The Inverse Variational Problem for Autoparallels

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Abstract

We study the problem of the existence of a local quantum scalar field theory in a general affine metric space that in the semiclassical approximation would lead to the autoparallel motion of wave packets, thus providing a deviation of the spinless particle trajectory from the geodesics in the presence of torsion. The problem is shown to be equivalent to the inverse problem of the calculus of variations for the autoparallel motion with additional conditions that the action (if it exists) has to be invariant under time reparametrizations and general coordinate transformations, while depending analytically on the torsion tensor. The problem is proved to have no solution for a generic torsion in four-dimensional spacetime. A solution exists only if the contracted torsion tensor is a gradient of a scalar field. The corresponding field theory describes coupling of matter to the dilaton field.

1 Introduction and motivations

In Riemann-Cartan spaces, a connection $\Gamma_{\mu\nu}^\sigma$ compatible with the metric $g_{\mu\nu}$ (meaning that $D_\mu g_{\nu\sigma} = 0$, with $D_\mu$ being the covariant derivative) may have nonvanishing antisymmetric components $S_{\mu\nu}^\sigma = \frac{1}{2}(\Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma)$ which are the torsion tensor components in a coordinate basis. A general affine connection compatible with the metric can always be represented in the form \[ \Gamma_{\mu\nu}^\sigma = \Gamma_{\mu\nu}^\sigma + g_{\sigma\alpha}(S_{\mu\nu\alpha} - S_{\nu\mu\alpha} + S_{\alpha\mu\nu}), \] where $\Gamma_{\mu\nu}^\sigma$ are the Christoffel symbols associated with the metric $g_{\mu\nu}$. As was first pointed out by Cartan, the existence of connections that are compatible with the metric and do not coincide with the natural Riemannian connection $\Gamma_{\mu\nu}^\sigma$ may lead to more general theories of gravity than Einstein’s general relativity (see, e.g., for a review [2] and references therein). Consequently, the actual motion of a particle may, in principle, deviate from the usual geodesic motion due to an interaction with torsion.

The torsion force cannot be arbitrary and its possible form should be obtained from some physical principles. It is natural to assume the actual motion of a particle to enjoy general coordinate covariance. A trajectory of the motion is determined by its tangent vector (or velocity). So to specify the corresponding equations of motion, one has to define the variation of the velocity along the trajectory. In a space with a general affine connection there exist two independent variation operators that involve a displacement and produce tensors out of tensors (i.e., variations covariant under general coordinate transformations): the Lie derivative and the covariant derivative [1], p.335. A physically acceptable variation should contain the displacement $d_{\alpha}u^\mu = \dot{u}^\mu$ of the velocity along

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1 On leave from Laboratory of Theoretical Physics, JINR, Dubna, Russia.
itself (acceleration). The Lie derivative does not provide us with such a displacement. Therefore the only possibility is

\[ D_u u^\mu = u^\nu D_\nu u^\mu = \dot{u}^\mu + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma = F^\mu(S, g, u), \]  

(1.1)

where \( F^\mu \) is a vector force. Next we require that the motion becomes geodesic when the torsion vanishes, that is, the vector \( u^\mu \) is transported parallel along itself with respect to a natural Riemannian connection \( \tilde{\Gamma}_{\mu\nu} \)

\[ \overline{D}_u u^\mu = \dot{u}^\mu + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma = 0. \]  

(1.2)

This implies the condition \( F^\mu(S = 0, g, u) = 0 \). The simplest possibility proposed first by Ponomarev [3] is to set \( F^\mu = 0 \). The corresponding curve is called the autoparallel. Its characteristic geometrical property is similar to that of geodesics. The tangent vector is transported parallel along itself with respect to a full affine connection. But it does not share another property of geodesics such as being the shortest line between two points of the manifold.

As follows from the comparison of Eqs. (1.2) and (1.1) with \( F^\mu = 0 \), the deviation of the autoparallel from the geodesic is caused by the torsion force \( 2 S_{\mu\nu\sigma} u^\nu u^\sigma \). The choice between the geodesic and the autoparallel motion can either be decided experimentally or on theoretical grounds following from the compatibility of the postulate \( F^\mu = 0 \) in (1.1) with other fundamental principles of physics. In [4] it is argued that the energy-momentum conservation law of a spinless point particle leads to geodesics rather than to autoparallels. The conclusion is based on the earlier work by Papapetrou [5] that prescribes a specific relation between the canonical momentum and the velocity of the particle. In general, the energy-momentum tensor is defined as the variational derivative of the particle Lagrangian with respect to the metric tensor. Its conservation law specifies the particle equations of motion that are the usual Euler-Lagrange equations. Hence, if equation (1.1) admits the Euler-Lagrange form, then the energy momentum conservation law may lead to the autoparallels as is shown in Appendix A with an explicit example.

Based on a physical analogy between spaces with torsion and crystals with topological defects [6], the attention has been brought again to the autoparallel motion in [7], Sec. 10, where it was also quantized by the path integral method. The approach gives a consistent quantum theory only for a special (“gradient”) torsion [7], Sec. 11. For a generic torsion it has lead to difficulties with the probabilistic interpretation of the corresponding quantum mechanics and with the correspondence principle [8].

The problem of coupling between matter and the spacetime geometry is undoubtedly of great importance. So far only the principle of minimal gauge coupling has been explored [9, 2], except, maybe, for the conformal coupling [10]. The aim of the present work is to approach the problem from a different and more general point of view. All models of the fundamental interactions are described by quantum field theory. Thus, if the autoparallels indeed describe the motion of a spinless point particle in a general Riemann-Cartan space, then they must follow from a local quantum scalar field theory in the semiclassical (eikonal) approximation. A conventional way to construct a quantum field
theory that satisfies the correspondence principle is first to quantize the relativistic particle motion, thus obtaining relativistic quantum mechanics, and then to apply the so called

Consider, for example, the geodesic motion (1.2). It follows from a least action principle for the action

$$S_g = \int L_g dt = -m \int \sqrt{g_{\mu\nu} v^\mu v^\nu} dt = -m \int ds,$$

where $v^\mu = dq^\mu / dt$. In (1.1) it has been set $\dot{u}^\mu = du^\mu / ds$ and $u^\mu = dq^\mu / ds$. To quantize the
system, one goes over to the canonical Hamiltonian formalism by means of the Legendre
transformation for $v^\mu$. Defining the canonical momentum $p^\mu = \partial L_g / \partial v^\mu$ we find that the
canonical Hamiltonian $H = p^\mu v^\mu - L_g = 0$ vanishes identically. This happens due to the
local time reparametrization symmetry of the action (1.3). It is not hard to be convinced
that the Hessian $H_{\mu\nu} = \partial^2 L_g / (\partial v^\mu \partial v^\nu)$ is degenerate (in particular, $H_{\mu\nu} v^\nu = 0$) and,
therefore, the system has a constraint. It has the well known form $\Pi = p^2 - m^2 = 0$. According to Dirac [12], after promoting $p^\mu$ and $q^\mu$ to self-adjoint operators satisfying the
Heisenberg algebra, the constraint $\hat{\Pi}$ has to annihilate physical states

$$\hat{\Pi} \psi = (\hat{p}^2 - m^2) \psi = 0,$$

where $-\hat{p}^2$ is the Laplace-Beltrami operator ($\hbar = 1$). In doing so, we have obtained a
relativistic quantum mechanics that leads to the geodesic motion of the wave packets
in the eikonal approximation. Note that the canonical Hamiltonian vanishes identically,
hence, the Schrödiger evolution $i \partial_t \psi = \hat{H} \psi \equiv 0$ is trivial. So, the constraint (1.4) entirely
specifies the evolution of relativistic quantum particle states. This latter property allows
one to construct a corresponding quantum field theory. If all solutions of (1.4) are labelled
by a set of parameters $k$, then a Heisenberg quantum field operator that carries quanta
(particles) with quantum numbers $k$ and wave functions $\psi_k(q)$ reads $\hat{\phi} = \Sigma_k \psi_k(q) \hat{a}_k + h.c.$
where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are destruction and creation operators of these quanta. The corresponding
action of such a field theory in $n$ dimensions is [10]

$$S = \int d^n q \sqrt{\mathcal{G}} \phi \hat{\Pi} \phi = \int d^n q \sqrt{\mathcal{G}} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right).$$

Thus the constraint occuring through the time reparametrization (gauge) symmetry specifies the sought-for quantum field theory obeying the correspondence principle.

The same strategy could be applied to build a relativistic quantum theory for the
autoparallel motion. That is, we need a Lagrangian for the equation (1.1). It has to
fulfill some additional physical conditions: (i) to be time reparametrization invariant, (ii)
to be invariant under general coordinate transformations (i.e., to be a scalar) and (iii)
to turn into (1.3) as the torsion approaches zero (analyticity in torsion). We remark
that the autoparallel equation (1.1) with $F_\mu \equiv 0$ exhibits the time reparametrization symmetry therefore it is natural to expect the Lagrangian to fulfill the condition (i). Yet,
as has been pointed out, the constraint occuring through this gauge symmetry entirely
determines the evolution of a relativistic quantum particle interacting with the spacetime
geometry. The second condition is the standard one: Physics can not depend on the choice of a coordinate system. The third one is natural since we expect a small deviation from the geodesic motion in the limit of small torsion. So, we have reduced our problem to the well-known and, in fact, long-standing problem of mathematical physics: given a set of equations of motion, find out whether they admit the Euler-Lagrange form. This is the inverse problem of the calculus of variations. Necessary and sufficient conditions for the solution to exist have been first formulated by Helmholtz [13].

2 The Helmholtz conditions for the autoparallel motion

Let the equations of motion be a system of differential equations of second order

\[ G_\mu (\dot{v}, v, q) = H_{\mu\nu}(v, q)\dot{v}^\nu + B_\mu(v, q) = 0. \]  

(2.1)

The question arises: Does there exist a Lagrangian whose Lagrange derivative \([L]_\mu\) coincides with the equation of motion? That is,

\[ G_\mu = [L]_\mu \equiv \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} \dot{v}^\nu + \frac{\partial^2 L}{\partial v^\mu \partial q^\nu} v^\nu - \frac{\partial L}{\partial q^\mu}. \]  

(2.2)

Helmholtz found as necessary and sufficient conditions on the functions \(G_\mu\) of the independent variables \(q, v, \dot{v}\) in order for the Lagrangian to exist [13]:

\[ \frac{\partial G_\mu}{\partial \dot{v}^\nu} = \frac{\partial G_\nu}{\partial \dot{v}^\mu}, \]  

(2.3)

\[ \frac{\partial G_\mu}{\partial v^\nu} + \frac{\partial G_\nu}{\partial v^\mu} = \frac{d}{dt} \left\{ \frac{\partial G_\mu}{\partial \dot{v}^\nu} + \frac{\partial G_\nu}{\partial \dot{v}^\mu} \right\}, \]  

(2.4)

\[ \frac{\partial G_\mu}{\partial q^\nu} - \frac{\partial G_\nu}{\partial q^\mu} = \frac{1}{2} \frac{d}{dt} \left\{ \frac{\partial G_\mu}{\partial \dot{v}^\nu} - \frac{\partial G_\nu}{\partial \dot{v}^\mu} \right\}. \]  

(2.5)

With respect to an arbitrary time parameter \(t\) the autoparallel equation (1.1) is

\[ G_\mu = [L]_\mu = 2S_{\mu\nu\lambda} \frac{v^\nu v^\lambda}{v^2} = 0. \]  

(2.6)

It is obvious that the geodesic term \([L_g]_\mu\) fulfills the Helmholtz conditions. The second term in (2.6) is the torsion force that causes a deviation of the trajectory from the geodesics \([L_g]_\mu = 0\). Due to the linearity in \(G_\mu\), the Helmholtz conditions yield restrictions on the torsion force only. From the second Helmholtz condition (2.4), the restriction \(S_{\mu(\nu\lambda)} = 0\) on torsion can be deduced. This implies vanishing the torsion force in (2.6). Thus, Eq. (2.2) does not have any solution for a non-vanishing torsion force.

The only possibility to find a Lagrangian formalism for the autoparallel is to look for an equivalent set of equations which may have the Euler-Lagrange form. This can be done by introducing a multiplier \(\Omega_{\mu\nu}(v, q)\) with \(\det \Omega_{\mu\nu} \neq 0\) which acts as an integrating factor in Eq. (2.2). We are then looking for a solution to the equation

\[ [L]_\mu = \Omega_{\mu\nu} G_\nu. \]  

(2.7)
The integrability conditions (2.3)-(2.5) become less restrictive for $G_\mu$ itself since some of them can be fulfilled by an appropriate choice of the multipliers. This procedure was first proposed in [14]. Although there has been much progress in this approach (see [15]) and some useful techniques have been invented to simplify the Helmholtz conditions, the problem still remains unsolved. Recently, the inverse variational problem for Eq. (1.1) with $F_\mu = 0$ has been solved in two dimensions [16]. However, in these works the proper time $s$ in the equation of motion (1.1) has been considered as the Lagrangian time $t$. Consequently, the actions obtained are not time reparametrization invariant and it would be difficult to give them a physical interpretation in the framework of a relativistic theory. However, they might be useful to study a nonrelativistic autoparallel motion on two-dimensional surfaces.

3 The gradient case

Here we show that the Helmholtz integrability conditions can be fulfilled for the generalized problem (2.7). In the special case when the trace of the torsion tensor is a gradient and the traceless part vanishes,

$$S_{\mu\nu\lambda} = \frac{1}{2} (\delta^\lambda_\mu \partial_\nu \sigma - \delta^\lambda_\nu \partial_\mu \sigma), \tag{3.1}$$

the corresponding autoparallel equation (2.6) follows from the least action principle $\delta S_{(\sigma)} = 0$ where [17]

$$S_{(\sigma)} = \int L_{(\sigma)} dt = -m \int e^{\sigma(q)} \sqrt{g_{\mu\nu} v^\mu v^\nu} dt = -m \int e^{\sigma(q)} ds. \tag{3.2}$$

Whereas the action (1.3) for geodesics is just an integral over proper time, in (3.2) a scalar factor $e^{\sigma(q)}$ occurs. The same Lagrangian was obtained in Brans-Dicke theory [18], where the masses of particles depend on position $m \rightarrow m(q) = m e^{\sigma(q)}$. The scalar field $\sigma$ can also be interpreted as the dilaton field [19] emerging in the low energy limit of the string theory together with the metric $g_{\mu\nu}$.

The Lagrange derivative of $L_{(\sigma)}$ reads

$$[L_{(\sigma)}]_\mu = e^\sigma \left( [L_{\sigma}]_\mu + (g_{\mu\lambda} \partial_\nu \sigma - g_{\nu\lambda} \partial_\mu \sigma) \frac{v^\nu v^\lambda}{\sqrt{v^2}} \right) = 0. \tag{3.3}$$

Note that the Lagrange derivative has the form (2.7) with the multiplier $\Omega^\nu_\mu = e^\sigma \delta^\nu_\mu$. Eq. (3.3) exhibits the time reparametrization symmetry. The motion can be specified in a gauge invariant way by defining the proper time. Since the theory has an extra scalar function $\sigma$ available, the gauge invariant time is not unique: $ds = f(\sigma) \sqrt{g_{\mu\nu} v^\mu v^\nu} dt$ with $f(\sigma)$ being a general positive function of $\sigma$. If we set $f = 1$, Eq. (3.3) turns into the autoparallel equation

$$g_{\mu\nu} \dot{u}^\nu + \left( \Gamma_{\nu\mu \lambda} + g_{\mu\lambda} \partial_\nu \sigma - g_{\nu\lambda} \partial_\mu \sigma \right) u^\lambda u^\nu = 0. \tag{3.4}$$
It should be stressed that the motion depends on the definition of the (proper) gauge invariant time. For instance, with the choice \( f = e^\sigma \) Eq. (3.3) turns into a geodesic equation. Indeed, under the conformal transformation,

\[
g_{\mu\nu} \longrightarrow g^{(\sigma)}_{\mu\nu} = e^{2\sigma} g_{\mu\nu} ,
\]

the action (3.2) goes over to the action (1.3) for geodesics associated with the new metric \( g^{(\sigma)}_{\mu\nu} \) and the new proper time \( ds^{(\sigma)} = e^\sigma ds \). Thus, a violation of Einstein’s equivalence principle due to the “dilaton” force in (3.4) can be observed, provided there is a possibility to distinguish experimentally between the measurements of distances and time intervals relative to the metrics \( g_{\mu\nu} \) and \( g^{(\sigma)}_{\mu\nu} \). We return to this issue later in the conclusions.

The metric rescaling (3.5) can be used to remove the force caused by the “gradient” part of the torsion tensor from the equation of motion:

\[
G_\mu (g_{\alpha\beta}, S_{\alpha\beta\gamma}) = e^\sigma G_\mu (e^{-2\sigma} g_{\alpha\beta}, S_{\alpha\beta\gamma} + S^{(\sigma)}_{\alpha\beta\gamma}) ,
\]

where \( S^{(\sigma)}_{\alpha\beta\gamma} \) is given by (3.1) and in both sides of Eq. (3.6) the proper time is defined with \( f = 1 \).

Now we make use of this symmetry to build up a quantum field theory which in a semiclassical approximation would lead to the autoparallel motion of the wave packets in the “gradient” torsion and metric background fields. To this end we go over to the Hamiltonian formalism for the action (3.2). The canonical momenta are \( p_\mu = \partial L_{(\sigma)}/\partial v^\mu = -m e^\sigma v_\mu/\sqrt{v^2} \), so the constraint is

\[
\Pi_{(\sigma)} = p^2 - m^2 e^{2\sigma} = 0 .
\]

To construct the corresponding quantum field theory we can simply adopt the field action (1.5) with the new metric \( g^{(\sigma)}_{\mu\nu} \) and subject it to quantization. The correspondence principle is automatically fulfilled. Indeed, in the semiclassical approximation for the quantum field theory associated with the action (1.5) the wave packets would follow geodesics with respect to the background metric \( g^{(\sigma)}_{\mu\nu} \) [20]. Making use of the symmetry (3.6) we see that the classical trajectories are autoparallels with respect to the metric \( g_{\mu\nu} \) and the “gradient” torsion generated by the background scalar field \( \sigma \). Thus the scalar field action that leads to a quantum scalar field theory compatible with the correspondence principle is

\[
S = \int d^n q \, e^{(n-2)\sigma} \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 e^{2\sigma} \phi^2 \right) .
\]

It yields the following equation of motion for the scalar field \( \phi \)

\[
\Box \phi + (n - 2) \partial_\mu \sigma \partial^\mu \phi + m^2 e^{2\sigma} \phi = 0 .
\]

where \( \Box \) is the Laplace-Beltrami operator: \( \Box \phi = (\sqrt{g})^{-1} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) \).

Eq. (3.9) can be regarded as the quantum version of the constraint (3.7). Note that a multiplication of the constraint (3.7) by some function of coordinates would lead to an equivalent constraint on the classical level. In quantum theory the ordering of operators
is generally not unique. Here we have promoted $\Pi_{(\sigma)}$ into an operator by multiplying it by $e^{-2\sigma}$ and postulating that $e^{-2\sigma}\hat{p}_{(\sigma)}^2$ is the Laplace-Beltrami operator with respect to the metric $g^{(\sigma)}_{\mu\nu}$. This ensures the hermiticity of the constraint with respect to a scalar product with the measure $\sqrt{g^{(\sigma)}}d^nq = e^{n\sigma}\sqrt{g}d^nq$, thus providing the unitarity of the time evolution.

4 Perturbation theory

Here we come to the conclusion that there is no Lagrangian formalism, except for the gradient case. We make use of our third physical assumption that the Lagrangian, if it exists, should be analytical in the torsion tensor. So far, no experimental observation of torsion has been made. Therefore the torsion force must be small as compared with the gravitational force induced by the metric. This, in turn, suggests solving the integrability conditions for Eq. (2.7) by the perturbation theory in the torsion tensor. We shall see that the integrability conditions are not fulfilled even in first order perturbation theory, thus leading to the conclusion of the nonexistence of a Lagrangian in general. We start with the ansatz

$$L(v,g,S) = L_g(v,g) + L_1(v,g,S) + O(S^2),$$

which contains the Lagragian $L_g$ (1.3) for geodesics and a perturbation $L_1$ linear in the torsion tensor. From Eq. (2.7) follows that the multiplier must also be analytic in torsion, so we set

$$\Omega^\lambda_{\mu}(v,g,S) = \delta^\lambda_{\mu} + \omega^\lambda_{\mu}(v,g,S) + O(S^2).$$

In this approximation, the substitution of (2.6) in (2.7) leads to

$$[L_1]_{\mu} = \omega^\lambda_{\mu}[L_g]_{\lambda} + 2S_{\mu\nu\sigma}v^{\nu}v^{\sigma}\sqrt{v^2}. \quad (4.3)$$

The variables $\dot{v}, v, q$ are considered as independent variables. The integrability conditions for (4.3) are still difficult to analyze because of the presence of the general functions $\omega^\lambda_{\mu}$. Therefore we first look for the integrability conditions in the velocity space assuming the configuration space point to be fixed. So we set $q^\mu = q_0^\mu$ after calculating all the derivatives $\partial_{\mu}$ in (4.3). Eq. (4.3) is covariant under general coordinate transformations as a consequence of our second assumption. In particular, we may assume a geodesic coordinate system [21] at $q_0^\mu$. The advantage of this is that the Christoffel symbols are zero at the origin $\Gamma^\lambda_{\mu\nu}(q_0) = 0$. Thanks to this property, the term $\omega^\lambda_{\mu}[L_g]_{\lambda}$ is proportional to the acceleration $\dot{v}^\mu$ and must cancel against the corresponding term contained in $[L_1]_{\mu}$. This leads to an equation for the multiplier which is not relevant for the subsequent analysis. For the remaining terms we obtain

$$v^n \frac{\partial^2 L_1}{\partial q^n \partial v^\mu} - \frac{\partial L_1}{\partial q^n} = 2S_{\mu\nu\sigma}v^n v^\sigma\sqrt{v^2}. \quad (4.4)$$

Next, in the vicinity of $q_0$ we apply the Fourrier transform $L_1(q,v) = \int dk e^{ika} \hat{L}_1(k,v)$, similarly for $\hat{S}_{\mu\nu\sigma}(k)$, so that

$$\left. \frac{\partial L_1}{\partial q^n} \right|_{q=q_0} = \int dk i k_\mu e^{ika} \hat{L}_1(k,v).$$

Substituting this into (4.4),
we obtain a first-order differential equation for \( \bar{L}_1(k, v) \) as a function of \( v^\mu \). This equation can be simplified by the ansatz \( \bar{L}_1 = k_\mu v^\mu c(k, v) \), leading to

\[
i \frac{\partial c}{\partial v^\mu} = \frac{2}{(k, v)\sqrt{v^2}} \bar{S}^\mu v^\sigma \sqrt{v^2}. \tag{4.5}
\]

The integrability conditions for Eq. (4.5) are now easy to derive. After multiplying them by the factor \((k, v)\sqrt{v^2})^3\), they turn into a set of vanishing linear combinations of the monomials \( v^\nu v^\sigma v^\alpha v^\beta \). Since \( v^\mu \) are independent variables we are left with the equation

\[
2k_{[\mu}\bar{S}_{\lambda]\nu\sigma} + k_{(\nu} \left\{ \bar{S}_{[\mu\lambda]\sigma} \eta_{\alpha\beta} + \bar{S}_{[\sigma\lambda]} \eta_{\alpha\beta} + \bar{S}_{[\lambda]_{\alpha\beta}} \eta_{\sigma\mu]}\right\} = 0. \tag{4.6}
\]

Here the indices \((\nu\sigma\alpha\beta)\) must be symmetrized, while the indices in the square brackets \([\mu\lambda]\) are antisymmetrized.

There are two cases where the integrability condition (4.6) is identically fulfilled and, hence, the Lagrangian always exists. First, we observe that \( v^\mu \partial_c/\partial v^\mu \equiv 0 \) since \( S_{\mu\nu\sigma} = -S_{\nu\mu\sigma} \). Therefore, \( c \) depends only on the angular variables in the velocity space, not on the modulus \( \sqrt{v^2} \). In two dimensions, Eq. (4.5) contains only one non-trivial equation which always has a solution. The Lagrangian can be constructed as proposed in Appendix B. The second case is \( \bar{S}_{\mu\nu\lambda} \sim \delta^n(k) \), i.e., when the torsion tensor is constant in the coordinate system chosen. It is easy to obtain a simple recursion relation for an explicit form of all orders of perturbation theory for the Lagrangian \( L \). However, the condition \( \partial_\mu S_{\nu\lambda\sigma} = 0 \) is not covariant under general coordinate transformations. So, the corresponding Lagrangian is not a scalar and can not be regarded as physically acceptable.

The torsion tensor can always be decomposed into a trace, a totally antisymmetric part and a traceless part \( Q_{\mu\nu\lambda} \) which is not totally antisymmetric. The totally antisymmetric part satisfies (4.6) identically because it does not contribute to the torsion force at all. So we set

\[
S_{\mu\nu\sigma} = \frac{1}{n-1} \left( S_{\mu} \delta^\sigma_{\nu} - S_{\nu} \delta^\sigma_{\mu} \right) + Q_{\mu\nu\sigma}, \tag{4.7}
\]

where \( S_{\mu} = S_{\mu\lambda} \). Contracting (4.6) with \( k^\nu k^\alpha k^\beta \eta_{\mu\sigma} \eta^{\alpha\beta} \) and \( k^\alpha k^\beta \eta^\mu \sigma \) we get a system

\[
\beta_{\mu\lambda} + 3\gamma_{\mu\lambda} = 0, \quad (2n+5)\alpha_{\mu\lambda} + (n+1)\beta_{\mu\lambda} = 0, \quad 3\alpha_{\mu\lambda} + (n+4)\beta_{\mu\lambda} + (2n+11)\gamma_{\mu\lambda} = 0, \tag{4.8}
\]

where \( \alpha_{\mu\lambda} = k_{[\mu} \bar{S}_{\lambda]\nu\sigma} \), \( \beta_{\mu\lambda} = k_{[\sigma} \bar{S}_{[\mu\lambda]\nu\sigma} + \bar{S}_{[\sigma]}_{[\nu\alpha\beta]} \eta_{\lambda\mu]} \) and \( \gamma_{\mu\lambda} = k_{[\mu} \bar{S}_{\lambda]\nu\sigma} k^\nu k^\sigma \). The determinant of the coefficients is \( 2(n-2)(n+1) \). So, for \( n > 2 \) we conclude \( \alpha_{\mu\nu} = 0 \) and \( \beta_{\mu\nu} = \gamma_{\mu\nu} = 0 \). The first relation gives rise to a restriction on the trace \( S_{\mu} \)

\[
k_{[\mu} \bar{S}_{\lambda]} = 0, \quad \text{hence,} \quad S_{\lambda}(q) \sim \partial_\lambda \sigma(q). \tag{4.9}
\]

That is, in any dimension greater than two the contracted torsion tensor must be a gradient. We conclude that for a *generic* torsion the inverse variational problem for the autoparallel equation has no solution. The “gradient” part of the torsion tensor (4.7) satisfies (4.6) identically, so that the integrability condition (4.6) applies to \( Q_{\mu\nu\sigma} \) only. We investigate it in three and four dimensions. Both cases are treated simultaneously.
The tensor $Q_{\mu\nu\sigma}$ can be parametrized in 3 and 4 dimensions respectively as

\begin{equation}
Q^{(3)}_{\mu\nu\lambda} = \epsilon_{\mu\nu\sigma} A^\sigma_{\lambda}, \quad Q^{(4)}_{\mu\nu\lambda} = \epsilon_{\mu\nu\sigma\rho} B^\sigma_{\rho\lambda},
\end{equation}

where $A_{\mu\nu}$ is a symmetric, traceless $3 \times 3$ matrix (since $Q^{(3)}_{\mu\nu\nu} = 0$ and $\epsilon_{\mu\nu\nu} Q^{(3)}_{\mu\nu\lambda} = 0$), and $B_{\sigma\rho\lambda}$ satisfies $\epsilon_{\mu\sigma\rho\lambda} B_{\sigma\rho\lambda} = 0$ and must be traceless $B^\sigma_{\rho\lambda} = 0$ (since $Q^{(4)}_{\mu\nu\nu} = 0$ and $\epsilon_{\mu\nu\nu\lambda} Q^{(4)}_{\mu\nu\lambda} = 0$). Thus, $A_{\mu\nu}$ contains 5 independent components, while $B_{\mu\nu\sigma}$ has 16. They are subject to the conditions

\begin{equation}
k_{\sigma} \tilde{A}^\sigma_{\rho} = 0, \quad k_{\sigma} \tilde{B}^{\mu\lambda}_{\rho\sigma} = 0,
\end{equation}

\begin{equation}
k_{(\nu} \tilde{A}_{\alpha\beta\delta\rho)}_{\sigma} = 0, \quad 2 \eta_{(\alpha\beta\delta} k_{\rho} \tilde{B}^{\rho}_{\sigma\delta\lambda)} + k_{(\nu} \tilde{B}_{\beta\delta\rho)}^{\mu\lambda\sigma} = 0.
\end{equation}

Eq. (4.11) is equivalent to $\beta_{\mu\nu} = 0$, while Eq. (4.12) stems from the integrability condition (4.6) where $\beta_{\mu\nu} = 0$ has been taken into account. It is possible to select 5 linearly independent equations for $A_{\mu\nu}$ and 16 for $B_{\mu\nu\sigma}$ out of these equations. So we conclude that $Q^{(3,4)}_{\mu\nu\lambda} = 0$, and the Lagrangian exists only for the “gradient” case.

5 Conclusions

We have proposed a rather general approach to study possible deviations from Einstein’s equivalence principle due to the coupling between matter and the torsion of spacetime. Our approach is based on the inverse problem of the calculus of variations and general principles of quantum field theory. It is far more general than the minimal gauge coupling principle which is typically used to construct a coupling between matter and spacetime geometry [9, 2]. We have shown that for a generic torsion force which makes the trajectories of classical spinless particles the autoparallels, no local quantum field theory exists in four dimensions that leads to the autoparallel motion in the semiclassical (eikonal) approximation. Only when the torsion tensor has a special form the above problem admits a solution. In this case the coupling between matter and torsion is equivalently described by the dilaton field whose existence is predicted by the string theory [19].

The Einstein equivalence principle is not violated by the coupling between matter and the dilaton field if the coupling obeys the universality principle [22], meaning that it is constructed by the replacement $g_{\mu\nu} \rightarrow g^{(\sigma)}_{\mu\nu} = e^{2\sigma} g_{\mu\nu}$ in the matter Lagrangian. Indeed, there would be no experiment that distinguishes the motion of test particles in the composite background metric $g^{(\sigma)}_{\mu\nu}$ from that in the background metric $g_{\mu\nu}$ and the dilaton field $\sigma$. A deviation from the Einstein general relativity can only be seen in cosmology which is affected by dynamics of the dilaton field [22, 23].

We remark that the minimal gauge coupling principle does not predict the dilaton and leads only to the coupling between spin and torsion. As has been stressed by some authors (see the discussion in [24]) such a coupling might pose a consistency problem since the spin of composed particles is not simply a sum of the spins of its constituents, but involves also the orbital angular momentum. For instance, a spinning particle could be a bound state of spinless particles (e.g., a vector boson composed of a few scalar bosons, etc.). Therefore,
such a spinning particle would not interact with torsion at all. Thus, when applying
the minimal gauge coupling principle, one has always to decide whether a given kind of
particles is truly elementary or composite. Such a drawback could be circumvented either
by allowing for the coupling of torsion to the angular momentum or by simply postulating
that any theory for composite spinning particles should be consistent with the minimal
gauge coupling principle, thus making a restriction on the future fundamental theories.
Given the difficulties of describing composite relativistic quantum fields, this latter option
does not seem easy to pursue in practice, as well as it does not admit a simple geometrical
interpretation.

Here we have explored the first possibility. The autoparallel equation (1.1) \( F_\mu \equiv 0 \)
can be rewritten as the matter energy-momentum conservation law [25]

\[
\overline{D}_\mu \overline{T}^{\mu\nu} + 2 S_\mu^\nu \overline{T}^{\mu\sigma} = 0,
\]

(5.1)

where \( \overline{T}_{\mu\nu} \) is the energy-momentum tensor in general relativity (see also Appendix A).
The second term in (5.1) contains an interaction between the torsion and the angular
momentum. We have proved that there exists no local quantum field theory whose dy-
namics complies with Eq. (5.1) except the special case when all the effects of torsion can be interpreted as those caused by the dilaton. One could also regard this result as an argument supporting the point of view that the spacetime geometry is specified only by the metric and possibly by the dilaton field, i.e., by the low energy modes of string theory.

We remark that the minimal price of incorporating Eq. (5.1) into quantum theory
is to give up locality [26]. This does not seem to us acceptable in quantum field theory
of fundamental interactions, but still may be possible in effective theories describing a quantum motion of interstitial particles in crystals with topological defects.

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Appendix A: Energy-momentum conservation for the autoparallels

The energy-momentum conservation law follows from the invariance of the action under
general coordinate transformations. As compared with the geodesic action (1.3), the
action (3.2) contains an extra scalar field describing the background spacetime geometry
so that its variation is determined by both the variations of the metric \( g_{\mu\nu} \) and the dilaton \( \sigma \). Thus, we get

\[
0 = \delta S = \int d^4q \sqrt{g} \left\{ \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + T^\mu_\mu \delta \sigma + \frac{\delta L}{\delta q^\mu} \delta q^\mu \right\},
\]

(A.1)

where, as usual,

\[
T^{\mu\nu}(q) \equiv \frac{\delta L}{\delta g_{\mu\nu}(q)} = \frac{1}{\sqrt{g(q)}} \int ds \delta^4(q - q(s)) p^\mu u^\nu
\]

(A.2)
is the energy-momentum tensor and $\mathcal{L}$ is the Lagrangian density defined by $L = \int d^3q \mathcal{L}$. We observe that $T^\mu_\nu = e^\sigma T^\mu_\nu$ where $T^\mu_\nu$ is the energy-momentum tensor for the geodesic motion. The difference occurs through the $\sigma$-dependence of the particle momentum: $p^\mu = -me^{\sigma(0)}u^\mu$. One can easily convince oneself that $\delta\mathcal{L}/\delta\sigma = T^{\mu\mu}$, which specifies the second term in (A.1). For the actual motion of the particle the third term in (A.1) vanishes and we get the energy-momentum conservation law (cf. (5.1))

$$\mathcal{D}_\mu T^{\mu\lambda} + T^{\mu\lambda} \partial_\mu \sigma - T^{\lambda\mu} \partial_\lambda \sigma = 0.$$  \hspace{1cm} (A.3)

So we see two additional terms occurring in the conservation law due to the torsion force. Integrating this equation over a three-dimensional spacelike hypersurface $q^0 = \text{const}$ we again recover the autoparallel equation for the “gradient” torsion (3.4).

**Appendix B: The autoparallel Lagragian in two dimensions**

In two dimensions the integrability conditions (4.6) yield no restriction on torsion because of the time reparametrization invariance. Indeed, by fixing the gauge $q^0 \equiv t$ the problem becomes one-dimensional. A general solution of the one-dimensional inverse variational problem was found by Darboux [27]. So, the Lagrangian always exists for the two-dimensional autoparallel motion. However, the constraint appears to be non-polynomial in the canonical momenta, thus leading to a nonlocal quantum field theory.

The torsion tensor can be parametrized in two dimensions by two scalar functions $\lambda$ and $\sigma$:

$$S_{\mu\nu}^\alpha = \frac{1}{2} \epsilon_{\mu\nu} \left( \partial^\alpha \lambda + e^{\alpha\beta} \partial_\beta \sigma \right).$$  \hspace{1cm} (B.1)

Setting $\varphi = (k,u)/\sqrt{k^2}$, we may decompose $u^{\mu} = [\varphi^1 k^\mu + (1 - \varphi^2)^{1/2} (\epsilon k)^\mu]/\sqrt{k^2}$. Solving Eq. (4.5) for $c = c(\varphi)$ we find

$$i\tilde{L}_1 = \sqrt{v^2} \left( \tilde{\sigma} + \varphi \ln[\varphi^{-1} + (\varphi^{-2} - 1)^{1/2}] \tilde{\lambda} \right).$$  \hspace{1cm} (B.2)

The first term is the linear part of the Lagrangian (3.2) for the “gradient” torsion. The second term is non-polynomial in $\varphi$. This fact is not changed by the higher orders of the expansion (4.1) as can be seen from a recursion relation for $\tilde{L}_i$. Because of this, the Lagrangian would lead to a constraint which is non-polynomial in $p$. Thus the corresponding quantum field theory would be non-local. So, we conclude that also in two dimensions only the “gradient” torsion leads to an acceptable theory.

It is certainly possible to find a Lagrangian for the generic torsion (B.1). However a complete discussion would be too involved and goes beyond the scope of this letter. Just to give an idea of how the Lagrangian would look like, we calculate it under the simplifying conditions $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\lambda \sigma = 0$. This latter condition obviously violates the general coordinate invariance, but will allow us to find an explicit form of the Lagrangian.

We also set $\sigma = 0$ since the gradient case has been already discussed. After fixing the gauge by $q^0 = t$ ($v^0 = 1$) and adopting the notations $v^1 \equiv v, \partial_1 \lambda \equiv \partial_x \lambda$ we get one simple equation out of the autoparallel equation (1.1)

$$\dot{v} + \partial_x \lambda (v - v^3) = 0.$$  \hspace{1cm} (B.3)
The associated Lagrangian can be found via the Hamiltonian formalism. The first Hamiltonian equation is set to be \( \dot{x} = \frac{p}{\sqrt{1 + p^2}} = \omega \partial_p H \). Then the second Hamiltonian equation can be derived from (B.3) as \( \dot{p} = -p \partial_x \lambda = -\omega \partial_x H \). These are equations for the Hamiltonian \( H \) and the symplectic structure \( \omega \) which can easily be solved. Next, choosing Darboux coordinates \( X = x \) and \( P \) such as \( \partial_P P = \omega^{-1} \), the Lagrangian is obtained by the Legendre transformation for \( P \): \( L(v, \lambda) = P\dot{x} - H \). The time reparametrisation invariance is restored by the rule \( L(\lambda) (v_0, v_1, \lambda) = v_0 L(v_1/v_0, \lambda) \) [12]. Therefore the Lagrangian is:

\[
L(\lambda) = -\sqrt{v^2} \cosh \lambda - v^1 \ln C \sinh \lambda,
\]

where \( C = |v_0/v_1 + \sqrt{(v_0/v_1)^2 - 1}| \). It is not difficult to see that the constraint resulting from \( u^2 = 1 \) is not polynomial in the canonical momenta \( p_\mu \) because \( p_\mu = p_\mu(u) \) contain \( \ln C \).

References


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