The Luminosity Distance, the Equation of State, and the Geometry of the Universe

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The second derivative of the luminosity distance with respect to the redshift is written in terms of the deceleration parameter $q_0$. We point out that the third derivative contains the information regarding the sound speed of cosmic matter as well as the curvature of the universe. We restrict physically possible parameter ranges of the coefficients. It is found that there is a relation between the coefficients in a flat universe model with matter such that $c_{s0}(1 + w_{x0}) = 0$ ($c_{s0}$ is the total sound speed of the matter component and $w_{x0} = p_{x0}/\rho_{x0}$).

§1. Introduction

There is growing observational evidence that the mean mass density of the universe is significantly less than the critical density$^{1,3}$. The cosmological constant is usually introduced to reconcile observations with the inflationary prediction of a spatially flat universe. However, the introduction of non-zero cosmological constant requires the fine-tuning of the vacuum energy, and at present we do not have any convincing explanation for the reason why such an extremely small value of the cosmological constant (in Planck units) is required.

Under these circumstances, several extensions of the standard $\Omega_{M0} = 1$ cold dark matter model have been proposed$^{4,6}$. Here $\Omega_{M0}$ is the present ratio of the mean mass density of the universe to the critical density. Essentially, these models are characterized by an “x-component” having two parameters; the present ratio of its pressure to its energy density $w_{x0} = p_{x0}/\rho_{x0}$ and the present sound speed $c_{s0}^2 = \delta p_{x0}/\delta \rho_{x0}$$^{5,7}$. Unfortunately, however, we do not presently have principles regarding the scalar field potential or the equation of state. Indeed, the situation concerning the dark matter component is becoming dark. However, it should be noted that any physically sensible attempts to include the x-component should introduce $c_{s0}^2 \geq 0$ independent of $w_{x0}$$^{6,7}$.

Recently, on the other hand, projects to discover high-redshift Type Ia supernovae are ongoing, and data are accumulating up to $z \simeq 1$$^{8,9}$. These data may indicate that the expansion of the universe is accelerating rather than decelerating$^{10,11}$. There is another indication for the existence of an x-component with $w_{x0} \lesssim -0.6$$^{12}$. In this letter, we consider the interpretation of data in more general context. To be specific, we attempt to extract useful information regarding the equation of state of the universe directly from observational data rather than to adjust

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a particular scalar field potential or equation of state to observations. The recent observations by the Type Ia supernovae cosmology project are providing the information regarding the third derivative (denoted as \(d_3\)) as well as the second derivative (\(d_2\)) of the luminosity distance with respect to the redshift\(^{13}\). First, we restrict possible parameter ranges of \(d_2\) and \(d_3\) using physical considerations. We then derive a relation between \(d_2\) and \(d_3\) for flat models. We discuss the implications of this consistency relation.

§2. Luminosity Distance and Equation of State of the Universe

We assume that our universe can be described by a homogeneous and isotropic (FRW) model. The luminosity distance is defined by

\[
d_L(z) = a_0(1 + z)f(\chi),
\]

\[
\chi = \frac{1}{a_0} \int_0^z \frac{du}{H(u)},
\]

where \(a_0\) is the present scale factor and \(f(\chi) = \chi, \sinh(\chi)\) and \(\sin(\chi)\) for flat, open and closed universes, respectively. We expand \(d_L(z)\) as \(H_0 d_L(z) = \sum_{i=1} d_i z^i\) and determine \(d_2\) and \(d_3\) (\(d_1 = 1\) by definition).

\[
H(z) = H_0 + \left. \frac{dH}{dz} \right|_0 z + \left. \frac{1}{2} \frac{d^2H}{dz^2} \right|_0 z^2 + \ldots,
\]

where “0′′ indicates that the value to be evaluated is that at the present epoch. By using the relation

\[
\frac{dt}{dz} = -\frac{1}{(1 + z)H(z)},
\]

we obtain

\[
\left. \frac{dH}{dz} \right|_0 = -\left. \frac{\dot{H}}{(1 + z)H} \right|_0 = (1 + q_0)H_0,
\]

\[
\left. \frac{d^2H}{dz^2} \right|_0 = -\left. \frac{\ddot{H}}{(1 + z)^2H^2} \right|_0 + \left. \frac{1}{(1 + z)^2H^2} \left( \frac{\dot{H}}{(1 + z)^2H^2} - \frac{\dot{H}^2}{(1 + z)^2H^3} \right) \right|_0
\]

\[
= (c_0 + 3q_0 + 2)H_0 - (2 + 3q_0 + q_0^2)H_0,
\]

where \(q_0\) and \(c_0\) are defined by\(^*\)

\[
q_0 \equiv \left. \left( \frac{\ddot{a}}{a^2} \right) \right|_0,
\]

\[
c_0 \equiv \left. \left( \frac{a^2}{a^3} \right) \right|_0.
\]

\(^*\) “c” stands for cubic, not to be confused with the sound speed, \(c_s\).
Then we have

\[ H_0 d_L(z) = \sum_{i=1}^{3} d_i z^i = z + \frac{1}{2} (1 - q_0) z^2 + \frac{1}{3!} (3q_0^2 + q_0 - c_0 - \Omega_{k0} - 1) z^3 + \ldots, \quad (2.9) \]

where \( \Omega_{k0} \equiv k/(a_0 H_0)^2 \) with \( k = 0, -1 \) and 1 for flat, open and closed universes, respectively.

Thus \( d_2 \) and \( d_3 \) are written as

\[
\begin{align*}
  d_2 &= \frac{1}{2} (1 - q_0) \\
  d_3 &= \frac{1}{6} (3q_0^2 + q_0 - c_0 - \Omega_{k0} - 1)
\end{align*}
\]

(2.10) (2.11)

We find that \( d_3 \) contains information concerning \( c_0 \) and the curvature of the universe. The meaning of \( c_0 \) will be clarified later.

We have not yet specified the properties of the cosmic matter. Now we restrict the possible ranges of \( d_2 \) and \( d_3 \) on the basis of physical requirements. We impose the following physically reasonable requirements on the cosmic matter: (1) the total density is non-negative; (2) the total density is currently not increasing as a function of time; (3) the present sound speed \( c_{s0} \) of the total system satisfies \( 0 \leq c_{s0}^2 \leq 1 \) for causality and local stability. The first and second requirements are equivalent to assuming the weak energy condition \(^{14}\). \(^*\)

The first requirement implies

\[ \Omega_{k0} \geq -1. \]

(2.12)

By use of the relation

\[ \dot{\rho} = -3H(\rho + p) = \frac{3H}{4\pi} \left( \dot{H} - \frac{k}{a^2} \right), \]

(2.13)

we find that the second requirement implies

\[ 1 + q_0 + \Omega_{k0} \geq 0. \]

(2.14)

The sound speed of the total system, \( c_{s0} \), can be written as

\[ \frac{c_{s0}^2}{\rho} \left|_{\dot{\rho}/\rho} \right. = \frac{c_0 - \Omega_{k0} - 1}{3(1 + q_0 + \Omega_{k0})}. \]

(2.15)

Then the third requirement, together with Eq.(2.14), implies

\[ 1 + \Omega_{k0} \leq c_0 \leq 4(1 + \Omega_{k0}) + 3q_0. \]

(2.16)

Now we rewrite these constraints in terms of \( d_2 \), \( d_3 \) and \( \Omega_{k0} \). The second constraint is rewritten as

\[ d_2 \leq 1 + \frac{1}{2} \Omega_{k0}. \]

(2.17)

\(^*\) In a \( \Lambda < 0 \) dominated universe, the first condition can be violated. We exclude such a case.
The third constraint is rewritten as
\[ 2d_2^2 - \frac{4}{3}d - \frac{2}{3} - \frac{5}{6} \Omega \leq d_3 \leq 2d_2^2 - \frac{7}{3}d + \frac{1}{3} - \frac{1}{3} \Omega. \] (2.18)

To conclude, the region bounded by the relations in Eq.(2.12) and Eq.(2.17) and Eq.(2.18) is allowed by three requirements. The allowed region is shown in Fig.1 for several \( \Omega \).

The quantities \( q_0 \) and \( c_0 \) can also be written in terms of the dark matter components as
\[
q_0 = \sum_i \frac{\Omega_i}{2} (1 + 3w_i),
\]
(2.19)
\[
c_0 = 1 + \Omega_{k0} + \frac{9}{2} c_0 \sum_i \Omega_i (1 + w_i),
\]
(2.20)
where \( i \) denotes the \( i \)-th component, and \( w_i = p_i / \rho_i \). We also note that the total sound speed \( c^2_{s0} \) can be written as

\[
c^2_{s0} = \frac{\sum_i c^2_{i0} \Omega_i (1 + w_i)}{\sum_i \Omega_i (1 + w_i)}. \]
(2.21)

Thus, we see that \( c_0 \) (or \( d_3 \)) contains information regarding the sound speed of the dark matter component. In the xCDM model (that is, CDM plus x-component; x can be \( \Lambda \) or anything else), \( q_0 \) and \( c_0 \) can be written as
\[
q_0 = \frac{\Omega_M}{2} + \frac{\Omega_x}{2} (1 + 3w_x),
\]
(2.22)
\[
c_0 = 1 + \Omega_{k0} + \frac{9}{2} c^2_{s0} \Omega_x (1 + w_x).
\]
(2.23)

For example, consider x to be \( \Lambda \) and take \((\Omega_M, \Omega_x) = (0.40, 0.70)\), that is, \((q_0, c_0) = (-0.50, 1.10)\) so that the universe is closed. However, in the xCDM model, even a flat model with \((\Omega_x, w_x, c^2_{s0}) \simeq (0.80, -0.83, 0.33)\) is allowed. The point is that the values of \( d_2 \) and \( d_3 \) can not determine the equation of state of the universe uniquely. Depending on the values of \( d_2 \) and \( d_3 \), an open or closed \( \Lambda \)CDM model may be interpreted by a flat xCDM model, for example. This degeneracy can be resolved only when \( \Omega_0 \) is determined independently, which may be done by observing the variation of the redshift in the absorption lines of quasars.\(^{15}\).

We would like to point out that there is an interesting “consistency relation” in flat models: If we assume \( c_{s0}(1 + w_{s0}) = 0 \) and a flat model, then \( d_3 \) can be written in terms of \( d_2 \) as
\[
d_2 = \frac{1}{2} (1 - q_0),
\]
(2.24)
\[
d_3 = \frac{1}{6} (3c^2_{s0} + q_0 - 2) = 2d_2^2 - \frac{7}{3}d_2 + \frac{1}{3}.
\]
(2.25)

Although this relation seems rather trivial, the contraposition of the above statement has important meanings. Namely, \textit{if the consistency relation Eq.}(2.25) \textit{did not hold}
in the observational data, then it would follow that the dark matter component has the property $c_{s0}(1+w_{x0}) \neq 0$, or the Universe is not flat. Note that for $\Lambda$-dominated universe models, the former possibility is automatically satisfied. The consistency relation Eq.(2.25) is represented in Fig.1 for $\Omega_{k0} = 0$ as a thick curve. Of course, the current observational data suffer from uncertainties, and it is premature to put the consistency relation into practice. However, if observations become sufficiently accurate, we should prepare ourselves for the possibility that the dark matter component has the property $c_{s0}(1+w_{x0}) \neq 0$, or the Universe is not flat. Both possibilities should be of great significance.

§3. Conclusion

We have shown that the third derivative of the luminosity distance with respect to redshift contains information regarding the sound speed of the cosmic matter as well as the curvature of the universe. We have also studied the possible parameter ranges of the expansion coefficients of the luminosity distance and have derived a consistency relation for the flat universe model with the cosmic matter of $c_{s0}(1+w_{x0}) = 0$. The number of data taken by the supernova cosmology project is increasing, and better statistics will be available in future. Then the more general possibility presented in this paper may be the case.

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**Figure Caption**

**Figure 1:** The allowed regions of $d_2$ and $d_3$ for several $\Omega_{k0}$. A region that is bounded by two curves and is on the left-hand-side of the vertical line is allowed. For $\Omega_{k0} = 0$, the consistency relation is indicated by the thick curve.