$$

Proposition *Let $\alpha : P \mapsto R^N$ be a proper s-Lipschitz map. Then*

$$\beta_t : S \mapsto SCl_0(P) \quad t \in [1, \infty)$$

by $\beta_t(f) = \beta(f_t)$ defines a Γ asymptotic morphism.

Proof: It is enough to show the following:

$$\lim_{t \rightarrow \infty} [f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} \alpha^*(C)), g] = 0 \quad g \in \Gamma. \quad *$$

S is generated by $e_t = \exp(-t^{-2}|x|^2)$ and $(xe)_t = t^{-1}x \exp(-t^{-2}|x|^2)$ as a C^* algebra. Thus it is enough to see the above for $f = \exp(-|x|^2)$ and $x \exp(-|x|^2)$. It is easily seen that:

$$\beta(e_t) = e_t \hat{\otimes} \exp(-t^{-2}|\alpha(p)|^2),$$

$$\beta((xe)_t) = (xe)_t \hat{\otimes} \exp(-t^{-2}|\alpha(p)|^2) + e_t \hat{\otimes} \alpha^*(C) \exp(-t^{-2}|\alpha(p)|^2).$$

For $f = \exp(-|x|^2)$, we have $[f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} \alpha^*(C)), g] = e_t \hat{\otimes} [\exp(-t^{-2}|\alpha(p)|^2), g]$,

$$[\exp(-t^{-2}|\alpha(p)|^2), g] \zeta(x) = (\exp(-t^{-2}|\alpha(p)|^2) - \exp(-t^{-2}|\alpha(gp)|^2)) \zeta(gx).$$

Thus $*$ follows from the following estimate for this case:

$$||\alpha(gx)|^2 - |\alpha(x)|^2| \leq (|\alpha(gx)| + |\alpha(x)|)|\alpha(gx) - \alpha(x)| \leq C(g).$$

For $f = x \exp(-|x|^2)$, since $t^{-1}\alpha(x) \exp(-t^{-2}|\alpha(x)|^2)$ is uniformly bounded, $*$ also follows from the same estimate as above. This completes the proof.

Later we will use a family of maps. Let us take a pair (P_0, P_1) and a family of proper s-Lipschitz maps $\alpha_1 : P_1 \times_{\Gamma} P_0 \mapsto R^{N_1}$, where Γ acts on $P_1 \times P_0$ diagonally. Then by Γ invariance, we have a Γ asymptotic morphism:

$$\beta_{p_0} : S \mapsto SCl_0(P_1), \quad f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} \alpha_1(\cdot, p_0)C)$$

for any fixed point $p_0 \in P_0$.

Let us consider a family of asymptotically proper s-Lipschitz maps:

$$\alpha_1^t : P_1 \times_{\Gamma} P_0 \mapsto R^N, \quad t \in [1, \infty)$$

with the increasing proper function $\mu : [0, \infty) \mapsto [0, \infty)$.

Corollary *There exists an increasing proper function $l : [1, \infty) \mapsto [1, \infty)$ such that for any choice of $p_0 \in P_0$, the following map is a Γ asymptotic morphism:*

$$\beta_{p_0} : S \mapsto SCl_0(P_1) \quad f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} \alpha_1^{l(t)}(\cdot, p_0)C).$$

Proof: Let us take l so that $t^{-1}\mu(l(t)) \rightarrow \infty$ ($t \rightarrow \infty$). Then it is enough to show:

$$|[f_t(\alpha_1^{l(t)}(\cdot, p)^*C), g]| \mapsto 0 \quad t \rightarrow \infty.$$

We will see the above for $f = \exp(-|x|^2)$. Another case $f = x \exp(-|x|^2)$ is similar. Let us show $\sup_{p, p'} |\exp(-t^{-2}|\alpha_1^{l(t)}(p, p')|^2) - \exp(-t^{-2}|\alpha_1^{l(t)}(gp, p')|^2)|$ approaches to 0 as $t \rightarrow \infty$. We consider the following cases:

$$(1) \quad d(p, p') \leq l(t), \quad d(p, gp') \leq l(t), \quad (2) \quad d(p, p') \geq l(t), \quad d(p, gp') \geq l(t),$$

$$(3) \quad d(p, p') \leq l(t), \quad d(p, gp') \geq l(t).$$

The case (1) follows from the proof of the above proposition. For the case (2), by the property of l , we have $t^{-1}|\alpha_1^{l(t)}(p, p')| \mapsto \infty$ ($t \rightarrow \infty$). Thus the above follows for this case. Let us consider the case (3). Let $g = g_m \dots g_1$ be a geodesic path for the word metric. Then there exists $g_{m'} \quad 1 \leq m' \leq m$ such that:

$$d(p, g_{m'-1} \dots g_1 p) \leq l(t), \quad d(p, g_{m'} \dots g_1 p) \geq l(t).$$

Thus the above follows for this case from the following:

$$\begin{aligned} & |\exp(-t^{-2}|\alpha_1^{l(t)}(p, p')|^2) - \exp(-t^{-2}|\alpha_1^{l(t)}(gp, p')|^2)| \leq \\ & |\exp(-t^{-2}|\alpha_1^{l(t)}(p, p')|^2) - \exp(-t^{-2}|\alpha_1^{l(t)}(g_{m'} \dots g_1 p, p')|^2)| + \\ & |\exp(-t^{-2}|\alpha_1^{l(t)}(g_{m'} \dots g_1 p, p')|^2) - \exp(-t^{-2}|\alpha_1^{l(t)}(gp, p')|^2)|. \end{aligned}$$

This completes the proof.

If a space X admits a selfcontracting F , then we can make Lipschitz constants of any map $f : X \mapsto Y$ arbitrarily small by composing with F . In the case when X is contractible space and satisfies some convexity, then [CGM] developed the method to construct self contracting maps on X , whose contracting constants become arbitrarily smaller and smaller as the point approaches to infinity. It is treated in [Ka] when X is not contractible, but satisfies some convexity.

When the target space is R^N , or Hilbert space, we can naturally deform any Lipschitz maps to s-Lipschitz ones, using selfcontracting maps on these convex spaces.

Lemma Any (proper) Lipschitz map $f : X \mapsto R^N$ is (properly) homotopic to a s -Lipschitz map.

Proof : The proof is a simple case of a general method, applied to R^N in [CGM], which is called symmetrization. Let us take a small constants $0 < \lambda < \mu \ll 1$, which are determined later. By modifying the identity: $R^N \mapsto R^N$, we have $i : R^N \mapsto R^N$ such that $i|_{B_\lambda(0)}$ is the identity and $i|_{R^N \setminus B_1(0)} = \mu^{-1} \text{id}$. Let us put $D_i = B_{\lambda^{-i}}(0) \setminus B_{\lambda^{-(i-1)}}(0)$, and define a contracting map $F : R^N \mapsto R^N$ by $F|_{D_i}(x) = \mu^{-i} i(\lambda^i x)$. F is clearly continuous and properly homotopic to the identity. Moreover there is a constant C such that $F|_{D_i}$ is $C(\mu^{-1}\lambda)^i$ Lipschitz, where C is determined only by a Lipschitz constant of $i|_{B_1(0)}$ and is independent of N .

Let us consider $F \circ f : X \mapsto R^N$ and take $f(p), f(p') \in D_i$, $d_X(p, p') \leq \delta$. Then

$$(1) \quad d(0, F \circ f(p)) \leq \mu^{-i}, \quad (2) \quad d(F \circ f(p), F \circ f(p')) \leq \text{Lip}(f) \delta C (\mu^{-1} \lambda)^i$$

where $\text{Lip}(f)$ is the Lipschitz constant of f . Thus by taking $\lambda < \mu^2$, we get the desired map $F \circ f$. This completes the proof.

Let $f : [1, \infty) \times X \mapsto R^N$ be an asymptotically proper Lipschitz map. Then the above proof shows that $F \circ f$ is an asymptotically proper s -Lipschitz map, with the obvious definition.

1.3 Local Bott periodicity: Let P be a complete Riemannian manifold and assume that TP is trivial. Then we have a natural Dirac operator D over

$$\eta(P) = L^2(P, \text{Cliff}(R^N)).$$

Let us regard $X \hat{\otimes} 1$ and $1 \hat{\otimes} D$ as essentially self adjoint multipliers over $SK(\eta(P)) \equiv S \hat{\otimes} K(\eta(P))$. We define a family of maps:

$$\varphi_t : SCl_0(P) \mapsto SK(\eta(P)) \quad t \in [1, \infty)$$

by $\varphi_t(f \hat{\otimes} h) = f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} D)M(h)$. $K(\eta(P))$ is graded corresponding to the one on $\eta(P)$. It is easily seen that ([HKT]):

$$\lim_{t \rightarrow \infty} \| [f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} D), 1 \hat{\otimes} M(h)] \| = 0.$$

Thus this defines an asymptotic morphism. When we consider Γ action, this becomes a Γ asymptotic morphism, since we may take Γ invariant Dirac operator.

Now let us take a small $\epsilon > 0$, and suppose a proper map $\alpha : P \mapsto R^N$ satisfies:

$$(1) \quad \alpha|_{B_\epsilon(p_0)} = \exp_{p_0}, \quad (2) \quad \alpha(P \setminus B_\epsilon(p_0)) \subset R^N \setminus \alpha(B_\epsilon(p_0)).$$

Proposition *Let $\zeta \in \eta(P) \neq 0$ be any even function, and $P(\zeta)$ be the rank 1 projection to $C\zeta$. Then the composition of the following:*

$$S \xrightarrow{\beta} SCl_0(P) \xrightarrow{\varphi_t} SK(\eta(P))$$

is asymptotically equivalent to $\text{id} \hat{\otimes} P(\zeta)$.

We may consider $\beta \in E(*, P)$ as a dual element of $\varphi_t \in E(P, *)$, since the composition $\varphi_t \circ \beta \in E(*, *)$ represents the identity.

Notice that for any non zero ζ_0 and ζ_1 , $\text{id} \hat{\otimes} P(\zeta_0)$ and $\text{id} \hat{\otimes} P(\zeta_1)$ are asymptotically equivalent, through a non vanishing path of projections $P(\zeta_t)$. Thus in order to see this proposition, it is enough to show for a particular ζ .

Sublemma *Let us take a small $0 < \delta < \epsilon$. Then $\beta : S \mapsto SCl_0(P)$ is asymptotically equivalent to another β_δ such that:*

$$(1) \quad \beta_\delta(f)|_{B_\delta(p_0)} = \beta(f)|_{B_\delta(p_0)}, \quad (2) \quad \beta_\delta(f)|_{P \setminus B_\epsilon(p_0)} = 0.$$

Proof: Let $\varphi_t : R_+ \mapsto [1, \infty)$ $t \in [1, \infty)$ be a continuous family of increasing maps such that:

$$(1) \quad \varphi_t|_{[0, \delta]} = 1 \text{ and } \varphi_1 \equiv 1,$$

$$(2) \quad \varphi_t|_{[0, \epsilon)} \text{ converges to a proper increasing function } \varphi_\infty : [0, \epsilon) \mapsto [1, \infty) \text{ uniformly on compact subsets.}$$

For example, $\varphi_t|_{[\delta, \epsilon]} = \min\{1, t-1\} \tan\{(\frac{\pi}{2} - t^{-1})(x-\delta)(\epsilon-\delta)^{-1}\} + 1$ is enough. Let us define:

$$\varphi_t^* : S \mapsto S, \quad \varphi_t^*(f)(x) = f(\varphi_t(|x|)x).$$

Then φ_t^* converges in S to $\varphi_\infty^* : S \mapsto S$:

$$\varphi_\infty^*(f)|_{[0, \epsilon)} = f(\varphi_\infty(|x|)x), \quad \varphi_\infty^*(f)|_{[\epsilon, \infty)} \equiv 0.$$

Thus the identity and φ_∞^* are homotopic $*$ homomorphisms, or asymptotically equivalent. Thus we get the desired $\beta_\delta = \beta \circ \varphi_\infty^*$.

Sublemma (HKT) *Let us put $P = R^N$ and $\alpha = \text{identity}$, and take the corresponding asymptotic morphisms:*

$$\beta^{st} : S \mapsto SCl_0(R^N), \quad \varphi^{st} : SCl_0(R^N) \mapsto SK(\eta(R^N)).$$

Then their composition $[\varphi^{\text{st}}] \circ [\beta^{\text{st}}]$ is asymptotically equivalent to $\text{id} \hat{\otimes} P$, where P is the rank 1 projection to $C \exp(-\frac{1}{2}|x|^2)1 \in \eta(R^N)$.

Proof of proposition: By slightly modifying the standard metric on R^N , we have another one d' such that $(B_\epsilon(p_0), d_P)$ and $(B_\epsilon(0), d')$ are isometric. Using d' , we have the corresponding β' and φ' . These are asymptotically equivalent to β^{st} and φ^{st} respectively. Let $\beta'_\delta : S \mapsto SCl_0(R^N, d')$ be constructed from (R^N, d') . By the construction, we may identify $\beta_\delta|_{B_\epsilon(p_0)}$ and $\beta'_\delta|(B_\epsilon(0), d')$.

Sublemma Suppose $\phi \in C_0(P)$ has its support in $B_\epsilon(p_0)$. Then for any $f \in S$, $f(t^{-1}D)\phi|_{P \setminus B_\epsilon(p_0)} \mapsto 0$ as $t \rightarrow \infty$ uniformly.

Proof: It is enough to see this for $f = \exp(-|x|^2)$, and $x \exp(-|x|^2)$. For $f = \exp(-|x|^2)$, the result follows from the next formula: $\exp(-t^{-2}D^2) = \sqrt{4\pi t^{-2}} \int_{-\infty}^{\infty} \exp(-\frac{1}{4}t^2\lambda^2) \exp(i\lambda D) d\lambda$, and the finite speed propagation of the solutions of the wave equation.

Proof of proposition: Let $\phi \in C_c(B_\epsilon(p_0))$ be a cut off function such that $\phi|_{B_\delta(p_0)} \equiv 1$. Then it follows from the asymptotic expansion of the heat kernel, that $\phi \exp(-t^{-2}D_P^2)\xi = \phi \exp(-t^{-2}(D')^2)\xi$ for $\xi \in C_c(B_\epsilon(p_0))$, where D' is the Dirac operator on (R^N, d') , and we identify $\xi \in \eta(P)$ and $\xi \in \eta(R^N, d')$. By the above lemma, we have $|(1 - \phi)\exp(-t^{-2}D_P^2)\xi|, |(1 - \phi)\exp(-t^{-2}D^2)\xi| \mapsto 0$ as $t \rightarrow \infty$.

Now let us denote asymptotic equivalence by $\sim$. Then

$$\begin{aligned} \varphi \circ \beta &\sim \varphi \circ \beta_\delta \sim M(\phi) \circ \varphi \circ \beta_\delta = M(\phi) \circ \varphi' \circ \beta'_\delta \\ &\sim \varphi' \circ \beta'_\delta \sim \varphi^{\text{st}} \circ \beta^{\text{st}} \sim \text{id} \hat{\otimes} P(\xi) \sim \text{id} \hat{\otimes} P(\phi\xi) \end{aligned}$$

where $\xi = \exp(-\frac{1}{2}|x|^2)1$. More precisely let us define a Hilbert space:

$$\eta \equiv \eta(P) \oplus \eta(R^N \setminus B_\epsilon(0)).$$

Roughly speaking this is a union of $\eta(P)$ and $\eta(R^N)$, identified $\eta(B_\epsilon(p_0)) \subset \eta(P)$ with $\eta(B_\epsilon(0)) \subset \eta(R^N)$. Then $M(\phi) \circ \varphi \circ \beta_\delta$ and $M(\phi) \circ \varphi' \circ \beta'_\delta$ are identified each other by the following form:

$$\begin{pmatrix} M(\phi) \circ \varphi \circ \beta_\delta & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} M(\phi) \circ \varphi' \circ \beta'_\delta & 0 \\ 0 & 0 \end{pmatrix}$$

In this sense, we may assume that $\varphi \circ \beta$ and $\varphi^{\text{st}} \circ \beta^{\text{st}}$ have the same target C^* algebra $SK(\eta)$, and have the above asymptotic equivalence. This completes the proof.

2 Poincaré duality

2.1 Poincaré duality: In general we have approximations of $B\Gamma$ by finite complexes for torsion free finitely generated groups, for example we may take Rips complexes. Thus there exists an increasing sequences of finite complexes:

$$P_0/\Gamma \subset P_1/\Gamma \subset \dots \subset B\Gamma$$

whose union $\cup_i P_i/\Gamma$ is of $K(\Gamma, 1)$ type. Let us embed P_0/Γ into R^{N_0} and take (open) tubular neighbourhoods $(\tilde{P}_0/\Gamma) \bar{P}_0/\Gamma$. We may assume that $\bar{P}_0/\Gamma \subset \tilde{P}_0/\Gamma$ and ϵ neighbourhood of $\bar{P}_0/\Gamma \subset R^{N_0}$ is also contained in $\tilde{P}_0/\Gamma$. Let us choose an induced Riemannian metric on $\bar{P}_0/\Gamma$ from the embedding $\bar{P}_0/\Gamma \subset R^{N_0}$, and a complete Riemannian metric on $\tilde{P}_0/\Gamma$. These lift to the ones $\bar{d}_0$ and $\tilde{d}_0$ on $\bar{P}_0$ and $\tilde{P}_0$ respectively. We may assume $\tilde{d}_0|_{N_\epsilon(\bar{P}_0/\Gamma)} = \bar{d}_0$ where $N_\epsilon(\bar{P}_0/\Gamma)$ is ϵ neighbourhood of $\bar{P}_0/\Gamma$.

Let $\alpha_0 : \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto R^{N_0}$ be a fiberwise proper map and take $p_0 \in \bar{P}_0$. Then we define

$$\sigma_{p_0} : S \mapsto SK(\eta(\tilde{P}_0))$$

by the following composition:

$$\begin{aligned} (1) \quad \beta_{p_0} : S &\mapsto SCl_0(\tilde{P}_0) \quad f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} \alpha_0(\cdot, p_0)C), \\ (2) \quad \varphi_t : SCl_0(\tilde{P}_0) &\mapsto SK(\eta(\tilde{P}_0)) \quad f \hat{\otimes} h \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} D)M(h). \end{aligned}$$

There is an obvious extension of the above. Let E be a complex vector bundle over $\tilde{P}_0/\Gamma$ and D_E be the corresponding Dirac operator on $Cliff(R^{N_0}) \hat{\otimes} E$. Let $\eta_E(\tilde{P}_0) = L_2(\tilde{P}_0, Cliff(R^{N_0}) \hat{\otimes} E)$. Then by 1.2 and 1.3, we have the following:

Proposition *The composition of the above:*

$$\sigma_{p_0}^E : S \mapsto SK(\eta_E(\tilde{P}_0))$$

defines an asymptotic morphism. If α_0 is fiberwise asymptotically proper s-Lipschitz, then it is a Γ asymptotic morphism.

Let us recall E theory description of K homology. Topological K homology of X , $K(X)$ is, by definition, generated by the set: $(E, F : M \mapsto X)$ where $E \mapsto M$ is a complex vector bundle, and M is a closed manifold equipped with a spin^c structure. Thus there exists a Z_2 graded complex spinor S over M . Let us assign to $(E, F : M \mapsto X)$ an asymptotic morphism $\varphi : SC(X) \mapsto SK(L^2(S \hat{\otimes} E))$ by

$$f \hat{\otimes} h \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} D_E)M(F^*h)$$

where D_E is the twisted Dirac operator over the twisted complex spinor $S \hat{\otimes} E = S_+ \otimes E \oplus S_- \otimes E$. This induces the canonical isomorphism $K_*(X) \cong E(X, *)$.

In the case when α_0 is fiberwise asymptotically proper s-Lipschitz, the above proposition provides us a natural map:

$$\sigma_{p_0} : E(\tilde{P}_0/\Gamma, *) \mapsto E_\Gamma(C, C) \cong E(C^*\Gamma, C).$$

2.1A Non degenerate pairing: Let us construct a natural pull-back operation:

$$\sim : E(\tilde{P}_0/\Gamma, *) \mapsto E(\tilde{P}_0 \times_\Gamma \bar{P}_0, \bar{P}_0/\Gamma).$$

Since $\tilde{P}_0/\Gamma$ is an open set in R^{N_0} , its tangent bundle is trivial, and thus we may take E' : complex bundle over $\tilde{P}_0/\Gamma$ and the natural Dirac operator over $\tilde{P}_0/\Gamma \times \text{Cliff}(R^{N_0})$ for any element of $K_*(\tilde{P}_0/\Gamma)$. Then for (E', D) , we assign an asymptotic morphism:

$$(E', D) : SC_0(\tilde{P}_0 \times_\Gamma \bar{P}_0) \mapsto S(C(\bar{P}_0) \otimes K(\eta_{E'}(\tilde{P}_0)))^\Gamma$$

by $f \hat{\otimes} h \mapsto [p] \in \bar{P}_0/\Gamma \rightarrow f(X \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{D}_{E'})h(\quad, p)$, where $\tilde{D}_{E'}$ is the lift of $D_{E'}$ to the operator on $\eta_{E'}(\tilde{P}_0)$.

Let $\alpha_0 : \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto R^{N_0}$ be a fiberwise proper degree 1 map such that on ϵ neighbourhood of diagonal $N_\epsilon(\Delta) = \{(p, p') \in \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mid \tilde{d}_0(p, p') < \epsilon\}$:

$$(1) \quad \alpha_0|_{N_\epsilon(\Delta)}(p, p') = \exp_p p', \quad (2) \quad \alpha_0(\tilde{P}_0 \times_\Gamma \tilde{P}_0 \setminus N_\epsilon(\Delta)) \subset R^{N_0} \setminus B_\epsilon(0).$$

Let $(\alpha_0)^*(\omega) \in E(*, \tilde{P}_0 \times_\Gamma \bar{P}_0)$ be the pull-back of the Bott generator $\omega \in E(*, R^{N_0})$. Then we get the following composition:

$$E(\tilde{P}/\Gamma, *) \xrightarrow{\sim} E(\tilde{P}_0 \times_\Gamma \bar{P}_0, \bar{P}_0/\Gamma) \xrightarrow{\alpha_0^*(\omega) \circ} E(*, \bar{P}_0/\Gamma).$$

Thus by composing, we have the following pairing:

$$E(\tilde{P}_0, *) \times E(\bar{P}_0/\Gamma, *) \mapsto Z.$$

Lemma *The above pairing is non degenerate over Q .*

In fact passing through Chern character, we get the equality (see 1.3):

$$\langle \sigma, \tau \rangle = \int_{ch_*(\tilde{\sigma}) \times_\Gamma ch_*(\tilde{\tau})} \alpha_0^*(\bar{\omega})$$

where $\bar{\omega} \in H_{\text{cp}}^{N_0}(R^{N_0})$ is the fundamental class. The lemma follows from the Poincaré duality.

2.2 The assembly map: Let $E = \bar{P}_i \times_{\Gamma} C^*\Gamma$ be the canonical $C^*\Gamma$ bundle over $\bar{P}_i/\Gamma$. It is a finitely generated projective $C(\bar{P}_i/\Gamma) \otimes C^*\Gamma$ module ([C] p 97). It corresponds to a projection e over $\text{Mat}(C(\bar{P}_i/\Gamma) \otimes C^*\Gamma)$. Thus by assigning $f \rightarrow f \otimes e$, e defines an element of $E(C, C(\bar{P}_i/\Gamma) \otimes C^*\Gamma)$. Then E theory description of the assembly map:

$$A : E(B\Gamma, *) \mapsto E(C, C^*\Gamma)$$

is the composition of the following:

$$(1) \quad \otimes 1 : E(\bar{P}_i/\Gamma, *) \mapsto E(C(\bar{P}_i/\Gamma) \otimes C^*\Gamma, C^*\Gamma),$$

$$(2) \quad E(C, C(\bar{P}_i/\Gamma) \otimes C^*\Gamma) \times E(C(\bar{P}_i/\Gamma) \otimes C^*\Gamma, C^*\Gamma) \mapsto E(C, C^*\Gamma).$$

Lemma (K) *If the assembly map is rationally injective for Γ , then Γ satisfies the Novikov conjecture.*

The assumption is known as the Kasparov's strong Novikov conjecture. It is known to be true for many classes of discrete groups. In order to show this, we will construct σ_{p_0} and the commutative diagram as follows:

$$E(\bar{P}_0/\Gamma, *) \times E(\tilde{P}_1/\Gamma, *) \mapsto Z$$

$$\downarrow A \times \sigma_{p_0} \qquad \parallel$$

$$E(C, C^*(\Gamma)) \times E(C^*(\Gamma), C) \mapsto Z$$

where the lower composition is E theory intersection product, and the upper composition is the restriction of the nondegenerate pairing to $E(\tilde{P}_1/\Gamma, *) \times E(\bar{P}_0/\Gamma, *)$. Notice that by shifting suffix, we will get similar diagrams for any pair (P_i, P_{i+1}) .

By definition, $E(B\Gamma, *) = \lim_i E(\bar{P}_i/\Gamma, *)$. Thus for any $a \in E(B\Gamma, *)$, there is some representative $a_i \in E(\bar{P}_i/\Gamma, *)$. Suppose $a \neq 0 \in E(B\Gamma, *) \otimes Q$. Then by non degeneracy, there is $b_{i+1} \in E(\bar{P}_{i+1}/\Gamma, *)$ such that $\langle a_{i+1}, b_{i+1} \rangle \neq 0$. Thus from commutativity of this diagram, we get a contradiction if $A(a) \otimes Q = A(a_i) \otimes Q = 0$.

Let $\alpha_1 : \tilde{P}_1 \times_{\Gamma} \bar{P}_1 \mapsto R^{N_1}$ be a fiberwise proper map with $\alpha_1|_{N_{\epsilon}(\Delta)} = \exp$, $\alpha_1(\tilde{P}_1 \times_{\Gamma} \bar{P}_1 \setminus N_{\epsilon}(\Delta)) \subset R^{N_1} \setminus B_{\epsilon}(0)$. Suppose we have a family of asymptotically proper s-Lipschitz maps $\alpha_1^t : \tilde{P}_1 \times_{\Gamma} \bar{P}_0 \mapsto R^{N_1}$ $t \in [1, \infty)$, $\alpha_1^1 = \alpha_1|_{\tilde{P}_1 \times_{\Gamma} \bar{P}_0}$. In later sections, we will construct a deformation of $\alpha_1|_{\tilde{P}_1 \times_{\Gamma} \bar{P}_0}$ so that the result

is an asymptotically proper Lipschitz map. The restriction comes from a fact that P_0 is not self contracting, but is contractible in P_1 .

From 2.1, we have the following asymptotic morphism:

$$\sigma_{p_0} : E(\tilde{P}_1/\Gamma, *) \mapsto E(C^*\Gamma, C).$$

Let us consider the following diagram:

$$\begin{array}{ccc} E(\tilde{P}_1/\Gamma, *) & \xrightarrow{\alpha_1^*(\omega) \circ \sim} & E(*, \bar{P}_0/\Gamma) \\ \downarrow \sigma_{p_0} & & \uparrow e \circ \\ E(C^*\Gamma, C) & \xrightarrow{\otimes 1} & E(C^*\Gamma \otimes C(\bar{P}_0/\Gamma), C(\bar{P}_0/\Gamma)) \end{array}$$

We see commutativity of this diagram as follows. Let us take $b = (E, D) \in E(\tilde{P}_1/\Gamma, *)$ and define:

$$\tilde{\sigma}_{p_0}^E : S \mapsto S(C(\bar{P}_0) \otimes K)^\Gamma$$

by $\tilde{\sigma}_{p_0}^E(f)(p_0) = \sigma_{p_0}(f)$, and on Γ orbit of p_0 , we define obviously. On each simplex spanned by Γp_0 , we extend linearly.

Then by (s-) Lipschitzness of α_1 , $\tilde{\sigma}_{p_0}^E$ is asymptotically equivalent to $(\alpha_1)^*(\omega) \circ \tilde{b}$. Let $i : (C(\bar{P}_0/\Gamma) \otimes C^*\Gamma)^n \cong E \oplus E'$ be an isomorphism as $C(\bar{P}_0/\Gamma) \otimes C^*\Gamma$ modules, and P be the projection to the first factor. Then the above follows from asymptotic equivalence between $f \mapsto P \otimes_{C^*\Gamma} \sigma_{p_0}^E(f)$ and $f \mapsto i^{-1} P \tilde{\sigma}_{p_0}^E(f) P i$, since the latter is another expression of an asymptotic morphism $\tilde{\sigma}_{p_0}^E : S \mapsto SC(\bar{P}_0/\Gamma) \hat{\otimes} K$. Thus we get commutativity of the above diagram.

From this we see commutativity of the first diagram as follows: Suppose we have a family of asymptotic proper Lipschitz maps $\alpha_1 : \tilde{P}_1 \times_\Gamma \bar{P}_0 \mapsto R^{N_1}$ as above. Then for any $a \in E(\bar{P}_0/\Gamma, *)$ and $b \in E(\tilde{P}_1/\Gamma, *)$, we have the following equalities:

$$\begin{aligned} \langle A(a), \sigma_{p_0}(b) \rangle &= e \circ a \hat{\otimes} 1 \circ \sigma_{p_0}(b) = e \circ \sigma_{p_0}(b) \hat{\otimes} 1 \circ a \\ &= \alpha_1^*(\omega) \circ \tilde{b} \circ a = \langle a, b \rangle. \end{aligned}$$

In general we have approximations of $B\Gamma$ by finite complexes $P_0/\Gamma \subset P_1/\Gamma \subset \dots \subset B\Gamma$. By iterating construction of asymptotically proper Lipschitz maps on each P_i , we will get a famiy of fiberwise asymptotically proper Lipschitz maps $\{\alpha_i^t : \tilde{P}_i \times_\Gamma \bar{P}_{i-1} \mapsto R^{N_i}\}_{i \geq 1}$. In the next section, we will construct such a family.

We will see later that one can construct naturally an asymptotic proper contracting maps from combable groups to their Rips complexes, using “radial projections” $F(\cdot)(t)$. On the other hand, in order to obtain proper maps, we need to deform it along combing paths.

Let us introduce the following notion for combing. A combing F is *proper* if there exists a proper increasing function $\mu : [0, \infty) \mapsto [0, \infty)$ satisfying the following: For any $R, M \gg 0$, there exists $L = L(M, R) \gg M, R$ such that for $\gamma, \gamma' \in \Gamma$, $|\gamma|, |\gamma'| \geq L$, $d(\gamma, \gamma') \leq M$, we have:

$$d(e, F(F(\gamma)(R'), F(\gamma')(R'))(t)) \geq \mu(R)$$

for any $R' \geq R$ and $t = 0, 1, \dots$.

3.2A Multiplicity of combings: As a first step, we will construct asymptotically proper contracting (Lipschitz constant $\ll 1$) maps for proper combings of bounded multiplicity. This is a model case for finite dimensional target space, R^N .

Let $K \subset \Gamma$ be a subset and denote:

$$CK_* = \{F(*, \gamma)(t) | \gamma \in K, \quad t = 0, 1, \dots\}$$

for a fixed $* \in K$. If we donot specify the fixed point, we only write CK . For n fold cone of K , we write $C^n K = C \circ \dots \circ CK$. Let us denote:

$$\Gamma(L) = \{\gamma \in \Gamma | |\gamma| \geq L\}, \quad X(L, R) = F(R)(\Gamma(L)).$$

An *asymptotic multiplicity* of the combing is a function $l = l(R, \delta)$ which satisfies the following: For each $n \in \mathbb{N}$, there exists $R(n) \gg 0$, and for $L = L(R, \delta, n)$, $R \geq R(n)$ and any δ , we have a covering:

$$X(L, R) = \cup_{k=1}^l Y_k$$

and another coverings:

$$Y_k = \cup_j B_{k,j}$$

such that for $F(R) : \Gamma(L) \mapsto \Gamma$ and $n(\delta, X) = \{\gamma \in \Gamma(L) | d(\gamma, X) \leq \delta\}$,

$$d(F(R)^{-1}(B_{k,j}), F(R)^{-1}(B_{k,j'})) \geq \delta, \quad j \neq j'$$

$$d(e, C^n F(R)(n(\delta, F(R)^{-1}(B_{k,j}))) \geq \mu(R).$$

We say that $(X(L, R), B_{k,j})$ is (δ, R) cover of the combing. If we can take l independent of R, δ and n , then it is called bounded multiplicity.

Proof : This follows from the definitions and a remark in 3.1. Notice that $\bar{d}_0$ and $\bar{d}_1|_{\bar{P}_0}$ are Lipschitz equivalent.

Example (Ka) *If Γ is torsion free quasi geodesic bicombing, then there exists a family of Rips complexes $\{P_i\}_{i \geq 0}$ such that we have a family of fiberwise proper contracting maps:*

$$F_i : \bar{P}_{i+1} \times_{\Gamma} \bar{P}_i \mapsto \bar{P}_{i+1} \times_{\Gamma} \bar{P}_{i+1} \quad i \geq 0$$

which are fiberwise properly homotopic to the embedding.

3.2 Combable groups: Let Γ be a finitely generated group. By fixing a generating set, we obtain a connected one dimensional simplicial complex, the Cayley graph $G(\Gamma)$. It admits a metric structure.

Definition Γ is combable if there exist constants C_1, C_2 and a map:

$$F : \Gamma \times [0, \infty) \mapsto \Gamma$$

such that $F(\gamma)$ is a C_2 quasi Lipschitz with $F(\gamma)(0) = e$, $F(\gamma)(t) = \gamma$ for large t . Moreover for any γ_1, γ_2 , $d(\gamma_1, \gamma_2) \leq 1$, the uniform distance between $F(\gamma_1)$ and $F(\gamma_2)$ is bounded by C_1 ; $\sup_t d(F(\gamma_1)(t), F(\gamma_2)(t)) \leq C_1$.

We say that C_1 is a combing constant, and C_2 a quasi Lipschitz constant. We will use the notation $F(\gamma_1, \gamma_2) : N \mapsto \Gamma$ by $F(\gamma_1, \gamma_2)(t) \equiv \gamma_1 F((\gamma_1)^{-1} \gamma_2)(t)$.

Let Γ be a finitely generated group, with a fixed generating set. *Rips complexes* $P_n(\Gamma)$ are finite dimensional simplicial complexes whose k simplices are spanned by:

$$\{(\gamma_0, \dots, \gamma_k); |\gamma_i - \gamma_j| \leq n, \gamma_i \in \Gamma\}.$$

All Rips complexes contain $G(\Gamma)$. These satisfy the following properties:

- (1) Γ acts on $P_n(\Gamma)$ simplicially,
- (2) If Γ is torsion free, then the action of Γ is also free,
- (3) $P_n(\Gamma)/\Gamma$ is compact,
- (4) $P_n(\Gamma) \subset P_{n+1}(\Gamma) \dots$,
- (5) ([Rips]) If Γ is word hyperbolic, then $P_n(\Gamma)$ is contractible for large n .

One of the important properties of combing is as follows:

Lemma (A1) *If Γ is combable, then for any i , there exists $i' \geq i$ such that $P_i(\Gamma)$ is contractible in $P_{i'}(\Gamma)$.*

3 Deformation of maps

3.1 Asymptotic contracting maps: Let $\tilde{P}_0$ and $\bar{P}_0$ be as before. Thus these are metric spaces $(\tilde{P}_0, \tilde{d}_0)$, $(\bar{P}_0, \bar{d}_0)$. We may assume that these are Lipschitz equivalent, $C^{-1}\bar{d}_0 \leq \tilde{d}_0 \leq C\bar{d}_0$ for some constant C . Since $\tilde{P}_0/\Gamma$ is an open set in R^{N_0} , there exists a radial projection:

$$P_0 : m \in \tilde{P}_0/\Gamma \mapsto P_0(m) : \tilde{P}_0/\Gamma \mapsto B_\epsilon(m) \subset R^{N_0}$$

where $B_\epsilon(m) \subset R^{N_0}$ is ϵ ball with center $m \in R^{N_0}$. Namely $P_0(m)$ maps $\tilde{P}_0/\Gamma \setminus B_\epsilon(m)$ in $\partial B_\epsilon(m)$ and it is the identity on $B_\epsilon(m)$. Let us define:

$$N_\epsilon(\Delta) = \{(p, p') \in \tilde{P}_0 \times_\Gamma \bar{P}_0 \mid \tilde{d}_0(p, p') < \epsilon\}.$$

Lemma *There exists a fiberwise proper map: $\alpha_0 : \tilde{P}_0 \times_\Gamma \bar{P}_0 \mapsto R^{N_0}$ such that:*

$$\alpha_0|_{N_\epsilon(\Delta)} = \exp, \quad \alpha_0(\tilde{P}_0 \times_\Gamma \bar{P}_0 \setminus N_\epsilon(\Delta)) \subset R^{N_0} \setminus B_\epsilon(0).$$

Let us define α_0 by the following:

$$\alpha_0|m \times B_\epsilon(m) = \exp, \quad \alpha_0|m \times \bar{P}_0 \setminus B_\epsilon(m)(x) = \epsilon^{-1}\tilde{d}_0(x, m)P_0(m)(x).$$

This is the desired map. Notice that there exists a constant C such that:

$$d(\alpha_0(x, y), \alpha_0(x, y')) \leq C\{d(x, y) + d(x, y')\}.$$

By shifting suffix, we have a family of maps: $\{\alpha_i : \tilde{P}_i \times_\Gamma \bar{P}_i \mapsto R^{N_i}\}_{i \geq 0}$.

In general Rips complexes are not selfcontracting. Thus one cannot expect to have asymptotically proper selfcontracting maps on these spaces. In fact the existence of the family implies contractibility of the space. On the other hand for large class of discrete groups, P_i is contractible in $P_{i'}$ for large $i' \gg i$. This is the case for combable groups (see 3.2).

Proposition *Suppose we have a family of asymptotically proper contracting maps:*

$$G_t : \tilde{P}_1 \times_\Gamma \bar{P}_0 \mapsto \tilde{P}_1 \times_\Gamma \bar{P}_1$$

such that $G_1 = \text{embedding}$, and the fiberwise contracting constants satisfy:

$$\text{Lip} G_t|_{N_t(\Delta)} \leq Ct^{-1}$$

for some constant C . Then the restriction $\alpha_1|_{\tilde{P}_1 \times_\Gamma \bar{P}_0}$ in 3.1 is fiberwise properly homotopic to a fiberwise asymptotically proper Lipschitz maps:

$$\alpha_t \equiv \alpha_1 \circ G_t : \tilde{P}_1 \times_\Gamma \bar{P}_0 \mapsto R^{N_1}.$$

Example *Hyperbolic groups and quasi geodesic bicomblings are all proper combings of bounded multiplicity.*

3.3 Non proper contracting maps: Let us take sufficiently large constants r and δ , and the corresponding $L = L(\delta, r, n)$ in 3.2, where n is large which is determined by r . Let us take a maximal δ net $n(\delta)$ in $\Gamma(L)$. A maximal δ net is a maximal subset of $\Gamma(L)$ whose distinct points have distance at least δ .

Let F be a combing on Γ and define:

$$G_r : n(\delta) \mapsto G(\Gamma)$$

by $G_r(\gamma) = F(\gamma)(r)$. A priori, this is a $2kr\delta^{-1}$ Lipschitz map, where $k > 0$ is determined by the combing of Γ . Here it is important to hold large distance of the images from the base point. Let us denote:

$$An(A, B) \equiv \{\gamma \in \Gamma \mid A \leq |\gamma| \leq B\}.$$

Then by properness of F , there exist a proper increasing map

$$\mu : [0, \infty) \mapsto [0, \infty)$$

and constants A and B such that:

$$F(r)(\Gamma(L)) \subset An(\mu(r), Ar + B).$$

The rest of 3.3 will be devoted to extend this non proper contracting map.

3.3A Contracting homotopies: For any subset $K \subset \Gamma$, we denote $K_i \subset P_i$ to be the set of simplices spanned by elements of K .

Let us take $K \subset \Gamma(L)$. By definition of combing, there is a constant C such that for any $\gamma, \gamma' \in K$, we have:

$$|F(r)(\gamma) - F(r)(\gamma')| \leq C|\gamma - \gamma'|.$$

Thus for each i and $i' = Ci$, $F(r) : K \mapsto An(\mu(r), Ar + B) \subset \Gamma$ extends simplicially to the next continuous map:

$$F(r) : K_i \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

where $An^i(A, B) \equiv \{x \in \bar{P}_i \mid A \leq d_i(p_0, x) \leq B\}$, and μ_i and μ are Lipschitz equivalent, $C_i^{-1}\mu_i \leq \mu \leq C_i\mu_i$. A_i, B_i and C_i all depend only on i .

Let us take $K \subset \Gamma(L)$ with $\text{diam} K \leq \delta$ and connect any element $k \in K$ with a base point $*$ in $F(r)(K)$ by the combing:

$$F(*, k) : \{0, 1, \dots\} \mapsto An(\mu(r), Ar + B).$$

By definition of properness, these paths are also contained in $An(\mu(r), Ar + B)$. Let us contract $F(r)(K)$ to $*$ along these paths. Then the contraction holds distances of each points at most by multiple C , where C is the same constant as above. Thus by the construction of Rips complexes, this contraction extends to a homotopy:

$$C_s(F(r)(K)_i) : [0, 1] \times F(r)(K)_i \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

such that $C_0(F(r)(K)_i) = id$, $C_1(F(r)(K)_i) = *$.

One may assume contractibility of the image of $C_s(F(r)(K)_i)$. In fact it follows from the Whitney's embedding theorem that by slightly perturbing, we can make $C_s(F(r)(K)_i)$ to be embedding, $0 \leq s < 1 - \epsilon$, when the dimension restriction is satisfied: $\dim \bar{P}_i \geq 2\dim K_i + 3$. In particular we may assume that the image of $C_s(F(r)(K)_i)$ is itself contractible in $An^{i'}(\mu_i(r), A_i r + B_i)$. We will use this when we remove the condition of bounded multiplicity.

3.3B Lipschitz extension: Let us recall some notions. A net constant for the embedding $\Gamma \subset \bar{P}_i$ is a positive number C_i such that for any $x \in \bar{P}_i$, there exists $\gamma \in \Gamma$ with $\bar{d}_i(x, \gamma) \leq C_i$.

A metric space V is uniformly locally bounded (ULB) iff there exists a covering $\{B_i\}$ of V such that:

$$(1) \sup_i \text{diam} B_i \leq \beta < \infty,$$

(2) for every $x \in Y$, the number of $\{i | B_R(x) \cap B_i \neq \emptyset\} \leq B(R)$ where $B(R)$ depends only on R .

Our basic lemma to construct Lipschitz maps is the following ([CGM], see[Ka] 3.3): Let (V, W) be a pair of ULB simplicial complexes with Lipschitz equivalent metrics, and take any metric space Y and its subspace Z of ϵ net in Y .

Proposition (CGM) *Suppose V is contractible in W . Then there exists a constant $C = C((V, W), \epsilon\lambda)$ such that any λ Lipschitz map $G_0 : Z \mapsto V$ extends to a $C\lambda$ Lipschitz map $G : Y \mapsto W$.*

For any subset $M \in \Gamma(L)$, let us denote:

$$n(\delta, M) = \{\gamma \in \Gamma(L) | d(\gamma, M) \leq \delta\}.$$

Similarly we denote $n_i(\delta, M) = \{x \in \bar{P}_i | \bar{d}_i(x, n(\delta, M)) \leq C_i\}$ where C_i is a net constant. Let $(X_{L,r} = \cup_k Y_k, B_{k,j})_{k=1}^l$ be a (δ, r) cover of the combing. Using contraction $C_s((B_{k,j})_i)$, let us extend $G_r : n(\delta) \cap F(r)^{-1}(B_{1,j}) \mapsto B_{1,j}$:

$$G_r : n_i(C_i, F(r)^{-1}(B_{1,j})) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

The Lipschitz constant increases at most by multiple $C = C(r)$, independent of δ , because $2kr\delta\delta^{-1}$ is independent of δ . Thus it is $C(r)\delta^{-1}$ Lipschitz map. Since the image itself is also contractible, using homotopy extension theorem, we have a homotopy $G_r^s : n_i(C_i, F(r)^{-1}(B_{k,j})) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$ between G_r and $F(r)$.

By constructing G_r on each $n_i(C_i, F(r)^{-1}(B_{1,j}))$, we have:

$$G_r : n_i(C_i, Y_1) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

and a homotopy G_r^s between G_r and $F(r)$. Let us take two points x and y in $n_i(C_i, Y_1)$ which are δ separated, $d(x, y) \geq \delta$. Then a priori, $G_r|_{\{x, y\}}$ is a $kr\delta^{-1}$ Lipschitz map. In particular, by the δ separation of each $F(r)^{-1}(B_{k,j})$ and $F(r)^{-1}(B_{k',j'})$, $j \neq j'$, G_r is itself $C_i(r)\delta^{-1}$ Lipschitz map. Notice that the image $G_r^s(n_i(C_i, F(r)^{-1}(B_{1,j})))$ is contained in $n_{i'}(C_{i'}, C_*(B_{1,j}))$.

3.3C Iteration: Let us continue this extension process to the next step. For each $F(r)(\gamma)$, $\gamma \in n(\delta, F(r)^{-1}(B_{2,j})) \setminus F(r)^{-1}(B_{2,j})$, let us connect each $F(r)(\gamma)$ and $*$ in $B_{2,j}$ by the combing, by choosing fixed points $*$. We may assume that these paths are disjoint from each other, and thus contractible to $*$. Suppose $n_i(C_i, Y_1) \cap n_i(\delta, F(r)^{-1}(B_{2,j})) = \emptyset$. Then we can iterate the same construction as above, and get the desired $C(r)\delta^{-1}$ Lipschitz map:

$$G_r : n_i(C_i, Y_1 \cup F(r)^{-1}(B_{2,j})) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

Next suppose $n_i(C_i, F(r)^{-1}(B_{1,j})) \cap n_i(\delta, F(r)^{-1}(B_{2,j})) \neq \emptyset$. Let us take $p \in G_r(n_i(\delta, F(r)^{-1}(B_{2,j})))$ and take a cone:

$$C_p[n_{i'}(C_{i'}, C_*(B_{1,j})) \cup F(r)(n_i(\delta, F(r)^{-1}(B_{2,j})))] \subset An^{i'}(\mu_i(r), A_i r + B_i). \quad *$$

Thus by embedding $P_{i'}$ into more larger space $P_{i''}$, $i' < i''$, we get a contraction C_s which contracts $*$ to any fixed point in $An^{i''}(\mu_{i''}(r), A_{i''}r + B_{i''})$. By iterating the above procedure, we get an extension of $C(r)\delta^{-1}$ Lipschitz map:

$$G_r : n_i(\delta, F(r)^{-1}(B_{2,j})) \mapsto An^{i''}(\mu_{i''}(r), A_{i''}r + B_{i''})$$

where we change its Lipschitz constant $C(r)$, if necessary, which only depends on r and i . Thus we get a $C(r)\delta^{-1}$ Lipschitz map:

$$G_r : n_i(C_i, Y_1 \cup Y_2) \mapsto An^{i''}(\mu_{i''}(r), A_{i''}r + B_{i''})$$

and a homotopy $G_r^s : [0, 1] \times n_i(Y_1 \cup Y_2) \mapsto An^{i''}(\mu_{i''}(r), A_{i''}r + B_{i''})$ between $G_r^0 = F(r)$ and $G_r^1 = G_r$.

We continue this procedure inductively on Y_k , $0 \leq k \leq l$, and finally get a $C(r)\delta^{-1}$ Lipschitz map:

$$G_r : \bar{P}_i \setminus B_L(p_0) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

and a homotopy G_r^s between $F(r)$ and G_r .

Let us take another $P_{i''}$ in which $P_{i'}$ is contractible. Using contractibility of $An^{i'}(\mu_i(r), A_i r + B_i)$ in $P_{i''}$, we have an extension of $C(r)\delta^{-1}$ Lipschitz:

$$G_r : \bar{P}_i \mapsto \bar{P}_{i''}$$

and an extension of homotopy $G_r^s : [0, 1] \times \bar{P}_i \mapsto \bar{P}_{i''}$. By choosing δ sufficiently large, we may assume $C(r)\delta^{-1} \ll r^{-1}$. Combined, we get:

Lemma G_r^s has the following properties:

- (1) G_r^1 is a $C(r)\delta^{-1}$ contracting map and $G_r^0 = F(r)$,
- (2) there exists a large constant $L \gg r$ such that

$$G_r^s(\bar{P}_i \setminus B_L(p_0)) \subset An^{i''}(\mu_i(r), A_i r + B_i).$$

3.4 Deformation along combing: Let us choose sufficiently large constants:

$$L \ll L' \ll L''.$$

and deform G_r to G'_r so that it satisfies:

$$(1) \quad G'_r|_{B_{L'}(p_0)} = G_r, \quad (2) \quad G'_r|_{\bar{P}_i \setminus B_{L''}(p_0)} = F(r).$$

Let us choose another constant M so that $\Gamma(M) \subset \bar{P}_i \setminus B_{L''}^i(p_0)$ and deform $G'_r|_{\Gamma(M)} = F(r)$ as follows: First we define $G''_r : \Gamma(M) \mapsto G(\Gamma)$ by:

$$G''_r(\gamma) = \begin{cases} F(2Ai + r)(\gamma) & |\gamma| = M + i \quad |2Ai + r| \leq b_\gamma, \\ \gamma & 2Ai + r \geq b_\gamma. \end{cases}$$

Notice that if $|\gamma|$ is sufficiently large, then $G''_r(\gamma)$ is γ . By the definition of combing, there exists a constant C , which only depends on the combing such that

$$d(G''_r(\gamma), G''_r(\gamma')) \leq Cd(\gamma, \gamma') \quad \gamma, \gamma' \in \Gamma(M)$$

By taking i' sufficiently large, we get a simplicial extension of G''_r :

$$G''_r : P_i \setminus B_{L''}^i(p_0) \mapsto P_{i'}.$$

Since $\bar{P}_i$ is a deformation retract of P_i , we get finally the desired map $G_r : \bar{P}_i \mapsto \bar{P}_{i'}$ with the following properties; there exist large constants $L_1 \ll L_2$ such that:

- (1) $G_r|_{B_{L_1}^i(p_0)}$ is $C(r)\delta^{-1} \ll L_1^{-1}$ contracting,
- (2) $G_r(\bar{P}_i \setminus B_{L_1}^i(p_0)) \subset \bar{P}_{i'} \setminus B_{\mu(r)}^{i'}(p_0)$,
- (3) $G_r|_{\bar{P}_i \setminus B_{L_2}^i}$ is the identity.

3.5 Continuous family of maps: Let us connect G_r continuously with respect to the parameter $r \in [r_0, \infty)$. Let $r_0 \in N$ be sufficiently large, and suppose we have G_r , $r = r_0 + k$, $k = 0, 1, \dots$ by the previous construction. We will use Lipschitz extension theory for families developed in [CGM] (see[Ka] 3.7).

Let $Y_1' \subset Y_1$ be a compact pair and Y_2 be a geodesic space. Suppose we have closed ϵ nets $X \subset X' \subset Y_2$ and put $Z = Y_1' \times X' \cup Y_1 \times X \subset Y_1 \times Y_2$. Let us consider a family of λ Lipschitz maps $G_0 : Z \mapsto V$, where we regard Y_1 as the parameter space.

Lemma (CGM) *Let (V, W) be in 3.3B. If V is contractible in W , then there is an extension of a family of $C\lambda$ Lipschitz maps $G : Y_1 \times Y_2 \mapsto W$, where C depends on $\epsilon\lambda$, Y_1 , Y_2 and W .*

This is a consequence of the metric extension lemma as follows: Let d be a metric on $Z = Y_1 \times X \cup Y_1' \times X'$ such that:

$$d|_{Y_1' \times X'} = d_{Y_2}|_{X'}, \quad d|_{Y_1 \times X} = d_{Y_2}|_X$$

for any $y_1' \in Y_1'$ and $y_1 \in Y_1$. Then there exists a metric $\bar{d}$ on $Y_1 \times Y_2$ such that:

$$\bar{d}|_{Y_1 \times Y_2} = d_{Y_2}, \quad 2\bar{d}|_Z \geq d.$$

Let d_0 be a product metric on $Y_1 \times Y_2$. Then by applying this metric extension lemma to another metric $\max\{d_0, \lambda^{-1}G_0^*d_W\}$ on Z , we get $\bar{d}$. Using this metric, the above lemma follows from 3.3B, since $G_0 : (Z, \bar{d}) \mapsto V$ is itself λ Lipschitz map.

Let us return to our G_r . During constructing each G_{r_α} , $r_\alpha = r_0 + \alpha$, $\alpha \geq 0$, we have large constants δ_α and L_α . By taking these inductively, we may assume:

$$\delta_\alpha \ll \delta_{\alpha+1}, \quad L_\alpha \ll L_{\alpha+1}, \quad n(\delta_{\alpha+1}) \subset n(\delta_\alpha), \quad \alpha \geq 0.$$

Let $r \in [r_\alpha, r_{\alpha+1}]$ and define $G_r : [r_\alpha, r_{\alpha+1}] \times n(\delta_{\alpha+1}) \mapsto G(\Gamma)$ by $G_r(\gamma) = F(r)(\gamma)$.

Let us make the previous construction for families. Let:

$$X(L_\alpha, r_\alpha) = \cup_{k=1}^l Y_k^\alpha, \quad Y_k^\alpha = \cup_j B_{k,j}^\alpha$$

be $(\delta_\alpha, r_\alpha)$ coverings, $\alpha = 0, 1, \dots$. By the previous construction, we have a non proper contraction:

$$G_\alpha : \bar{P}_i \mapsto \bar{P}_{i'}$$

for each r_α . Notice that:

$$G_\alpha(n_i(C_i, F(r_{\alpha+1})^{-1}(B_{k,j}^{\alpha+1})) \cup G_{\alpha+1}(n_i(C_i, F(r_{\alpha+1})^{-1}(B_{k,j}^{\alpha+1})))$$

is contained in $An^{i'}(\mu_i(r_\alpha), A_i r_{\alpha+1} + B_i)$. By taking $i'' \geq i'$, it can be contracted in $An^{i''}(\mu_i(r_\alpha), A_i r_{\alpha+1} + B_i)$. Thus we have the following map:

$$G_\tau^s : \{r_\alpha, r_{\alpha+1}\} \times n_i(C_i, F(r_{\alpha+1})^{-1}(B_{1,j}^{\alpha+1})) \cup [r_\alpha, r_{\alpha+1}] \times \{n(\delta_{\alpha+1}) \cap F(r_{\alpha+1})^{-1}(B_{1,j}^{\alpha+1})\} \mapsto \bar{P}_{i'}$$

where $G_\tau^s|_{r_\tau \times n_i(C_i, F(r_{\alpha+1})^{-1}(B_{1,j}^{\alpha+1}))} = G_\tau^s$, $\tau = \alpha, \alpha + 1$. Thus by taking cones, and using the above proposition, we get an extension:

$$G_\tau^s : [r_\alpha, r_{\alpha+1}] \times n_i(C_i, F(r_{\alpha+1})^{-1}(B_{1,j}^{\alpha+1})) \mapsto \bar{P}_{i''}$$

which is fiberwise $C(r_{\alpha+1})\delta_\alpha^{-1}$ contracting.

Next let us take another j' , and make the same construction. We can construct extensions on each $F(r_{\alpha+1})^{-1}(B_{1,j'}^{\alpha+1})$ individually, since we have the following bound:

$$d(F(r_{\alpha+1})^{-1}(B_{1,j}^{\alpha+1}), F(r_{\alpha+1})^{-1}(B_{1,j'}^{\alpha+1})) \geq \delta_{\alpha+1}.$$

Thus we get an extension: $G_\tau^s : [r_\alpha, r_{\alpha+1}] \times n_i(C_i, F(r_{\alpha+1})^{-1}(Y_1^{\alpha+1})) \mapsto \bar{P}_{i''}$. By doing the same thing inductively on $k = 1, 2, \dots$, we get a family of $C(r_{\alpha+1})\delta_\alpha^{-1}$ contracting maps:

$$G_\tau^s : [r_\alpha, r_{\alpha+1}] \times n_i(C_i, \Gamma(L_{\alpha+1})) \mapsto \bar{P}_{i'}.$$

Again by extending:

$$G_\tau^s|_{\{r_\alpha, r_{\alpha+1}\} \times \bar{P}_i \cup [r_\alpha, r_{\alpha+1}] \times n_i(C_i, \Gamma(L_{\alpha+1}))}$$

to $G_\tau^s : [r_\alpha, r_{\alpha+1}] \times \bar{P}_i \mapsto \bar{P}_{i'}$, we get the desired family.

Next let us make deformation as before. Let us choose large constants as before $L_\alpha \ll L'_\alpha \ll L''_\alpha$. By taking linear combinations $L_r = (r_{\alpha+1} - r)L_\alpha + (1 - r_{\alpha+1} + r)L_{\alpha+1}$, we can construct a deformation of family C'_r which satisfies:

$$G'_r|B^i_{L'_r}(p_0) = G_r, \quad G'_r|\bar{P}_i \setminus B^i_{L''_r}(p_0) = F(r).$$

The rest is the same as before, using the ordinary homotopy extension theorem. Thus we get the desired deformation:

$$G_r : [r_\alpha, r_{\alpha+1}] \times \bar{P}_i \mapsto \bar{P}_{i'}.$$

3.6 Family of maps: We have constructed asymptotic Lipschitz maps for one variable case. We need more construction, namely for families.

Definition *Let us take i . A family of maps:*

$$G_t : [1, \infty) \times \bar{P}_{i+1} \times_\Gamma \bar{P}_i \mapsto \bar{P}_{i+1} \times_\Gamma \bar{P}_{i+1}$$

is called Γ properly as-contracting if there exist a constant C and proper increasing functions:

$$\mu, l : [0, \infty) \mapsto [0, \infty) \quad \mu(t) \ll t \ll l(t),$$

such that:

- (1) $G_t|N_t(\Delta) \mapsto \bar{P}_{i+1} \times_\Gamma \bar{P}_{i+1}$ is fiberwise $C_i t^{-1}$ Lipschitz,
- (2) $G_t(\bar{P}_{i+1} \times_\Gamma \bar{P}_i \setminus N_t(\Delta)) \subset (N_{\mu(t)}(\Delta))^c \subset \bar{P}_{i+1} \times_\Gamma \bar{P}_{i+1}$,
- (3) $G_t|\bar{P}_i \times_\Gamma \bar{P}_i \setminus N_{l(t)}(\Delta)$ is the identity.

From 3.1, if we have a Γ properly as-contracting map, then we can construct naturally a family of asymptotic proper Lipschitz maps. In order to construct such maps, we will take a family version of the previous steps. We will use Proposition in 3.5.

Let us take a covering $\{Y_l\}$, $1 \leq l \leq k$ of $\bar{P}_{i+1}$ such that:

$$\gamma(Y_l) \cap Y_l = \emptyset, \quad \gamma \neq id, \quad \gamma(Y_l) = Y_{l'}, \quad \text{diam } Y_l < \epsilon.$$

These are called Γ covering of $\bar{P}_{i+1}$. Then by constructing a family of contracting maps $G_l : Y_l \times \bar{P}_i \mapsto \bar{P}_{i+1}$ inductively on l , we have the following family (see [CGM], [Ka]): Let us denote:

$$\begin{aligned} \bar{N}_i(A, B) &= \{(p, p') \in \bar{P}_i \times_\Gamma \bar{P}_i | A \leq \bar{d}_i(p, p') \leq B\}, \\ \bar{N}_A^i(\Delta) &= \{(p, p') \in \bar{P}_i \times_\Gamma \bar{P}_i | \bar{d}_i(p, p') \leq A\}. \end{aligned}$$

Lemma For each t and any large δ , there exists a family of maps:

$$G_t^s : [1, \infty) \times \bar{P}_{i+1} \times_{\Gamma} \bar{P}_i \mapsto \bar{N}_{A_i t + B_i}(\Delta)^i$$

such that (1) $G_t^0 = F(t)$, G_t^1 is $C(t)\delta^{-1}$ contracting,
(2) $G_t^s(\bar{P}_{i+1} \times_{\Gamma} \bar{P}_i \setminus \bar{N}_t(\Delta)) \subset \bar{N}_i(\mu_i(t), A_i t + B_i)$.

The rest of the process is the same as before. Thus we get a Γ properly as-
contracting map:

$$G_t : [1, \infty) \times \bar{P}_{i+1} \times_{\Gamma} \bar{P}_i \mapsto \bar{P}_{i+1} \times_{\Gamma} \bar{P}_{i+1}.$$

3.7 Modification near infinity: In order to obtain a family of asymptotic
proper Lipschitz maps:

$$\alpha_i : [1, \infty) \times \tilde{P}_{i+1} \times_{\Gamma} \tilde{P}_i \mapsto R^{N_{i+1}}$$

we need slightly technical modification near infinity of $\tilde{P}_{i+1}$. This is done in [Ka]
p357. For convenience, we briefly sketch this. Let us recall that $\tilde{P}_{i+1}/\Gamma \subset R^{N_{i+1}}$
is an open tubular neighbourhood. By reembedding in $R^{N_{i+1}} \subset R^{N_{i+1}} \oplus R^2$, we
have another open tubular neighbourhood $\tilde{P}'_{i+1}/\Gamma$, which contains the original
 $\tilde{P}_{i+1}/\Gamma$. Let $r : \tilde{P}'_{i+1} \mapsto \tilde{P}_{i+1}$ be the retraction. Then we obtain a family of
asymptotic proper Lipschitz maps:

$$\alpha'_i : [1, \infty) \times \tilde{P}'_{i+1} \times_{\Gamma} P_i \mapsto R^{N_{i+1}} \oplus R^2$$

by $\alpha'_{i+1}(t, p, p') = \alpha_{i+1}(t, F(p, p')) + l(p, p')v$, where $v = (0, (1, 0)) \in R^{N_{i+1}} \oplus R^2$, F is the identity near the diagonal, and is equal to $(r(p), p')$ except small
neighbourhood of the diagonal. Thus $\alpha'_i([1, \infty) \times \partial \tilde{P}'_{i+1} \times_{\Gamma} P_i) \subset R^{N_{i+1}} \oplus 0$. l
has the following property: For sufficiently small ϵ' ,

$$l(p, p') = 0 \quad \text{if } d'(p, p') \equiv \frac{d(p, p')}{d(\partial \tilde{P}'_{i+1}, p)} \leq \epsilon', \quad = \frac{1}{d(\partial \tilde{P}'_{i+1}, p)} \quad \text{if } d'(p, p') \geq 2\epsilon'.$$

α'_i is the desired family of asymptotic proper Lipschitz maps.

3.8 Maps into R^∞ : In order to remove multiplicity condition, we must be
careful about the number of times of extensions. In fact in this situation it
may increase with respect to t . Then the previous argument does not work
for this case, because if we do that, the target space P_i may become larger
and larger when r increases. Thus we need more careful method. We will use

the property that the image of contracting homotopies, in the previous section, itself is contractible. This is done by slightly perturbing the original contracting homotopies. However here we need dimension restriction, $\dim \mathcal{P}_i \geq 2\dim K_i + 3$. In order to use these contractions iteratedly, we take infinite dimensional tubular neighbourhood of P_i . In this section, we will construct a family of maps:

$$G_r : \tilde{P}_{i+1} \times_\Gamma \bar{P}_i \mapsto \tilde{P}_{i+1} \times_\Gamma \bar{P}_{i+1} \times R^\infty$$

such that by letting R^∞ factor to be zero, we get a Γ properly as-contracting map, which is the desired one. In fact we may assume that R^∞ factor of the image of G_r is contained in the unit cube, because our perturbation is sufficiently small. Then by composing with α_{i+1} , we have a family of asymptotically proper Lipschitz maps:

$$\alpha'_{i+1} : \tilde{P}_{i+1} \times_\Gamma \bar{P}_i \mapsto R^{N_{i+1}}.$$

3.8A Combing of unbounded multiplicity: Let $F : N \times \Gamma \mapsto G(\Gamma)$ be a proper combing with an increasing proper map $\mu : [0, \infty) \mapsto [0, \infty)$. Our step starts from constructing non-proper contracting maps from $\bar{P}_i$ to $R^\infty \times \bar{P}_{i+1}$ for a subfamily of Rips complexes $\{P_i\}_{i \geq 0}$.

Let us choose large constants r, δ and the corresponding $L = L(r, \delta)$. Let $n(\delta) \subset \Gamma(L)$ be a maximal δ net, and define $G_r : n(\delta) \mapsto An(\mu(r), Ar + B)$ to be $F(r)$ as before. For any $p \in G_r(n(\delta))$, let us define:

$$n_f(\delta, p) = \{\gamma \in \Gamma(L) \mid d(\gamma, G_r^{-1}(p)) \leq \delta\}.$$

We also use a notation $n_f^i(\delta, p) = \{x \in \bar{P}_i \mid d_i(x, n_f(\delta, p)) \leq C_i\}$ where C_i is a net constant of the embedding $\Gamma \subset \bar{P}_i$.

Sometimes we replace L by larger ones L' so that $L' \geq L(r, k\delta)$, where k may depend on r , but independent of δ . Now by taking L larger, we can contract $F(r)(n_f(2\delta, p))$ to $F(r)(p)$ in $An(\mu(r), Ar + B)$. By the definition of combing, we have an extension of contracting maps:

$$C_s(p) : [0, 1] \times n_i(C_i, F(r)(n_f(2\delta, p))) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

From the Whitney's embedding theorem, we may assume that the image of C_s itself is contractible if the dimension of $\bar{P}_i$ is larger than $2\dim \bar{P}_i + 3$. Now from 3.3B, we have a $C(r)\delta^{-1}$ contracting map G_r and a homotopy between G_r and $F(r)$:

$$G_r^s : n_f^i(2\delta, p) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

Since δ is sufficiently large, we have a $C(r)\delta^{-1}$ contracting map and a homotopy as above:

$$G_r^s : n(\delta) \cup n_f^i(\delta, p) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

We will continue this step. Let us order points in $F(r)(n(\delta))$ arbitrarily as $\{p_0, p_1, \dots, p_l\}$, $p_0 = p$. Suppose $n_f^i(2\delta, p_1) \cap n_f^i(\delta, p_0) = \phi$. Then by the same step, we have an extension of $C(r)\delta^{-1}$ contracting map and a homotopy:

$$G_r^s : n_f^i(\delta, p_0 \cup p_1) \cup n(\delta) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

Suppose the intersection is not empty, and take $x \in n_f^i(2\delta, p_1) \cap n_f^i(\delta, p_0) \neq \phi$. Then there exist $x' \in G_r^{-1}(p_0) \cap n(\delta)$ and $x'' \in G_r^{-1}(p_1) \cap n(\delta)$ satisfying:

$$d_i(x, x') \leq \delta + C_i, \quad d_i(x, x'') \leq 2\delta + C_i, \quad d_i(x', x'') \leq 3\delta + 2C_i.$$

In particular by taking L larger, we can extend the contraction to:

$$C_s(p_0) : [0, 1] \times n_i(C_i, F(r)(n_f(2\delta, p_0)) \cup p_1) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

Let μ be a positive number and put:

$$n_f(\mu, p_0, p_1) = \cup_{\gamma} \{B_{\mu}(\gamma) \subset \Gamma | \gamma \in G_r^{-1}(p_1) \cap n(\delta), \quad B_{2\delta}(\gamma) \cap n_f(\delta, p_0) = \phi\}.$$

First of all we have another $C(r)\delta^{-1}$ contracting maps and a homotopy:

$$G_r^s : n_i(C_i, n_f(2\delta, p_0, p_1)) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

using contraction:

$$C_s(p_1) : n_i(C_i, F(r)(n_f(2\delta, p_1))) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

3.8B Deformation of contractions: Next we want to extend this to a map from $n_f^i(2\delta, p_1)$. In order to success this process iteratedly, we need to be careful when using contracting homotopies. Let us call a family of maps $C_{s,t} : [0, 1]^2 \times K \mapsto K$ is a *deformation of contractions* if:

$$(1) \quad C_{0,t} = id, \quad (2) \quad C_{1,t} = p(t) : \text{constants}.$$

Let us put:

$$n_f^1(\mu, p_0, p_1) = \cup_{\gamma} \{B_{\mu}(\gamma) \subset \Gamma | \gamma \in G_r^{-1}(p_1) \cap n(\delta), \quad B_{2\delta}(\gamma) \cap n_f(\delta, p_0) \neq \phi\}$$

and $K = F(r)(n_f^1(2\delta, p_0, p_1))$, $K_l = F(r)(n_f(2\delta, p_l))$ $l = 0, 1$. Let us take $p' \in F(r)(n_f^1(2\delta, p_0, p_1))$. Then we have a contracting homotopy:

$$C_s(p') : [0, 1] \times n_i(C_i, K_0 \cup K_1) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

Let us construct deformations of contractions $C_{s,t}(p', p_l)$, $l = 0, 1$ between $C_s(p_0), C_s(p')$ and $C_s(p_1), C_s(p')$. Let us take a path $p_l : [0, 1] \mapsto An(\mu(r), Ar + B)$ between p_0, p' and p_1, p' respectively, consisted by the combing paths $F(p_0, p')$ and $F(p', p_1)$. We may assume $\cup_{0 \leq t \leq 1} p_l(t)$ is contained in K_l . Then we have a family of maps $F(r)(\cdot, p_l(t)) : [0, \infty) \times [0, 1] \times K_l \mapsto An(\mu(r), Ar + B)$. By changing parameter, we have the initial condition:

$$C_{s,t}(p', p_l) : \partial[0, 1]^2 \times n_i(C_i, K_l) \mapsto An^{i'}(\mu_i(r), A_i r + B_i) \quad l = 0, 1.$$

We may assume that the image is contained in $C_s(p')([0, 1] \times n_i(C_i, K_l)) \cup C_s(p_l)([0, 1] \times n_i(C_i, K_l))$, which is contractible in $An^{i'}(\mu_i(r), A_i r + B_i)$. Thus we have an extension of the map, which is by definition, a deformation of contractions $C_{s,t}(p', p_l) : [0, 1]^2 \times n_i(C_i, K_l)$ between $C_s(p')$ and $C_s(p_l)$. By the same way as above, in fact we have deformations of contractions:

$$C_{s,t}(p', p_l) : [0, 1]^2 \times n_i(C_i, K_0 \cup K_1) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

such that $C_{s,0}(p', p_l)|_{n_i(C_i, K_l)} = C_s(p_l)$, $C_{s,1}(p', p_l) = C_s(p')$. Thus we have shown:

Lemma *There exists a deformation of contractions:*

$$C_{s,t}(p_0, p_1) : [0, 1]^2 \times n_i(C_i, K_0 \cup K_1) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

such that $C_{s,l}(p_0, p_1)|_{n_i(C_i, K_l)} = C_s(p_l)$, $l = 0, 1$. Moreover we may assume that its image is contractible.

Thus we have an extension of $C(r)\delta^{-1}$ contracting maps and a homotopy between the extension and $F(r)$:

$$G_r^s : [0, 1] \times n(\delta) \cup n_f^i(\delta, p_0 \cup p_1) \mapsto An^{i'}(\mu_i(r), A_i r + B_i).$$

3.8C Iterated extensions 1: Let us take $p_2 \in F(r)(n(\delta))$. In this step, the dimension of the target space which we need increases. Let us put:

$$n_f(\mu, p_0, p_1, p_2) = \cup \{B_\mu(\gamma) \subset \Gamma(L) \mid \gamma \in G_r^{-1}(p_2) \cap n(\delta), \\ B_{2\delta}(\gamma) \cap n_f(\delta, p_0 \cup p_1) = \emptyset\}.$$

First we have $G_r^s : n_i(C_i, n_f(2\delta, p_0, p_1, p_2)) \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$, using a contracting homotopy $C_s(p_2)$. Next let us put:

$$n_f^1(\delta, p_0, p_1, p_2) = \cup \{B_\mu(\gamma) \subset \Gamma \mid \gamma \in G_r^{-1}(p_2) \cap n(\delta), \\ B_{2\delta}(\gamma) \cap \{n_f(\delta, p_0) \cup n_f^1(2\delta, p_0, p_1)\} = \emptyset, \quad B_{2\delta}(\gamma) \cup n_f(\delta, p_0, p_1) \neq \emptyset\}.$$

$$n_f^2(\mu, p_0, p_1, p_2) = \cup \{B_\mu(\gamma) \subset \Gamma \mid \gamma \in G_\tau^{-1}(p_2) \cap n(\delta), \\ B_{2\delta}(\gamma) \cap n_f(\delta, p_0) = \phi, \quad B_{2\delta}(\gamma) \cap n_f^1(\delta, p_0, p_1) \neq \phi\}.$$

We can extend G_τ on $n_i(C_i, n_f^1(2\delta, p_0, p_1, p_2))$ using $C_{s,t}(p_1, p_2)$ similarly as before.

Let us extend G_τ on $n_f^2(2\delta, p_0, p_1, p_2)$. If $n_f^2(2\delta, p_0, p_1, p_2) \neq \phi$, then there exist $\gamma(p_i) \in G_\tau^{-1}(p_i) \cap n(\delta)$ with:

$$d(\gamma(p_i), \gamma(p_1)) \leq 3\delta, \quad i = 0, 2, \quad d(\gamma(p_0), \gamma(p_2)) \leq 6\delta.$$

Let $\Delta(p_0, p_1, p_2)$ be a triangle spanned by $F(r)(p_0, p_1)$, $F(r)(p_1, p_2)$, $F(r)(p_2, p_0)$. Then there exists a domain $\bar{\Delta}(p_0, p_1, p_2)$ in $An^i(\mu_i(r), A_i r + B_i)$ which contains $\Delta(p_0, p_1, p_2)$, and itself is contractible. $\bar{\Delta}(p_0, p_1, p_2)$ is parametrized by $[0, 1]^2$. Making $\bar{\Delta}(p_0, p_1, p_2)$ as a family of contracting points, we have a family of contractions:

$$C_{s,t,u}(p_0, p_1, p_2) : [0, 1] \times [0, 1]^2 \times n_i(C_i, F(r)(n_f(2\delta, p_0 \cup p_1 \cup p_2))) \\ \mapsto An^{i'}(\mu_i(r), A_i r + B_i)$$

such that $C_{0,t,u} = id$, $C_{1,t,u} = p(t, u)$: constants, and $C_{s,0,0} = C_s(p_0)$, $C_{s,0,1} = C_s(p_1)$ and $C_{s,1,0} = C_s(p_2)$. Notice that to use the Whitney's embedding theorem, we need more 2 dimensional target space; $\dim \bar{P}_{i'} \geq 2\dim \bar{P}_i + 6$. Using this contracting homotopies, we get an extension of G_τ on $n_i(C_i, n_f^2(\delta, p_0, p_1, p_2))$.

Finally let:

$$n_f^3(\mu, p_0, p_1, p_2) = \cup \{B_\mu(\gamma) \subset \Gamma \mid \gamma \in G_\tau^{-1}(p_2) \cap n(\delta), \\ B_{2\delta}(\gamma) \cap n_f(\delta, p_0) \neq \phi, \quad B_{2\delta}(\gamma) \cup n_f(\delta, p_1) = \phi\}.$$

Notice that we have $\cup_{0 \leq i \leq 3} n_f^i(\delta, p_0, p_1, p_2) = n_f(\delta, p_2)$, where $n_f^0(\delta, p_0, p_1, p_2) = n_f(\delta, p_0, p_1, p_2)$. If $n_f^2(2\delta, p_0, p_1, p_2) \neq \phi$, then this case is contained in the one for $n_f^2(\delta, p_0, p_1, p_2)$. Thus suppose $n_f^2(2\delta, p_0, p_1, p_2) = \phi$. We may have the situation:

$$n_f^3(2\delta, p_0, p_1, p_2) \cap n_f^1(\delta, p_0, p_1, p_2) \neq \phi, \quad n_f^3(2\delta, p_0, p_1, p_2) \cap n_f(\delta, p_0) \neq \phi.$$

The situation is similar to extend G_τ on $n_f^2(2\delta, p_0, p_1, p_2)$. In this case, there exist $\gamma(p_i) \in G_\tau^{-1}(p_i) \cap n(\delta)$, $i, j = 0, 1, 2$ such that

$$d(\gamma(p_0), \gamma(p_1)) \leq 9\delta, \\ d(\gamma(p_1), \gamma(p_2)) \leq 3\delta, \quad d(\gamma(p_0), \gamma(p_2)) \leq 6\delta.$$

By taking L larger, this case is also the same as the one for $n_f^2(2\delta, p_0, p_1, p_2)$. If we don't have such situation, then using a contracting homotopy $C_{s,t}(p_0, p_2)$, we have an extension as usual. Thus we complete this step by obtaining:

$$G_r^s : [0, 1] \times n_f^i(\cup_{0 \leq k \leq 2p_k}) \mapsto An^{i'}(\mu_i(r), A_i r + E_i).$$

3.8D Iterated extensions 2: In order to continue this procedure inductively, we describe the next step. From this case, it becomes clear how to proceed the construction for the general case. Let us take $p_3 \in F(r)(n(\delta))$ and put:

$$n_f^0(\mu, p_0, \dots, p_3) = \cup \{B_\mu(\gamma_3) \in G_r^{-1}(p_3) \cap n(\delta) | B_{2\delta}(\gamma_3) \cap n_f(\delta, p_0 \cup p_1 \cup p_2) = \phi\},$$

$$n_f^k(\mu, p_0, \dots, p_3) = \cup \{B_\mu(\gamma_3) \in G_r^{-1}(p_3) \cap n(\delta) | B_{2\delta}(\gamma_3) \cap n_f(\delta, p_0 \cup p_1) = \phi, B_{2\delta}(\gamma_3) \cap \cup_{j=k}^3 n_f^j(\delta, p_0, p_1, p_2) = \phi \quad 1 \leq k \leq 4\}.$$

Let us proceed the extension step as follows: We can use contracting homotopies $C_s(p_3)$, $C_{s,t}(p_2, p_3)$ on $n_i^k(2\delta, p_0, \dots, p_3)$, $k = 0, 1$, respectively.

Now let us consider $n_i^2(2\delta, p_0, \dots, p_3) \neq \phi$. Then there are $\gamma(1), \gamma(2), \gamma(3) \in \Gamma$ such that $\gamma(i) \in F(r)^{-1}(p_i) \cap n(\delta)$ and:

$$d(\gamma(1), \gamma(2)) \leq 3\delta, \quad d(\gamma(1), \gamma(3)) \leq 6\delta, \\ d(\gamma(2), \gamma(3)) \leq 3\delta.$$

As before, by taking a triangle $\Delta(p_1, p_2, p_3) \subset An(\mu(r), Ar + B)$, and using a family of contracting homotopies:

$$C(p_1, p_2, p_3) : [0, 1] \times [0, 1]^2 \times n_i(C_i, F(r)(n_f(2\delta, p_1 \cup p_2 \cup p_3))) \\ \mapsto An^{i'}(\mu_i(r), A_i r + B_i),$$

we have an extension on $n_i^2(2\delta, p_0, p_1, p_2)$. We can use contracting homotopies $C(p_0, \dots, p_3)$ and $C(p_0, p_2, p_3)$ on $n_i^3(2\delta, p_0, \dots, p_3)$ and $n_i^4(2\delta, p_0, \dots, p_3)$.

At each stage, the extension process is determined by $\Delta(p_i, \dots)$, family of contracting points. Let us put $\Delta(p_i) = p_i$: point. Then we have the following indication for the process to extend G_r on $n_i^k(\delta, p_0, \dots, p_3)$:

$$\Delta(p_3) \ (k = 0), \quad \Delta(p_2, p_3) \ (k = 1), \\ \Delta(p_1, p_2, p_3) \ (k = 2), \quad \Delta(p_0, \dots, p_3) \ (k = 3), \\ \Delta(p_0, \dots, p_3) \text{ or } \Delta(p_0, p_2, p_3) \ (k = 4).$$

We will take similar steps to extend the rest domains in $n_f(2\delta, p_3)$. Thus first we determine each subdomain in $n_f(2\delta, p_3)$ on which we extend G_τ inductively, and next find families of contracting points $\Delta(p_i, \dots)$ at each step.

We proceed to the next step. For $0 \leq l \leq 5$, let us define:

$$n_f^{5+l}(\mu, p_0, \dots, p_3) = \cup \{B_\mu(\gamma_3) \in G_\tau^{-1}(p_3) \cap n(\delta) \mid B_{2\delta}(\gamma_3) \cap n_f(\delta, p_0) = \phi, \\ B_{2\delta}(\gamma_3) \cap n_f(\delta, p_1) \neq \phi, \quad B_{2\delta}(\gamma_3) \cap \cup_{j=l}^4 n_f^j(\delta, p_0, \dots, p_3) = \phi\}.$$

Then we have the next families of contracting points:

$$\begin{aligned} \Delta(p_1, p_3) \quad (l = 0, 1), \\ \Delta(p_1, p_2, p_3) \quad (l = 2, 3), \\ \Delta(p_0, \dots, p_3) \quad (l = 4, 5). \end{aligned}$$

The next is all the rest case. Let us put:

$$n_f^{10+m}(\mu, p_0, \dots, p_3) = \cup \{B_\mu(\gamma_3) \in G_\tau^{-1}(p_3) \cap n(\delta) \mid B_{2\delta}(\gamma_3) \cap n_f(\delta, p_0) \neq \phi, \\ B_{2\delta}(\gamma_3) \cap \{\cup_{j=m}^9 n_f^j(\delta, p_0, \dots, p_3)\} = \phi\} \quad 0 \leq m \leq 9.$$

$$\begin{aligned} \Delta(p_0, p_3) \quad (m = 0, 1), \quad \Delta(p_0, p_2, p_3) \quad (m = 2), \\ \Delta(p_0, \dots, p_3) \quad (m = 3, 4), \\ \Delta(p_0, \dots, p_3) \text{ or } \Delta(p_0, p_2, p_3) \quad (m = 5), \\ \Delta(p_0, p_1, p_3) \quad (m = 6), \quad \Delta(p_0, \dots, p_3) \quad (m = 7, 8, 9). \end{aligned}$$

This completes the construction of G_τ^s on $n_f^i(\delta, p_0 \cup p_1 \cup p_2 \cup p_3)$. We can iterate similar process inductively on p_i . Now we have got a map:

$$G_\tau^s : \bar{P}_i \mapsto An^i(\mu_i(r), A_i r + B_i)$$

which has the same as the one at the end of 3.3. The family version of the above construction is completely parallel, and we omit it. By doing the same process as 3.4, 3.5, 3.6 and 3.7, we have the following:

Theorem *Let Γ be a torsion free proper combing. Then for each i , there exists a family of asymptotically proper Lipschitz maps:*

$$\alpha_i : [1, \infty) \times \tilde{P}_{i+1} \times_\Gamma \bar{P}_i \mapsto R^{N_{i+1}}$$

such that α_i^1 is the one in 3.1.

In particular, the assembly map for Γ is rationally injective.

II Locally combable groups

In part II, we consider a more general class of discrete groups. A group Γ is *asynchronously combable* if it admits a function which assigns paths from the identity to γ for each element $\gamma \in \Gamma$. If two elements has distance less than 1 from each other, then the corresponding paths have uniformly bounded Hausdorff distance. It is known that there are examples which are not combable, but asynchronously combable. In fact these examples admit asynchronously combing, which look like combing locally in some sense. Later we will introduce a geometric class of discrete groups, *locally combable groups*. A local combing admits combing structure on each compact subset of Γ , whose paths are consisted by reparametrizations of the asynchronously combing. We will extend the method in part I in order to applicable to locally proper combable groups. In this class, we donot have the property that each Rips complex is contractible in more larger one. Thus to treat these, we need to use all approximating spaces of $B\Gamma$ at the same time. At this point, we have some infinite dimensional situation which is much interesting.

In order to treat this class, we need some algebraic topological condition on groups. This is related to the so-called Mittag-Leffler condition on E or K theory of $B\Gamma$. Troughout part II, we will assume the following: Suppose $b_i = (E, D) \in E(\tilde{P}_i/\Gamma, *)$ is non zero over rational, where E is a complex vector bundle and D is the Dirac operator. If $\langle a_i, b_i \rangle \neq 0$ for some $a_i \in E(B\Gamma, *)$, then $E \mapsto \tilde{P}_i/\Gamma$ has extension stably over any $\tilde{P}_{i'}/\Gamma$ for $i' \geq i$.

4 Infinite number of families of maps

4.1 Deformation of family of maps: Let $P_0/\Gamma \subset P_1/\Gamma \subset \dots \subset B\Gamma$ be approximations of $B\Gamma$ by finite complexes and take tublar neighbourhoods of $P_i/\Gamma \subset R^{N_i}$ as before, denote them by $\tilde{P}_i/\Gamma$, $\bar{P}_i/\Gamma$. We may assume that each inclusion $\tilde{P}_i \subset \tilde{P}_{i+1}$ is a proper embedding. Let us put $m_i = N_{i+1} - N_i$. Then by taking another open tublar neighbourhoods, we have open, (but non proper) embeddings:

$$R^{m_i} \times \tilde{P}_i/\Gamma \hookrightarrow \tilde{P}_{i+1}/\Gamma.$$

We will embed all these spaces into a Hilbert space. This will allow us to treat maps on these at the same time. Let us embed R^∞ into a separable Hilbert space H , e.g. H is the closure of R^∞ for the standard metrics. Then we have

a family of embeddings:

$$(R^{N_0})^\perp \times \tilde{P}_0/\Gamma \subset (R^{N_1})^\perp \times \tilde{P}_1/\Gamma \subset \cdots \subset (R^{N_i})^\perp \times \tilde{P}_i/\Gamma \subset \cdots \subset H.$$

We take Riemannian metrics $\tilde{d}_i$ and $\bar{d}_i$ on $\tilde{P}_i$ and $\bar{P}_i$ respectively as before, equip metrics on $R^m \times \tilde{P}_i$ by taking product with the standard one on R^m . We may assume that there exists $\epsilon > 0$, independent of $i \geq 0$ such that:

$$B_\epsilon(0) \times \tilde{P}_i/\Gamma \hookrightarrow \tilde{P}_{i+1}/\Gamma$$

is a locally isometric embedding, where $B_\epsilon(0) \subset R^{m_i}$ is ϵ ball.

Let us take a family of maps: $\{\alpha_i : \tilde{P}_i \times_\Gamma \bar{P}_i \mapsto R^{N_i} \quad i \geq 0\}$ such that:

$$(1) \quad \alpha_i|_{N_\epsilon^i(\Delta)} = \exp, \quad (2) \quad \alpha_i(\tilde{P}_i \times_\Gamma \bar{P}_i \setminus N_\epsilon^i(\Delta)) \subset R^{N_i} \setminus B_\epsilon(0),$$

$$(3) \quad \alpha_{i+1}|_{\tilde{P}_i \times_\Gamma \bar{P}_i} = \alpha_i.$$

We will use the same notations to denote the obvious extensions of α_i :

$$\alpha_i : (R^{N_i})^\perp \times \tilde{P}_i \times_\Gamma \bar{P}_i \mapsto H \quad i \geq 0.$$

A deformation of α_0 constructed from a family $\{\alpha_i\}_{i \geq 0}$ is a family of proper maps α_t with parameter $t \in [1, \infty)$ such that: (1) $\alpha_0^1 = \alpha_0$,

(2) there exist an exhausting increasing family $[1, t_0] \subset [1, t_1] \subset \cdots \subset [1, \infty)$ and the corresponding positive numbers $l_i \in \mathbb{N}$ such that $\alpha_t|_{[1, t_i]}$ factors through $\alpha_{l(i)}$,

$$\alpha_t|_{[1, t_i]} : \tilde{P}_{l_i} \times_\Gamma \bar{P}_0 \rightarrow \tilde{P}_{l_i} \times_\Gamma \bar{P}_{l_i} \xrightarrow{\alpha_{l(i)}} R^{N_{l(i)}}.$$

A Γ as-proper Lipschitz family is a family of maps above with the additional property:

(3) there exist an increasing proper function $\mu : [0, \infty) \mapsto [0, \infty)$ and a constant C such that $\alpha_t|_{[1, t_i]} : \tilde{P}_{l_i} \times_\Gamma \bar{P}_0 \mapsto R^{N_{l_i}}$ is a family of asymptotic proper Lipschitz maps with respect to (μ, C) .

We have some umbiguity to choose $\{l_i\}$. For example we may extend:

$$\alpha_t|_{[1, t_i]} : \tilde{P}_{l_{i+1}} \times_\Gamma \bar{P}_0 \mapsto \tilde{P}_{l_{i+1}} \times_\Gamma \bar{P}_{l_{i+1}} \mapsto R^{N_{l_{i+1}}}.$$

When we extend as above, we may assume the following properties (3.7):

- (1) $\alpha_t|_{[1, t_i]} : B_\epsilon(0) \times \tilde{P}_{l_i} \times_\Gamma \bar{P}_0 : (v, (p, p')) \mapsto v + \alpha_t(p, p')$,
- (2) $\alpha_t|_{[1, t_i]}(B_\epsilon(0)^c \times \tilde{P}_{l_i} \times_\Gamma \bar{P}_0) \subset B_\epsilon(0)^c \subset R^{N_{l_{i+1}}}$.

In the following, we will assume (1) and (2), without reffering about these.

4.2 Asymptotic equivalence: Let $\alpha_0 : \tilde{P}_0 \times_\Gamma \bar{P}_0 \mapsto R^{N_0}$ be in 3.1 and $b = (E, D) \in E(\tilde{P}_0/\Gamma, *)$. Then we have the corresponding asymptotic morphism:

$$\sigma_0 = (\alpha_0(\omega))^* \times \tilde{b} \in E(*, \bar{P}_0/\Gamma) : S \mapsto S(C(\bar{P}_0) \otimes K(\eta(\tilde{P}_0)))^\Gamma.$$

Let us take α_t , a deformation of α_0 constructed from the family $\{\alpha_i\}_{i \geq 0}$. During deformation of α_0 , we have used much larger spaces compared with P_0 . Thus in order to construct asymptotic morphism, compatible with the deformation, we need to use a sufficiently large Hilbert space H , which contains all $\eta(\tilde{P}_i)$. In 4.4, we construct an asymptotic morphism:

$$\sigma_t = (\alpha_t)^*(\omega) \times \tilde{b} : S \mapsto S(C(\bar{P}_0 \otimes K(H)))^\Gamma$$

using α_t , and show the asymptotic equivalence between σ_0 and σ_t . In fact since $\eta(\tilde{P}_0)$ is contained in H , we may regard σ_0 with the target space $S(C(\bar{P}_0 \otimes K(H)))^\Gamma$. We will show the following in 4.3 and 4.4:

Proposition σ_0 and σ_t are asymptotically equivalent.

Moreover suppose α_t is a Γ as-proper Lipschitz family. Then the same construction as σ_t with fixed parameter will give us a Γ asymptotic morphism:

$$\sigma_t(p_0) : S \mapsto SK(H).$$

Here we need the condition that E extends on any $\tilde{P}_i/\Gamma$. Recall that H contains all $\eta(\tilde{P}_i)$.

These morphisms form the following commutative diagram, which is a variant of the previous one:

$$E(\bar{P}_0/\Gamma, *) \times E(\tilde{P}_0/\Gamma, *) \mapsto Z$$

$$\downarrow A \times \sigma_t(p_0) \qquad \parallel$$

$$E(C, C^*(\Gamma)) \times E(C^*(\Gamma), C) \mapsto Z$$

Lemma Suppose a torsion free Γ admits a Γ as-proper Lipschitz family. Then under the condition of Γ in 4.1, Γ satisfies Novikov conjecture.

The rest of this section is devoted to construct σ_t , asymptotic equivalence between σ_0 and σ_t , and Γ properly as-Lipschitz family. We have already prepared enough to do them in the previous section, except some facts about infinite dimensional Bott periodicity developed by [HKT].

4.3 Bott periodicity on Hilbert space: To construct σ_t and asymptotic equivalence between σ_0 and σ_t , we need Hilbert space Bott periodicity constructed in [HKT]. For convenience, we recall a part of their content.

Let $E_a \subset E_b \subset H$ be any pair of finite dimensional linear subspaces, and let us define a $*$ homomorphism:

$$\beta(b, a) : SCl_0(E_a) \mapsto SCl_0(E_b)$$

by $f \otimes h \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{E_a^\perp \cap E_b}) \hat{\otimes} h$, where $C_{E_a^\perp \cap E_b} : E_a^\perp \cap E_b \rightarrow E_a^\perp \cap E_b$ is the identity. Then $\{\beta(b, a), E_a\}$ forms a directed family, and thus we get a direct limit C^* algebra and a $*$ homomorphism:

$$SCl_0(H) = \lim_{E_a \subset H} SCl_0(E_a)$$

$$\beta : S \mapsto SCl_0(H).$$

Clearly $SCl_0(H)$ is isomorphic to $Cl_0(E_a) \hat{\otimes} SCl_0((E_a)^\perp)$. Thus we have an extension $\beta : SCl_0(E_a) \mapsto SCl_0(H)$.

Let $E_a \subset E_b \subset H$ be as above, and define linear isometries:

$$T(b, a) : \eta(E_a) \mapsto \eta(E_b)$$

by $T(\zeta)(e, x) = \pi^{-\frac{m}{4}} \exp(-\frac{|e|^2}{2}) \zeta(x)$, $e \in E_a^\perp \cap E_b$, where m is $\dim E_a^\perp \cap E_b$. Using this system, let us define a Hilbert space:

$$\eta(H) = \lim_{E_a \subset H} \eta(E_a).$$

[HKT] constructed natural asymptotic morphisms $\varphi : SCl_0(H) \mapsto SK(\eta(H))$ and $\psi_a : SK(\eta(E_a)) \mapsto SK(\eta(H))$.

Lemma (HKT) *The following diagram asymptotically commutes:*

$$\begin{array}{ccccc} \varphi_a \circ \beta_a : S & \mapsto & SCl_0(E_a) & \mapsto & SK(\eta(E_a)) \\ & & \parallel & & \downarrow \psi_a \\ & & \downarrow & & \\ \varphi \circ \beta : S & \mapsto & SCl_0(H) & \mapsto & SK(\eta(H)) \end{array}$$

Moreover for each E_a , there exists a one dimensional projection $P((E_a)^\perp)$ in $K(\eta((E_a)^\perp))$ such that ψ_a is asymptotically equivalent to $\text{id} \hat{\otimes} P((E_a)^\perp)$. In particular $\varphi \circ \beta : S \mapsto K(\eta(H))$ is asymptotically equivalent to $\text{id} \hat{\otimes} P$.

4.4 σ_t and σ_0 : Let α_t be a deformation of the family, and $[1, t_0] \subset [1, t_1] \subset \dots [1, \infty)$ be an exhausting family such that $\alpha_t|_{[1, t_i]}$ factors through $\alpha_i|_{\tilde{P}_i \times_\Gamma \tilde{P}_i}$. By changing indices, we may assume $l_i = i$. Let $m_i = \dim \tilde{P}_{i+1} - \dim \tilde{P}_i = N_{i+1} - N_i$. For simplicity of the notation, we denote $\alpha_t|_{[1, t_i]} \equiv \alpha_i$ or α_i^t .

Let us consider two asymptotic morphisms (see 1.3 sublemma1):

$$\begin{aligned} SCl_0(\tilde{P}_i \times_\Gamma \bar{P}_0) &\xrightarrow{\beta(i+1, i) \otimes 1} SCl_0(R^{m_i} \times \tilde{P}_i \times_\Gamma \bar{P}_0) \\ &\xrightarrow{\beta(i+1, i) \otimes 1} SCl_0(\tilde{P}_{i+1} \times_\Gamma \bar{P}_0) \end{aligned}$$

where $\beta(i+1, i) = \beta(R^{N_{i+1}}, R^{N_i})$. Let us denote by $D(X)$ the Dirac operator over spin^c manifold X . By composing the above with $(E, D(\tilde{R}^{m_i} \times \tilde{P}_i))'$ and $(E, D(\tilde{P}_{i+1}))'$, we get the following diagram:

$$\begin{aligned} SCl_0(\tilde{P}_i \times_\Gamma \bar{P}_0) &\xrightarrow{\sigma_i} S(C(\bar{P}_0) \otimes K(\eta(R^{m_i} \times \tilde{P}_i)))^\Gamma \\ &\xrightarrow{\sigma'_i} S(C(\bar{P}_0) \otimes K(\eta(\tilde{P}_{i+1})))^\Gamma. \end{aligned}$$

By the assumption, $\alpha_{i+1}|_{B_\epsilon(0) \times \tilde{P}_i \times_\Gamma \bar{P}_i} = \alpha_i$, for the embedding $B_\epsilon(0) \times \tilde{P}_i \hookrightarrow \tilde{P}_{i+1}$. Thus using the identification in 1.3, it is easily seen that the above morphisms are asymptotically equivalent. Let us consider:

$$\sigma'_i \circ \alpha_i^*(\omega), \sigma_{i+1} \circ \alpha_{i+1}^*(\omega) : S \mapsto S(C(\bar{P}_0) \otimes K(\eta(\tilde{P}_{i+1})))^\Gamma.$$

By a similar way as 1.3, using φ_t , it is easy to see that these are asymptotically equivalent. Thus $\sigma_i \circ \alpha_i^*(\omega)$ and $\sigma_{i+1} \circ \alpha_{i+1}^*(\omega)$ are asymptotically equivalent.

Turning into infinite dimensional case, let:

$\beta : SCl_0(\tilde{P}_i \times_\Gamma \bar{P}_i) \mapsto SCl_0((R^{N_i})^\perp \times \tilde{P}_i \times_\Gamma \bar{P}_i) \equiv SCl_0((R^{N_i})^\perp) \hat{\otimes} Cl_0(\tilde{P}_i \times_\Gamma \bar{P}_i)$ be as above. Thus by composing, we get asymptotically equivalent morphisms:

$$\begin{aligned} SCl_0(\tilde{P}_i \times_\Gamma \bar{P}_0) &\xrightarrow{\sigma_i} S(C(\bar{P}_0) \otimes K(\eta((R^{N_i})^\perp \times \tilde{P}_i)))^\Gamma \\ &\xrightarrow{\sigma'_i} S(C(\bar{P}_0) \otimes K(\eta((R^{N_{i+1}})^\perp \times \tilde{P}_{i+1})))^\Gamma. \end{aligned}$$

More precisely these are asymptotically equivalent by regarding these with their target space $S(C(\bar{P}_0) \otimes K(H))^\Gamma$ for a Hilbert space constructed as follows: Let $N_\delta(\tilde{P}_i) \subset \tilde{P}_{i+1}$ be the δ neighbourhood of $\tilde{P}_i$ in $\tilde{P}_{i+1}$. It can be also regarded as a subset in $R^{m_i} \times \tilde{P}_i$. Let us denote $\eta_i = \eta((R^{N_i})^\perp \times \tilde{P}_i)$ and decompose:

$$\begin{aligned} \eta_i &= \eta((R^{N_i})^\perp) \hat{\otimes} \eta(N_\delta(\tilde{P}_{i-1})) \oplus \eta((R^{N_i})^\perp) \hat{\otimes} \eta(N_\delta(\tilde{P}_{i-1})^c) \\ &\equiv \eta_i^1 \oplus \eta_i^2. \end{aligned}$$

Now we have a Hilbert space H spanned by the direct sum of:

$$\eta_0, \quad \eta_i^2 \quad i \geq 1.$$

Notice that for each $i \geq 0$, we have a natural isometric embedding $\eta_i \hookrightarrow H$.

Then precisely:

$$\sigma_i, \sigma'_i : Cl_0(\tilde{P}_i \times_\Gamma \bar{P}_0) \mapsto S(C(\bar{P}_0) \otimes K(H))^\Gamma$$

are asymptotically equivalent. Similarly to the finite dimensional case, we have asymptotically equivalent morphisms:

$$\sigma_i \circ \alpha_i^*(\omega), \sigma_{i+1} \circ \alpha_{i+1}^*(\omega) : S \mapsto S(C(\bar{P}_0) \otimes K(H))^\Gamma.$$

Let us consider a family $\{(\alpha_t|[1, t_i])^*(\omega) : S \mapsto SCl_0(\tilde{P}_i \times_\Gamma \bar{P}_0)\}_{i \geq 0}$. Then by composing, $\sigma_i \circ (\alpha_t|[1, t_i])^*(\omega)$ and $\sigma_{i+1} \circ (\alpha_t|[1, t_{i+1}])^*(\omega)$ are asymptotically equivalent. Let us denote by τ_i^s their asymptotic equivalence. Let us connect a family $\{\sigma_i \circ (\alpha_t|[1, t_i])^*(\omega) : S \mapsto S(C(\bar{P}_0) \otimes K(H))^\Gamma\}_{i \geq 1}$ so that it becomes a continuous asymptotic morphisms:

$$\sigma \circ (\alpha_t)^*(\omega) : S \mapsto S(C(\bar{P}_0) \otimes K(H))^\Gamma.$$

Since we may assume $t_i + 1 \leq t_{i+1}$, we define the above morphism for $t = t_i + s \in [t_i, t_i + \frac{1}{2}]$ by τ_i^{2s} , and for $t \in [t_i + \frac{1}{2}, t_{i+1}]$, $\sigma_i \circ (\alpha_t)^*(\omega) = \sigma_{i+1} \circ (\alpha_t|[1, t_{i+1}])^*(\omega)$. Thus we get a continuous family of asymptotic morphisms $\sigma_t \equiv \sigma \circ (\alpha_t)^*(\omega)$.

By the condition $\alpha_t|_{t=1} = \alpha_0$, σ_t is asymptotically equivalent to σ_0 . This completes the proof of proposition.

4.5 Locally combable groups: Let $F : N \times \Gamma \mapsto \Gamma$ be an asynchronously combing. F is *locally combable* if for each large L , there are reparametrizations:

$$l_\gamma^L : N \mapsto N, \quad l_\gamma^{L+1}|\{0, \dots, L\} = l_\gamma^L|\{0, \dots, L\}$$

such that $F \circ l^L : \{0, \dots, L\} \times \Gamma \mapsto \Gamma$, by $F \circ l^L(t, \gamma) = F(l_\gamma^L(t), \gamma)$ is a combing, where the quasi Lipschitz constants k_L and combing constants C_L may depend on L .

A local combing is *proper* if there exists an increasing proper function $\mu : [0, \infty) \mapsto [0, \infty)$ with the following property; for any $R, \delta \gg 0$, there exist $T = T(\delta, R)$ and $C = C(\delta)$ such that for $\gamma, \gamma' \in \Gamma$, $|\gamma|, |\gamma'| \geq T$, $d(\gamma, \gamma') \leq \delta$, we have:

$$d(e, F \circ l^L(F \circ l^L(\gamma)(R'), F \circ l^L(\gamma')(R'))(t)) \geq \mu(R)$$

for $R, C(\delta) \leq R' \leq L$ and $t = 0, 1, \dots$.

Example The Baumslag-Solitar groups admit locally, proper combings. The groups have presentation by $G(p, q) = \{x, y | yx^p = x^qy\}$, $p, q \in \mathbb{N}$. It is known that they admit automatic structure if $p = q$, and otherwise only admit asynchronously automatic structure.

These groups admit more geometric structure. They are locally *quasi geodesic* combings in the following sense. A locally proper combing is quasi geodesic if the structure satisfies the followings; there are positive constants A , B and C such that:

$$(1) \quad |F \circ l_\gamma^L(R)| \leq AR + B, \quad (2) \quad \limsup_{L \rightarrow \infty} \frac{C_L}{L} = 0, \\ (3) \quad \mu(R) \geq A^{-1}R - B.$$

These additional conditions will be used when we construct families of radial projections, with small contracting constants.

4.6 Asymptotic Lipschitz family: Let $\{\alpha_i : \tilde{P}_i \times_\Gamma \bar{P}_i \mapsto R^{N_i}\}$ be a family of fiberwise proper degree 1 maps with $\alpha_{i+1}|_{B_\epsilon(0) \times \tilde{P}_i \times_\Gamma \bar{P}_i} = \alpha_i$.

In order to make a Γ as-proper Lipschitz family from α_i , we will construct the following maps. An *asymptotic contracting family* for $\{\bar{P}_i\}_{i \geq 0}$ is a family of proper continuous map:

$$g_t : [1, \infty) \times \bar{P}_0 \mapsto \bar{P}_\infty \equiv \cup_i \bar{P}_i$$

with the following properties; there are a family of small constants $1 \gg \lambda_i > 0$, $i = 0, 1, \dots$, a partition of $[1, \infty)$:

$$1 \ll t_0 \ll t_1 \ll \dots \ll t_i \ll \dots$$

and the corresponding family of numbers $l(i) = l(t_i) \in \mathbb{N}$ such that:

- (1) $g_t|_{[0, t_i]}$ factors through $\bar{P}_{l(i)} \subset \bar{P}_\infty$,
- (2) $g_t|_{B_t(p_0)} : B_t(p_0) \subset \bar{P}_0 \mapsto \bar{P}_{l(i)}$ is λ_i contracting with respect to the metrics $(\bar{P}_0, \bar{d}_0)$ and $(\bar{P}_{l(i)}, \bar{d}_{l(i)})$,
- (3) there is a positive constant $\epsilon > 0$ such that $g_t(B_t(p_0)^c) \subset B_\epsilon(p_0)^c$.

Let us take a fixed point $p_0 \in \bar{P}_0$ and g_t be an asymptotic contracting family. By the compatibility of the family $\{\alpha_i|_{p_0 \times \bar{P}_i}\}_{i \geq 0}$, we get a continuous map:

$$\alpha \circ g_t : [1, \infty) \times \bar{P}_0 \mapsto R^\infty$$

by $\alpha \circ g_t|_{[1, t_i]} = \alpha_{l(i)} \circ g_t : \bar{P}_0 \mapsto R^{N_{l(i)}}$. As before we may assume $l(i) = i$.

Let H be a real Hilbert space containing R^∞ . Then we may regard $\alpha \circ g_t$ with the target space H .

Lemma *There is a family of small constants:*

$$1 \gg \lambda_0 \gg \lambda_1 \gg \dots$$

such that for any asymptotic contracting family g_t with the contracting constants $\lambda'_i \leq \lambda_i$, there is an asymptotic proper Lipschitz map $\alpha_t : \bar{P}_0 \mapsto H$, which is a deformation of α'_0 for $\{\alpha'_i\}_{i \geq 0}$, where α'_i is properly homotopic to α_i .

Proof: Let us take a family of proper increasing functions:

$$u : [1, \infty) \times [0, \infty) \mapsto [1, \infty)$$

such that $u|_{t \times [0, t]} \equiv t$. Then let:

$$u\alpha \circ g_t : [1, \infty) \times \bar{P}_0 \mapsto H$$

by $u\alpha \circ g_t|_{[1, t_i]}(p) = u(t, d_0(p, p_0))\alpha_i(g_t(p))$. Then we have the following bound:

$$|t\alpha_i \circ g_t(p)| \geq t\epsilon, \quad d_0(p, p_0) \geq t.$$

Thus $u\alpha \circ g_t$ is a proper map. If λ_i satisfies $\lambda_i \leq C_i^{-1}t^{-2}$ for a suitable family of constants C_i , then the Lipschitz constants on $u\alpha_i \circ g_t|_{B_t(p_0)}$ are bounded by:

$$tC_i t \lambda_i \leq 1.$$

Thus we get the desired asymptotic Lipschitz family. This completes the proof.

Let us introduce the parameter space $\tilde{P}_i$ to the family of maps. A Γ *as-contracting family* is a family of fiberwise asymptotic contracting family:

$$g_t|_{[1, t_i]} : \tilde{P}_i \times_{\Gamma} \bar{P}_0 \mapsto \tilde{P}_i \times_{\Gamma} \bar{P}_i.$$

A family version of the above lemma reads as:

Corollary *Suppose we have a Γ as-contracting family. Then there is a Γ as-proper Lipschitz family $\alpha_t|_{[1, t_i]} : \tilde{P}_i \times_{\Gamma} \bar{P}_0 \mapsto R^{N_i}$.*

4.7 Construction for local combing: A proper map $f : X \mapsto Y$ is a (t, δ) *contraction* if (1) $f|_{B_t(x_0)}$ is a δ contraction, (2) $f(B_t(x_0)^c) \subset B_{\epsilon}(y_0)^c \subset Y$ for a positive ϵ .

Proposition *Let Γ be a locally quasi geodesic combing. Then there exists a Γ as-proper contracting family with arbitrary small contracting constants $\{\lambda_i\}$.*

Combined with 4.6 and 4.2, we get the following:

Corollary *Let Γ be a torsion free locally quasi geodesic combing. Then the analytic assembly map is rationally injective.*

Proof of Proposition: We have already seen how to make (t, δ) contraction. The different point from before is that there is no uniform bound on the local combing constants. This is why we cannot make asymptotic proper contracting maps. However if we give up to get proper maps, then the construction in the previous section works for this case and get (t, δ) contraction $G : \bar{P}_0 \mapsto \bar{P}_i$ for any larger t and small δ .

Let us take a maximal δ net $n(\delta) \subset \Gamma(T)$ as before, and define $G_r : n(\delta) \mapsto G(\Gamma)$ by $G_r(\gamma) = F \circ l^L(\gamma)(r)$. In order to make the image away from the identity, we need a lower bound on r , with respect to δ . Thus we have no controll of Lipschitz constants of G_r .

If we embed $G(\Gamma)$ into $P_i(\Gamma)$, then the Lipschitz constant of G_r becomes smaller as i increases. On the other hand, $d_i(e, G_r(p))$ becomes also small when i is large. We need some controll between i and this distance, when we construct a family of (t, δ) contractions. This is the place where we require quasi geodesic property.

Let us use the same notation as 4.5 for data of a locally quasi geodesic combing. Let us fix a sufficiently small $\epsilon > 0$, and take a large $L = r$ so that:

$$C_L \leq \epsilon L.$$

Then we may assume $A^{-1}L - B \gg \epsilon L$, and there is a constant $C > 0$, which may be arbitrarily large by taking smaller ϵ such that:

$$A^{-1} \geq C\epsilon.$$

Let us choose $i = \epsilon L$ and regard:

$$G_r : n(\delta) \mapsto P_i.$$

By the condition, there are positive constants C_1 and C_2 such that image of G_r is contained in $An^i(C_1, C_2)$. Again we may choose C_1 and C_2 arbitrarily large by taking i large (and ϵ small). By the previous construction, there is a constant C , which depends only on P_0 such that there is an extension of $C(r)\delta^{-1}$ contracting map:

$$G_r : P_0 \setminus B_T(e) \mapsto An^{C_i}(C_1, C_2) \subset P_{Ci}$$

and a homotopy G_r^s such that:

$$(1) \quad G_r^0(P_0 \setminus B_L(e)) \subset An^{C_i}(C_1, C_2), \quad (2) \quad G_r^0|_{B_L(e)} \text{ is the embedding.}$$

In fact we may construct G_r^0 naturally by using the local combing and extending it simplicially.

By changing indices, we may assume that the target space is P_i . Let us extend $G_r^s|_{P_0 \setminus B_T(e)}$ for families. Our first step is to construct:

$$G_r^s : \bar{P}_i \times_\Gamma \bar{P}_0 \setminus N_T(\Delta) \mapsto \bar{P}_i \times_\Gamma \bar{P}_i \setminus N_{C_1}(\Delta). \quad *$$

We regard $G_r^s|_{P_0 \setminus B_T(e)} = G_r^s|_e \times P_0 \setminus B_T(e)$. Then $G_r^s|_{\Gamma \times P_0 \setminus B_T(e)}$ is determined automatically.

Suppose $|\gamma| \leq i$. Then for $p \in P_0 \setminus B_L(e) \cap \Gamma$:

$$d((\gamma, \gamma), G_r^0(\gamma, p)) \geq A^{-1}L - B - i = A^{-1}L - B - \epsilon L \geq \delta L$$

where $0 < \epsilon \ll \delta < 1$. Thus by changing C_1 and C_2 if necessarily, we have the following bound:

$$G_r^s(\gamma \times P_0 \setminus B_L(e)) \subset An^{C_i}(C_1, C_2). \quad **$$

Let us take a Γ covering of $\bar{P}_i$, $\{Y_\alpha\}$. We may assume $\{Y_\alpha\} = \{\gamma Y_l | \gamma \in \Gamma, d_i(Y_l, e) \leq 1\}_{l=1}^m$. In order to extend G_r^s for family, it is enough to extend G_r^s on each $Y_l \times \bar{P}_0$. From **, we may assume:

$$G_r^s(Y_l \cap \Gamma \times P_0 \setminus B_L(e)) \subset An^{C_i}(C_1, C_2).$$

Thus using the previous method in §3, we get the desired *. Notice that G_r^0 has a natural extension on $\bar{P}_i \times_\Gamma \bar{P}_0$. Then by changing C_1 if necessarily, we have the following bound:

$$G_r^0(P_i \times_\Gamma P_0 \setminus N_L(\Delta)) \subset P_i \times_\Gamma P_i \setminus N_{C_1}(\Delta).$$

Then the extension:

$$G_r^0 : \bar{P}_i \times_\Gamma \bar{P}_0 \mapsto \bar{P}_i \times_\Gamma \bar{P}_i$$

satisfies that there is a compact subset $\Delta \subset K \subset \bar{P}_i \times_\Gamma \bar{P}_0$ such that:

- (1) $G_r^0|_K$ is the embedding,
- (2) $G_r^0(\bar{P}_i \times_\Gamma \bar{P}_0 \setminus K) \subset \bar{P}_i \times_\Gamma \bar{P}_i \setminus N_{C_1}(\Delta)$.

By the previous construction for families, we get the desired family of $(t = T, \delta)$ contractions:

$$G_r^s : \bar{P}_i \times_\Gamma \bar{P}_0 \mapsto \bar{P}_i \times_\Gamma \bar{P}_i.$$

By a similar way as 3.5, we can make G_r^s continuous with respect to r . This is the desired Γ as-contracting family. This completes the proof.

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