Fisher information in quantum statistics

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Abstract.
Braunstein and Caves (1994) proposed to use Helstrom’s quantum information number to define, meaningfully, a metric on the set of all possible states of a given quantum system. They showed that the quantum information is nothing else than the maximal Fisher information in a measurement of the quantum system, maximized over all possible measurements. Combining this fact with classical statistical results, they argued that the quantum information determines the asymptotically optimal rate at which neighbouring states on some smooth curve can be distinguished, based on arbitrary measurements on \( n \) identical copies of the given quantum system.

We show that the measurement which maximizes the Fisher information typically depends on the true, unknown, state of the quantum system. We close the resulting loophole in the argument by showing that one can still achieve the same, optimal, rate of distinguishability, by a two stage adaptive measurement procedure.

When we consider states lying not on a smooth curve, but on a manifold of higher dimension, the situation becomes much more complex. We show that the notion of “distinguishability of close-by states” depends strongly on the measurement resources one allows oneself, and on a further specification of the task at hand. The quantum information matrix no longer seems to play a central role.

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1. Introduction

Braunstein and Caves (1994) have clarified the relation between the classical Fisher expected information number $i(\theta)$, for the unknown parameter $\theta$ of a probability distribution $p(x; \theta)$, and the analogous concept of expected quantum information $I(\theta)$ for a quantum system in state $\rho = \rho(\theta)$ on some Hilbert space. They showed that $I(\theta)$ is the maximal Fisher information $i(\theta; M)$ in the distribution of the outcome of a measurement $M$, over all measurements of the state. Thereby they supplied a new proof of Helstrom’s (1976) quantum Cramér-Rao bound: no unbiased estimator of $\theta$, based on any measurement, has variance smaller than $I(\theta)^{-1}$. Recall that the classical bound states that no unbiased estimator of $\theta$ based on the outcome of the measurement $M$ has variance smaller than $i(\theta; M)^{-1}$.

For $n$ identical copies of a quantum system, and for $n$ independent and identically distributed observations from a probability distribution, quantum and Fisher information are both $n$ times the corresponding quantities for $n = 1$. By classical statistical theory, the quantum bound is therefore asymptotically achieved, as $n \to \infty$, by the maximum likelihood estimator of $\theta$ based on the outcomes of the measurement maximizing the Fisher information for $n = 1$, applied to each of $n$ copies of the quantum system separately.

In the present paper, we analyse the conditions for equality of the quantum and Fisher information. We show that in general there does not exist a measurement $M$ such that $i(\theta; M) = I(\theta)$ for all $\theta$ simultaneously, studying an elementary spin-half model in detail (pure state, density matrix known to lie on a circle). We show how adaptive measurements still allow one to asymptotically achieve the quantum information bound for a scalar parameter, though not in the vector case, where the picture is rather complicated and the quantum information matrix inadequate to describe what is possible.

In Section 2 of the paper we recapitulate some of the theory of classical and quantum information. Next, in Section 3, we specialize the conditions for attainability of the information bound, first to pure states, then further to spin-half models, and finally to the case of a state lying on a given circle on the surface of the Poincaré sphere. Unless the circle is a great circle, no measurement achieves the bound uniformly in the parameter $\theta$. In Section 4 we explore the consequences of this result. We show that one can in effect achieve $i(\theta; M_n) \approx nI(\theta)$ for all $\theta$ simultaneously, when we measure $n$ identical copies of the quantum system in one joint measurement $M_n$. This result gives support to Braunstein and Cave’s interpretation of the quantum information number $I(\theta)$ as a measure of statistical distinguishability between neighbouring quantum states. Finally we turn to the case when the parameter is a vector. Both quantum and classical information numbers have matrix generalizations, and inequality between them still holds, in the sense of positive semi-definite matrices. The inequality is sharp but however no longer attainable. For a completely unknown spin-half pure state we show that the optimal rate at which one can distinguish between different states does not follow from
the quantum information matrix in the way one would expect from analogy with classical Fisher information. Moreover it depends on some weighting of the different aspects of the states which one wants to distinguish. Major open problems remain, and the role of the quantum information does not appear to be primary.

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2. Expected classical and quantum information

On a given Hilbert space, consider a quantum state (density operator) \( \rho = \rho(\theta) \), which depends on an unknown scalar parameter \( \theta \). Consider also a generalised measurement (operator-valued probability measure, POVM) \( M \) with outcomes in a measurable space \((\mathcal{X}, \mathcal{A})\). Thus the outcome of a measurement of \( M \) on \( \rho \) is a random variable \( X \) taking values in \( \mathcal{X} \), such that for each measurable subset \( A \) of \( \mathcal{X} \), i.e., for each \( A \in \mathcal{A} \), we have \( \Pr_\theta \{ X \in A \} = \text{trace} \rho(\theta)M(A) \). Suppose that \( M \) is dominated by a sigma-finite measure \( \mu \) on \((\mathcal{X}, \mathcal{A})\), i.e., for each \( A \in \mathcal{A} \),

\[
M(A) = \int_A m(x)\mu(dx)
\]

where the operator \( m(x) \) is, for each \( x \), nonnegative and selfadjoint, and \( \int_{\mathcal{X}} m(x)\mu(dx) = 1 \). Then the outcome \( X \) of a measurement of \( M \) on \( \rho \) has probability density, with respect to \( \mu \), given by

\[
p(x; \theta) = \text{trace} \rho(\theta)m(x).
\]

Under some smoothness assumptions (in particular, assuming that \( \{ x : p(x; \theta) > 0 \} \) does not depend on \( \theta \)), the expected Fisher information number, for \( \theta \), from this measurement, is defined by

\[
i(\theta; M) = E\dot{i}(\theta)^2 = \int_{\mathcal{X}} (\dot{i}(x; \theta))^2 p(x; \theta)\mu(dx)
\]

where

\[
l(\theta) = l(X; \theta) = \log p(X; \theta)
\]

is the log likelihood and

\[
\dot{i}(\theta) = \frac{\partial}{\partial \theta} l(X; \theta)
\]

is the score function for \( \theta \).

Now, let \( \lambda = \lambda(\theta) \) denote the symmetric logarithmic derivative of \( \rho \) with respect to \( \theta \), that is, the self-adjoint operator given implicitly by

\[
\dot{\rho} = \frac{1}{2}(\rho\lambda + \lambda\rho).
\]
We call $\lambda$ the quantum score for $\theta$. From the relation $\text{trace } \rho = 1$ one finds, by differentiating, $\text{trace } \rho \lambda = 0$. The expected quantum information number for $\theta$ is defined by

$$I(\theta) = \text{trace } \rho \lambda^2.$$ 

Note that this quantity is defined without reference to any particular measurement $M$.

For future reference, we mention that when $\theta$ is a vector parameter, the Fisher information matrix is defined in the obvious way, while the quantum information matrix has $ij$-component $\frac{1}{2} \text{trace } \rho (\lambda_i \lambda_j + \lambda_j \lambda_i)$ where $\lambda_i(\theta)$ is the quantum score for $\theta_i$ keeping the other components of $\theta$ fixed.

One can express the Fisher information $i(\theta; M)$ in terms of the quantum score for $\rho$:

$$i(\theta; M) = \int_{\mathcal{X}^+} p(x; \theta)^{-1} (\Re \text{trace} (\rho \lambda m(x)))^2 \mu(dx).$$

where

$$\mathcal{X}^+ = \mathcal{X}^+_+ (M, \theta) = \{ x : p(x; \theta) > 0 \}. \tag{2}$$

This follows on noting that

$$\dot{i}(\theta) = p(x; \theta)^{-1} \text{trace } \dot{\rho} m(x)$$

$$= p(x; \theta)^{-1} \frac{1}{2} \text{trace } ((\rho \lambda + \lambda \rho) m(x))$$

$$= p(x; \theta)^{-1} \Re \text{trace } (\rho \lambda m(x)).$$

Helstrom’s (1982) original derivation of the quantum Cramér-Rao bound followed closely the lines of the usual proof of the classical bound: write down the unbiasedness relation, differentiate under the integral sign, and apply the Cauchy-Schwarz inequality; see Holevo (1982) for a more precise theorem and proof. Braunstein and Caves (1994) noted how the quantum bound could be obtained from the classical bound together with the new inequality $i(\theta; M) \leq I(\theta)$ for all measurements $M$. Their derivation, which we give in a moment, includes two inequality steps and therefore yields a pair of necessary and sufficient conditions for equality. We are going to analyse these conditions in detail in the sequel. Actually a third inequality is involved and hence a third condition has to be added. The first inequality step is the trivial $(\Re(z))^2 \leq |z|^2$ with equality if and only if $\Im(z) = 0$. The second is the Cauchy-Schwarz inequality $|\text{trace} (A^* B)|^2 \leq \text{trace}(A^* A) \text{trace}(B^* B)$ with equality if and only if $\text{trace}(B^* B) A = B \text{trace}(B^* A)$. The third inequality step follows from the fact that $M(\mathcal{X}^+) \leq 1$, i.e., $1 - M(\mathcal{X}^+)$ is a nonegative operator. Here is the proof:

$$i(\theta; M) = \int_{\mathcal{X}^+} p(x; \theta)^{-1} (\Re \text{trace} (\rho \lambda m(x)))^2 \mu(dx)$$
\[
\begin{align*}
\int_{X_+} p(x; \theta)^{-1} |\text{trace}(\rho \lambda m(x))|^2 \mu(dx) &\leq \int_{X_+} |\text{trace}(m(x) \frac{1}{2} \rho^2 \frac{1}{2})^* (m(x) \frac{1}{2} \lambda \rho^2 \frac{1}{2}) |^2 (\text{trace}(\rho m(x)))^{-1} \mu(dx) \\
&= \int_{X_+} \text{trace}(m(x) \lambda \rho \lambda) \mu(dx) \\
&\leq \text{trace}(\rho \lambda^2) = I(\theta) . 
\end{align*}
\]

The necessary and sufficient conditions for equality at the first two inequality steps in (3) can be recombined, following Braunstein and Caves, as the following two conditions: for \( \mu(dx) \) almost all \( x \) in \( X_+ \)

\[
\Im \text{trace}(\rho \lambda m(x)) = \Im \text{trace}((m(x) \frac{1}{2} \rho^2 \frac{1}{2})^* (m(x) \frac{1}{2} \rho^2 \frac{1}{2})) = 0 \tag{4}
\]

and

\[
(m(x) \frac{1}{2} \lambda \rho^2 \frac{1}{2}) \propto \Re (m(x) \frac{1}{2} \rho^2 \frac{1}{2}) \tag{5}
\]

where \( \propto \Re \) denotes ‘proportional to, with real constant of proportionality’. As Braunstein and Caves remark, these conditions are satisfied when each \( m(x) \) is proportional to a projector onto an eigenstate of \( \lambda \). Our third condition

\[
\text{trace}(M(X_+) \lambda \rho \lambda) = \text{trace}(\rho \lambda^2) \tag{6}
\]

is also satisfied in that case. In particular, one can achieve the bound by a von Neumann measurement of the observable \( \lambda \). However this is not the only way that the conditions (4) and (5) are satisfiable, and we therefore take a closer look at them.

### 3. Attainability of the quantum information bound

In the sequel we will be specially interested in models for pure states, \( \rho = \vert \psi \rangle \langle \psi \vert \), where the vector state is given by \( \vert \psi \rangle = \vert \psi(\theta) \rangle \). In that case, the quantum score can be computed explicitly and the conditions (4) and (5) simplify. Define the (unnormalized) vector \( \vert a \rangle = 2 \vert \psi \rangle \). Since \( \rho^2 = \rho \), we have \( \hat{\rho} = \rho \hat{\rho} + \hat{\rho} \rho \). The defining equation (1) for the quantum score therefore tells us that \( \lambda = 2 \hat{\rho} = \vert a \rangle \langle \psi \vert + \vert \psi \rangle \langle a \vert \). Now let \( \vert 1 \rangle = \vert \psi \rangle \) and let \( \vert 2 \rangle \) be a normalized orthogonal state such that \( \vert a \rangle \) is in the subspace spanned by \( \vert 1 \rangle \) and \( \vert 2 \rangle \); write \( \vert a \rangle = a_1 \vert 1 \rangle + a_2 \vert 2 \rangle \) where \( a_1 = \langle 1 \vert a \rangle \) and \( a_2 = \langle 2 \vert a \rangle \). (Note that all these definitions are relative to a given value of the parameter \( \theta \).) We find that

\[
\frac{1}{\rho^2 \lambda} = \rho \lambda = 2 \Re a_1 \vert 1 \rangle \langle 1 \vert + \overline{a}_2 \vert 1 \rangle \langle 2 \vert
\]

and the conditions (4) and (5) reduce to

\[
\Im \text{trace}\left( (2 \Re a_1 \vert 1 \rangle \langle 1 \vert + \overline{a}_2 \vert 1 \rangle \langle 2 \vert) m(x) \right) = 0
\]
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and

\[ m(x) \frac{1}{2} (2 \Re a_1 |1 \rangle \langle 1 | + a_2 |2 \rangle \langle 1 |) \propto \Re m(x) \frac{1}{2} |1 \rangle \langle 1 |. \]

Each can be again simplified, resulting in the conditions

\[ \Im (\bar{a}_2 \langle 2 | m(x) | 1 \rangle) = 0 \]

or equivalently, on conjugation,

\[ \Im \left( 1 | m(x) \frac{1}{2} | a_2 m(x) \frac{1}{2} | 2 \right) = 0, \]

and

\[ a_2 m(x) \frac{1}{2} |2 \rangle \propto \Re m(x) \frac{1}{2} |1 \rangle \]

or equivalently

\[ a_2 m(x) \frac{1}{2} |2 \rangle \propto \Re m(x) \frac{1}{2} |1 \rangle. \]  \hfill (7)

We see that for pure states, conditions (4) and (5) reduce to (7).

For spin-half models, thus a Hilbert space of dimension 2, a further simplification occurs. One can take \(|2 \rangle = |\psi\rangle^\perp\), forming an orthonormal basis (depending on \(\theta\)) with \(|1 \rangle = |\psi\rangle\). From (7) it follows that if \(m(x)\) achieves equality in (3), it must have less than full rank, and hence in the spin-half case both it and its square root must be proportional (with real constant of proportionality) to \(|\xi\rangle \langle \xi|\) for some state \(|\xi\rangle = |\xi(x)\rangle\).

A minor rewriting yields the following: for a pure spin-half state \(\rho = |\psi\rangle \langle \psi|\), the first two necessary and sufficient conditions for equality in (3) are equivalent to: for \(\mu(dx)\) almost all \(x \in X_+\), \(m(x)\) is proportional to a one-dimensional projector \(|\xi(x)\rangle \langle \xi(x)|\) satisfying

\[ \langle \xi | 2 \rangle \langle 2 | a \rangle \propto \Re \langle \xi | 1 \rangle \]

where \(|1 \rangle = |\psi\rangle\), \(|2 \rangle = |\psi\rangle^\perp\), and \(|a\rangle = 2|\hat{\psi}\rangle\).

Finally, let us consider an example. Consider a spin-half particle in the pure state \(|\psi\rangle = |\psi(\eta, \theta)\rangle\) given by

\[ |\psi\rangle = \begin{pmatrix} e^{-i\theta/2} \cos(\eta/2) \\ e^{i\theta/2} \sin(\eta/2) \end{pmatrix}. \]

As is well known, this pure state has density matrix \(\rho = \frac{1}{2} (1 + u_x \sigma_x + u_y \sigma_y + u_z \sigma_z) = \frac{1}{2} (1 + \vec{u} \cdot \vec{\sigma})\) where \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) are the three Pauli spin matrices and \(\vec{u} = (u_x, u_y, u_z) = \vec{u}(\eta, \theta)\) is the point on the unit sphere in \(\mathbb{R}^3\) with polar coordinates \((\eta, \theta)\). Suppose the colatitude \(\eta \in [0, \pi]\) is known and exclude the degenerate cases \(\eta = 0\) or \(\eta = \pi\); the longitude \(\theta \in [0, 2\pi]\) is the unknown parameter.

We have a pure state so \(\lambda = 2\dot{\rho} = 2\ddot{u} \cdot \vec{\sigma} = 2\sin(\eta) \ddot{u}(\eta/2, \theta + \pi/2) \cdot \vec{\sigma}\). Using the familiar relations \(\sigma_x^2 = 1\), \(\sigma_x \sigma_y = -\sigma_y \sigma_x = i \sigma_z\), and their cyclic permutations, and
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the fact that the spin matrices are traceless, one finds that the quantum information is $I(\theta) = \text{trace} \rho \lambda^2 = \sin^2 \eta$.

In order to achieve this information, almost all $m(x)$ must be of the form $|\xi \rangle \langle \xi|$ where $\xi = \xi(x)$ must satisfy (8). Now $|\psi\rangle = \psi(\pi - \eta, \theta + \pi)\rangle$ and $|\alpha \rangle = 2|\psi\rangle = |\psi(\eta, \theta + \pi)\rangle$. One finds $a_2 = \langle 2 | a \rangle = \sin \eta$, a nonzero real. To emphasize their dependence on $\theta$ let us write $|1\rangle = |1\rangle_\theta$, $|2\rangle = |1\rangle_\theta$. The conditions for equality are that $\xi_1 \propto \xi_2$, where $\xi_1 = \langle 1_\theta | \xi(x) \rangle$ and $\xi_2 = \langle 1_\theta | \xi(x) \rangle$.

Is it possible that this is true for for all $\theta$ simultaneously? Equivalently, can we have $|\xi \rangle = e^{i\phi(\theta)}(\alpha(\theta) |1\rangle + \beta(\theta) |2\rangle)$ constant in $\theta$ for real functions $\alpha$, $\beta$ and $\phi$? If so, then $\langle \xi | \xi \rangle = \alpha^2 + \beta^2$ is constant, and $| \langle \xi | \psi(0,0) \rangle |^2 = \alpha^2 \cos^2(\eta/2) + \beta^2 \sin^2(\eta/2)$ is constant. This implies, as long as $\eta \neq \pi/2$ so that these two equations are linearly independent, that $\alpha^2$ and $\beta^2$ are constant. Thus the pair $(\alpha, \beta)$ takes on at most four different values as $\theta$ varies continuously, and hence one can find two (indeed, uncountably many) different values of $\theta$ for which $(\alpha, \beta)$ is constant. But then $|\xi \rangle \langle \xi|$ is not constant over these same values of $\theta$. Consequently, for $\eta \neq \pi/2$, no measurement $M$ exists with Fisher information $i(\theta; M)$ equal to the quantum information $I(\theta)$ whatever the value of the unknown parameter $\theta$.

If $\eta = \pi/2$ it is possible to achieve the bound uniformly in $\theta$. Any measurement with all components proportional to projector matrices for spin directions in the plane $\eta = \pi/2$ will do the job; in particular, any simple measurement of spin in that plane.

4. Asymptotic attainability and vector parameters

We have shown, for the case of a one-dimensional parameter as considered by Braunstein and Caves, that there need not exist a measurement $M$ such that $i(\theta; M) = I(\theta)$ for all parameter values $\theta$ simultaneously. It is on the other hand possible to find a measurement $M$ such that at a given parameter-value, $i(\theta; M) = I(\theta)$, as Braunstein and Caves indicate: take each $m(x)$ proportional to a projector onto an eigenspace of the quantum score $\lambda(\theta)$. They do not remark on the possible dependence of $M$ on $\theta$. However, if all we know is that $\rho = \rho(\theta)$ for some $\theta$, we do not know which measurement to use. (This is a common oversight; see Fujiwara and Nagaoka (1995) section 4, formula (17) for another instance.) The eigenspace decomposition of $\lambda$ generally depends on $\theta$ so this does not define a measurable $M$ which achieves the bound uniformly in $\theta$. This is not the only way to achieve the bound, but our example in the previous section shows that one should not expect there to be a uniformly (in $\theta$) attaining measurement.

Note that the classical information based on $n$ independent and identically distributed realisations from a given density $p(x, \theta)$ is equal to $n$ times the information for one realisation. Similarly, the quantum information in the state $\rho(\theta)^\otimes n$ corresponding to $n$ identical particles each in state $\rho(\theta)$ is $n$ times the quantum information for one particle.

Braunstein and Caves’ aim was to define a statistical distinguishability metric
between quantum states. Suppose the measurement $M$ on a single particle satisfies $i(\theta; M) = I(\theta)$. Then the maximum likelihood estimator of $\theta$ based on $n$ separate measurements of $M$ on identical copies of the given quantum system, by classical results in mathematical statistics, is generally an asymptotically unbiased estimator with asymptotic variance $(ni(\theta; M))^{-1} = (nI(\theta))^{-1}$. By the quantum Cramér-Rao bound applied to the joint system of $n$ particles, no estimator based on any measurement whatsoever on $\rho^\otimes n(\theta)$ can do better. Thus $I(\theta)$ appears to exactly characterize the rate at which one can determine $\theta$.

However this argument is flawed since the measurement $M$ involved will be a different measurement for each $\theta$, and the whole point is that $\theta$ is not known in advance. The question therefore remains: does there exist a measurement procedure not depending on $\theta$ on the state $\rho^\otimes n$, on the basis of which an estimator of $\theta$ can be constructed having asymptotic variance $(nI(\theta))^{-1}$? If the answer is ‘yes’, then Braunstein and Caves’ proposed role for the quantum information $I(\theta)$ in defining a statistical distinguishability metric is well motivated.

It seems rather natural to try a two-stage procedure: first estimate the parameter using a perhaps inefficient procedure on a vanishing proportion of the particles, say $n_0 = n^\alpha$ ($0 < \alpha < 1$) out of the total of $n$; now carry out the ‘estimated optimal measurement’ on the remaining ones. In both stages only simple or von Neumann measurements (measurements of classical observables) on separate particles are needed.

In our example this would reduce to the following. Measure the spin $\sigma_x$ on $k = \frac{1}{2}n_0$ of the copies. The number of +1’s observed is binomially distributed with parameters $k$ and $p = \frac{1}{2}(\sin \eta \cos \theta + 1)$. Similarly for another $k$ measurements of the spin $\sigma_y$ we get a binomial number of ‘+1’ with parameters $k$ and $p = \frac{1}{2}(\sin \eta \sin \theta + 1)$. This allows us consistent estimation of both $\sin \theta$ and $\cos \theta$ and hence of $\theta \in [0, 2\pi)$. Denote such an estimator by $\tilde{\theta}$. We saw that $\lambda$ in this example was proportional to the spin in the direction $(\pi/2, \theta + \pi/2)$. Let us use the remaining $n' = n - n_0$ particles to measure this spin with $\theta$ replaced by $\tilde{\theta}$. Given $\tilde{\theta}$, this results in a binomial number $X$ of ‘+1’ with parameters $n'$ and $p = \frac{1}{2}(1 - \sin \eta \sin(\theta - \tilde{\theta}))$. Let

$$\hat{\theta} = \tilde{\theta} + \arcsin((n' - 2X)/(n' \sin \eta)).$$

Analysis of this ‘final’ estimator shows that $\hat{\theta}$ has asymptotically the $N(\theta, (n \sin^2(\eta))^{-1})$ distribution (the normal distribution with indicated mean and variance), whatever $\theta$, so that the quantum information bound is asymptotically achievable by our two stage procedure.

This approach will work in wide generality in problems with a one-dimensional parameter $\theta$. Suppose, as typically will be possible, that one can construct a consistent estimator $\tilde{\theta}$ based on certain measurements on a vanishing proportion of the particles. Compute the quantum score at $\theta = \tilde{\theta}$, and measure it on each of the remaining particles. Compute the maximum likelihood estimator $\hat{\theta}$ of $\theta$ based on the new data, whose probability distribution depends on the unknown $\theta$ (as well as on $\tilde{\theta}$, which is at this stage fixed). We argue as follows that $\hat{\theta}$ has approximately the $N(\theta, (nI(\theta))^{-1})$ distribution,
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thus this estimator asymptotically achieves the quantum information bound. Let \( i(\theta; \tilde{\theta}) \) denote the Fisher information for \( \theta \) in a measurement, on one particle, of the quantum score at \( \tilde{\theta} \); thus \( i(\tilde{\theta}; \tilde{\theta}) = I(\tilde{\theta}) \) for all values of \( \tilde{\theta} \), but generally \( i(\theta; \tilde{\theta}) < I(\tilde{\theta}) \). Now for \( n \) large, \( \tilde{\theta} \) is close to \( \theta \). By the classical results for maximum likelihood estimators, given \( \tilde{\theta} \), \( \tilde{\theta} \) has approximately the \( \mathcal{N}(\theta, (ni(\theta; \tilde{\theta}))^{-1}) \) distribution. So if \( \rho \) depends on \( \theta \) smoothly enough that \( i(\theta; \tilde{\theta}) \) is close to \( i(\tilde{\theta}; \tilde{\theta}) = I(\tilde{\theta}) \) for \( \tilde{\theta} \) close to \( \theta \), we have that unconditionally \( \tilde{\theta} \) has approximately the \( \mathcal{N}(\theta, (nI(\theta))^{-1}) \) distribution, hence asymptotically achieves the bound.

Consider now the case of vector parameters. Both quantum and Fisher information numbers are naturally generalized to information matrices; see Helstrom (1976), Holevo (1982). The Braunstein and Caves result generalizes to the following result: the quantum information matrix is larger (in the sense that the difference is positive semi-definite) than the Fisher information matrix based on the outcome of any measurement \( M \). However the bound is no longer attainable. As we saw above, the best measurement for each parameter separately is the measurement of the quantum score operator. Typically these do not commute and hence cannot be measured simultaneously. On the other hand, consideration of all smooth one-dimensional sub-models of a given model \( \rho = \rho(\theta) \) shows that the quantum information matrix \( I_\theta \) is the smallest matrix larger than the Fisher information matrix of any measurement on a single particle.

For instance, suppose we want to simultaneously estimate both parameters \( \eta, \theta \) of the pure-state, spin-half system; in other words, we have a completely unknown pure state. Rename \( \theta \) as \( \phi \), and let \( \theta \) from now denote the vector parameter with elements \( \eta, \phi \). Suppose we may dispose of a large number of identical copies of this system. Let \( I(\theta) \) denote the 2×2 quantum information matrix, and \( i(\theta; M) \) denote the Fisher information matrix based on the outcome of a measurement \( M \), both for a single copy of the quantum system. The quantum scores, for a single particle, for the two parameters \( \eta \) and \( \phi \) are \( \sigma_{\eta + \pi/2, \phi} \) and \( \sin \eta \sigma_{\pi/2, \phi + \pi/2} \) respectively. After a small proportion of measurements we know roughly the location of the parameter, and it is sufficient to investigate optimal measurement at a ‘known’ parameter value.

Without loss of generality let this be the special point \( \eta = \pi/2, \phi = 0 \). At this point the quantum scores are \( \sigma_y \) and \( -\sigma_z \), and the quantum information matrix is the identity \( \mathbf{1} \). However there is no measurement (on a single particle) whose probability distribution has Fisher information matrix for \( \theta \) equal to the quantum information matrix, since by our results it would have to be of the form \( m(x) = |\xi\rangle\langle\xi| \) with \( |\xi\rangle \), up to a phase, equal to \( |\alpha\rangle \uparrow + |\beta\rangle \downarrow \) with \( \alpha \) and \( \beta \) real and \( \alpha \) non-zero for attainability of the \( \phi \) component of the information, while by a similar calculation for \( \eta \) (for which the quantum score is \( i|\uparrow\rangle\langle\uparrow| - i|\downarrow\rangle\langle\downarrow| \) \( |\xi\rangle \), up to a phase, of the form \( \alpha'|\uparrow\rangle + \beta'|\downarrow\rangle \) with \( \alpha' \) and \( \beta' \) real and \( \alpha' \) non-zero, which is only possible if \( \beta = \beta' = 0 \). Though \( m(x) \) can have this form for some \( x \) it is impossible for it to be true for all, since \( \int m(x)\mu(dx) = 1 \).

Since different components of the parameter vector have incompatible quantum scores, it is clear that for different loss functions, different measurements will be optimal.
No single procedure will (asymptotically) dominate all others. Moreover, since we cannot achieve the quantum information bound by measurements on single particles, it is possible that joint measurements on several particles simultaneously could give larger Fisher information (per particle) than measurements on separate particles.

In some very special cases, an optimal procedure is known. An appealing loss function in the completely unknown pure spin-half model is one minus the squared inner-product between the true state vector and its estimate. This equals one minus the squared cosine of half the angle between the points on the Poincaré sphere representing the two states. At the special point under consideration therefore, the loss function is asymptotically equivalent to one quarter times the sum of the squares of the errors in $\eta$ and $\phi$. Massar and Popescu (1995), in response to a problem posed by Peres and Wootters (1991), exhibited a measurement, optimal in the Bayes sense, with respect to this loss function and a uniform prior distribution. It had an asymptotic mean square error $4/n$. This was a genuine generalised measurement of the composite system $\rho^{\otimes n}$.

They showed that for the case of $n = 2$ there were no measurement methods of the two particles separately which were as good as the optimal method, and this is expected to hold for all $n$.

Instead of this exactly optimal procedure (with respect to the given loss function and under a uniform prior) consider taking with probability half measurements of $\sigma_y$ and $\sigma_z$, independently on each particle. We find that the Fisher information matrix (based on one observation) for $\eta, \phi$, at $\eta = \pi/2, \phi = 0$, is $\frac{1}{2} \mathbf{1}$, or one half of the quantum information matrix. The inverse of this matrix, $2\mathbf{1}/n$ is an asymptotically achievable lower bound to the covariance matrix of (asymptotically unbiased) estimators of $\eta, \phi$ based on $n$ of such measurements. The maximum likelihood method would provide estimators asymptotically achieving this bound. The sum of the variances is $4/n$, the same as what is achieved by the Massar and Popescu procedure.

Thus the following two-stage procedure, similar to what we proposed in the one-parameter case, should have asymptotically equivalent covariance matrix to that of the Massar and Popescu procedure, and will also be optimal with respect to a uniform prior distribution and any smooth loss function, invariant under rotations of the sphere. First carry out measurements of each of $\sigma_x, \sigma_y$ and $\sigma_z$ on a small proportion of separate particles. Compute from the results a consistent estimate of $(\eta, \phi)$. With respect to a rotated coordinate system putting the estimated value at $(\eta, \phi) = (\pi/2, 0)$, measure alternately $\sigma_y$ and $\sigma_z$ on the remaining particles. Estimate $(\eta, \phi)$ in the new coordinate system by the method of maximum likelihood using the second stage observations. Finally rotate back to the original coordinate system.

This conjecture has been confirmed by recent further work of Gill and Massar (1999). Consider a sequence of measurements on $n$ identical copies of a spin-half state, on which is based a sequence of estimators $\hat{\theta}$. The parameter $\theta$ might be one- or two-dimensional for a pure state model, one-, two- or three-dimensional for a mixed state. Suppose that the estimators are asymptotically unbiased and have covariance matrices asymptotically of the form $V(\theta)/n$. Then it is now known that the collection
of attainable $V$ is precisely $\{V : \text{trace } I(\theta)^{-1}V(\theta)^{-1} \leq 1 \text{ for all } \theta \}$, in any of the following cases: the parameter is one-dimensional, or the state is pure, or the measurement can be implemented by separate measurements on separate particles. These optimal limiting covariance matrices can all be achieved using a two stage adaptive procedure of the type described above. The collection of attainable $V$ corresponds to the collection of attainable inverse Fisher information matrices for measurements on single particles. Using joint measurements on mixed states with more than one unknown parameter, one can attain strictly smaller asymptotic covariance matrices. But a clean description of what is attainable is not known. One would like to describe the collections of scaled information matrices $\{i(\theta; M^m)/m\}$ for $m = 1, 2, \ldots$, where $M^m$ is an arbitrary joint measurement on $m$ particles. These sets are all convex, they grow with $m$; we know the set for $m = 1$; and each set is included in the set of matrices less than or equal to $I(\theta)$. The inverses of these information matrices will be the achievable (scaled) asymptotic covariance matrices based on measuring a large number $n$ of particles in groups of $m$ at a time.

To conclude, in the multiparameter case, the bound implied by the quantum information matrix is not even asymptotically achievable. The rate at which one can distinguish between more than two neighbouring quantum states does not correspond to the rate at which one can distinguish between just two; it depends on what aspect of the quantum states is important, and it depends on whether one may use joint measurements or only separate measurements. The quantum information matrix only plays a role in special cases.

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