Note on a Positronium Model from Flow Equations in Front Form Dynamics

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Abstract

In this note we address the problem of solving for the positronium mass spectrum. We use front form dynamics together with the method of flow equations. For a special choice of the cutoff function, the calculations can be simplified by analytically integrating over the azimuthal angle. One obtains an effective Hamiltonian and we solve numerically for its spectrum. Comparing our results with different approaches we find encouraging properties concerning the cutoff dependence of the results.
1 Introduction

Solving for QCD bound states from first principles remains a yet unsolved problem. There is some hope of simplifications in the framework of front form dynamics [1], because of a simpler vacuum state. At least three different methods have been developed over the last years to cope with the stunning problem of constructing a 'finite' (effective) Hamiltonian out of the 'infinite' canonical QCD Hamiltonian [2]. The method of flow equations as developed by Wegner [3] is closely related to the similarity transformations of Wilson and Glazek [4]. These methods are based on unitary transformations to block- or band-diagonalize the Hamiltonian. Another way to deal with QCD bound states is to first explicitly solve the many body part of the problem. This method of iterated resolvents is advocated by Pauli [5] and results on this way can be found in Refs. [6, 7]. So far, calculations in the first two formalism were performed using light-cone perturbation theory and the non-relativistic limit to extract eigenvalues. The results obtained for QED, in particular the positronium spectrum, agree in this approximation with standard methods and are very encouraging [8]. But so far no calculations using the fully relativistic and covariant effective light-cone Hamiltonian was performed in the manner of Refs. [6, 9]. Moreover, it is clear that methods for solving QCD problems have to be essentially non-perturbative. It is therefore necessary to scrutinize the methods and look at their features obscured so far by approximations on the way. We present numerical results in the similarity flow scheme of Wegner, with a special choice of the cutoff function, applied to the positronium system of QED. Analogous calculations in a model inspired by the method of iterated resolvents have been performed in Ref. [6]. We compare the results and point out the specific virtues of the methods.

The paper is organized as follows. In the following section we define the formalism we are using by presenting the ideas of the flow equation and similarity renormalization method as special cases of a more general framework. Section 3 gives a sketch of the analytic calculations on the way to an effective Hamiltonian. The actual matrix elements were calculated in Refs. [10, 11] and are listed for completeness in the appendix. In Section 4 we present our numerical results, based on the code developed in Ref. [6]. A discussion of the results follows.

2 Flow Equations and Similarity Renormalization

The aim of this paper is to present a non-perturbative calculation of the spectrum of an effective Hamiltonian. To interpret the results it is necessary to understand the formalism in which the Hamiltonian was constructed. As mentioned above, two of the proposed methods to derive an effective Hamiltonian are closely related. In fact, it turns out that both the flow equations of Wegner [3] and the similarity renormalization of Wilson and Glazek [4], can be retrieved as special cases of the more general similarity flow framework [12]. The idea behind it is as follows.
We want to construct an effective Hamiltonian $H'$ from the canonical Hamiltonian $H$ (regularized at some scale $\Lambda$) by a transformation

$$H' = U H U^\dagger.$$  \hfill (1)

The generator of the transformation is anti-hermitian, $\eta^+ = -\eta$. The Hamiltonian is considered a function of the flow parameter $l$, with the bare and effective Hamiltonians being $H(l = 0)$ and $H(l \to \infty)$, respectively. Its change with respect to the flow parameter is given as usual by the commutator with the generator $\eta$ of these unitary transformations

$$\frac{dH(l)}{dl} = [\eta(l), H(l)].$$  \hfill (2)

The goal is to choose an $\eta$ such that the transformed Hamiltonian has the form

$$H(\lambda)_{ij} = f \left( \frac{E_i - E_j}{\lambda} \right) H_{ij}(\lambda).$$  \hfill (3)

Here, the flow parameter is connected to the scale $\lambda$ by $l = 1/\lambda^2$, and the free particle energies $E_i$ are defined by the free Hamiltonian

$$H_d^{(0)} |\bar{v}\rangle = E_i |\bar{v}\rangle.$$  \hfill (4)

If the so-called similarity function $f(\lambda)$ has the properties

$$f(\lambda \to \infty) \to 1, \quad f(\lambda \to 0) \to 0,$$  \hfill (5)

then $H(l)_{ij}$ will become a (block) diagonal operator when $\lambda \to 0$, because its off-diagonal matrix elements are transformed to zero in this limit. The generator $\eta$ of the unitary transformations is by construction the commutator of the diagonal part of the Hamiltonian with its complement

$$\eta = [H_d, V].$$  \hfill (6)

The two schemes of Wegner and Wilson-Glazek are defined by the choice of the generator $\eta$. Wegner applied his scheme to condensed matter problems and found it preferable to choose the particle number conserving part of the Hamiltonian as diagonal. The scheme was constructed as a method to cope mainly with the many body aspects of a Hamiltonian theory rather than with the renormalization issues. The latter play a minor role in this field due to the inherent cutoffs in condensed matter problems. In the limit $\lambda \to 0$ one obtains a Hamiltonian diagonal in particle number space, i.e. sectors with different particle numbers are totally decoupled. This is clearly advantageous.

On the other hand, Wilson and Głazek [4] tried to find a renormalization scheme for divergences in light-cone field theory. The idea was to set up a scheme which reveals a version of the constituent quark model as lowest approximation of the effective Hamiltonian. Following this point of view, one wants to integrate out high energy modes
and to band-diagonalize the Hamiltonian operator in such a way as to keep only the effective degrees of freedom relevant for a special scale of interest.

Note that the similarity function $f(\lambda)$ is still arbitrary in both schemes. In particular, we obtain the flow equation method proper, a subset of the Wegner scheme, by setting

$$f(\lambda) = \exp \left\{ -\frac{(E_i - E_j)^2}{\lambda^2} \right\}.$$  \hfill (7)

Contrary to the Wegner scheme, one runs into conceptual difficulties applying the Wilson-Glazek scheme in the limit $\lambda \to 0$. This is due to the fact that one is supposed to integrate out only the higher modes of the problem and leave the interactions at the interesting scale in the problem. For instance, when considering positronium, it makes no sense to integrate out also the Coulomb, fine and hyperfine scales, yielding a free electron-positron pair. As a result one has a rather restrictive condition on the scale [8]

$$m^2 \alpha^2 \ll \lambda^2 \ll m^2 \alpha,$$  \hfill (8)

with $\alpha$ being the fine structure constant. Here, one has to consider $m^2 \alpha^2$ as 'almost zero', while $m^2 \alpha$ is pretended to be 'close to infinity'. For small couplings this seems to make sense, but it is likely to cause troubles if $\alpha$ is of order one. We are therefore on the safe side with the Wegner scheme, when choosing $\alpha = 0.3$ in the calculations to numerically trace rotational invariance breaking.

## 3 Positronium Model

To actually solve Eq. (2), we expand the Hamiltonian, and by this the generator $\eta$, into a power series in the bare coupling $g$

$$H = H_d^{(0)} + V^{(1)} + H_d^{(2)} + V^{(2)} + H_d^{(3)} + V^{(3)} + \ldots,$$  \hfill (9)

$$\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)} + \ldots$$  \hfill (10)

where the operators with a superscript ($n$) are proportional to $g^n$. $H_d^{(0)}$ is the free part of the Hamiltonian, $V^{(1)}$ is the usual vertex interaction, $H_d^{(2)}$ contains (among others) the seagull interactions. Up to second order we have

$$\frac{dH(l)}{dt} = [H_d^{(0)}, V^{(1)}] + [H_d^{(0)}, V^{(2)}] + [H_d^{(0)}, V^{(1)}, H_d^{(1)}] + [H_d^{(0)}, V^{(2)}, H_d^{(1)}] + [H_d^{(1)}, V^{(1)}, H_d^{(0)}] + [H_d^{(1)}, V^{(2)}, H_d^{(0)}] + O(g^3),$$  \hfill (11)

$$\eta = [H_d^{(0)}, V^{(1)}] + [H_d^{(0)}, V^{(2)}] + [H_d^{(1)}, V^{(1)}] + O(g^3).$$  \hfill (12)

With the energies $E_i$ depending on the flow parameter only in second order, we can solve the differential equation for the Hamiltonian in the energy basis defined by Eq. (4) order by order. With the definition (cf. Eq.(3))

$$V_{ij}^{(2)}(l) = f(l)V_{ij}^{(2)}(l),$$  \hfill (14)
we get the solutions in the second order

\[
H_{d,ij}^{(2)}(l) = H_{ij}^{(2)}(l = 0) + \int_0^l [\eta^{(1)}_{ij}, H^{(1)}_{ij}]^{(d)}(\lambda')d\lambda',
\]

\[
\hat{V}_{ij}^{(2)}(l) = \hat{V}_{ij}^{(2)}(l = 0) + \int_0^l f(l')[\eta^{(1)}_{ij}, H^{(1)}_{ij}]^{(V)}(l')dl',
\]

where the superscripts \((d)\) and \((V)\) denote the diagonal and the particle number changing part, respectively. We are supposed to take the bare cutoff \(\Lambda\), which defines the terms at \(l = 0\), to infinity.

After this formal manipulations, we now have to evaluate Eqs. (15) and (16) to obtain the matrix elements of the Hamiltonian. We are interested in the lowest particle-diagonal block, i.e. in the electron-positron sector. When choosing the particle number conserving part of the Hamiltonian as diagonal, we reduce the particle number violating blocks of the Hamiltonian to zero with the flow equations. This solves en passant also the multi-particle problem. Instead of having to truncate the Fock space a là Tamm-Dancoff, we are now dealing with isolated blocks of definite particle number. The effective matrix elements are thus obtained by the process

\[
V_{\text{eff}} = \lim_{\Lambda \to 0} \left( V^{\text{gen}} + V^{PT} \right).
\]

The interaction is given by the structure of Eqs. (15) and (16): the part without the integral \((V^{PT})\) is the one obtain by usual perturbation theory. The second one \((V^{\text{gen}})\) is generated by the flow of the Hamiltonian. In other words, by reducing the off-diagonal matrix elements, we are inducing changes on the diagonal.

To calculate the matrix elements, one evaluates the associated diagrams, Fig. 1, applying light-cone perturbation theory [13]. The electron and positron momenta and
currents are
\[ l_e^\mu = (k_e' - k_e)^\mu, \quad l_x^\mu = (k_x' - k_x)^\mu, \]
\[ j(l_e)^\mu = \bar{u}(k_e')\gamma^\mu u(k_e), \quad j(l_x)^\mu = \bar{u}(k_x')\gamma^\mu u(k_x), \]
where \( k_e' \) and \( k_x' \) are the electron momenta before and after the interaction. For the numerical calculations we use relative coordinates: \( x = \frac{p^x}{P^+} \) is the longitudinal momentum fraction, and \( \vec{k}_\perp \) is the transverse momentum. We obtain
\[
V^{\text{ren}}_\lambda = \frac{j(l_e)\mu j_\mu(l_x)}{\Delta_1} \int d\lambda \frac{d f_\lambda(\Delta_1)}{d\lambda} f_\lambda(\Delta_2) d\lambda + \frac{j(l_e)\mu j_\mu(l_x) f_\lambda(\Delta_2)}{D f_\lambda(\Delta_1)} d\lambda,
\]
\[
V^{\text{PT}}_\lambda = \frac{j(l_e)\mu j_\mu(l_x)}{D} f_\lambda(\Delta_1) f_\lambda(\Delta_2) + \frac{j(l_x)\mu j_\mu(l_e)}{|x - x'|^2} (T^+ - \omega).
\]
Here, \( T^+ = \frac{1}{2}(k_e' + k_x')^2 + \frac{1}{2}(k_e + k_x)^2 \) is the average kinetic energy before and after the interaction and \( \omega \) is one of the (unknown) eigenvalues of the full Hamiltonian. This ambiguity is no problem in the formalism considered here, because the perturbative term vanishes when the scale \( \lambda \) goes to zero. The energy denominator is given by
\[
\mathcal{D} = |x - x'| (T^+ - \omega) - \frac{1}{2} (l_e^2 + l_x^2),
\]
and we used the definitions
\[
\hat{\Delta}_1 = m_f^2 \frac{(x - x')^2}{x x'} + \frac{x'}{x} k_\perp^2 + \frac{x}{x'} k_\perp^2 - 2k_\perp k'_\perp \cos(\varphi - \varphi'),
\]
\[
\hat{\Delta}_2 = m_f^2 \frac{(x - x')^2}{1 - x} + \frac{1 - x'}{1 - x} k_\perp^2 + \frac{1 - x}{1 - x'} k_\perp^2 - 2k_\perp k'_\perp \cos(\varphi - \varphi'),
\]
\[
\Delta_i = \frac{\hat{\Delta}_i}{x - x'}.
\]
The similarity function \( f_\lambda(\Delta) \) in the interaction is still at our disposal. In the electron-positron sector we have the integral equation
\[
0 = \left( \frac{m_f^2 + \vec{k}_\perp^2}{x(1 - x)} - \Lambda_f^2 \right) \psi_n(x, \vec{k}_\perp; \lambda_1, \lambda_2) \]
\[
+ \frac{g^2}{16\pi^3} \sum_{\lambda_1, \lambda_2} \int d\lambda' d\lambda'' \mathcal{D}(\vec{k}_\perp; \lambda_1, \lambda_2) V_{\text{eff}}(x', \vec{R}_\perp; \lambda_1') \psi_n(x', \vec{R}_\perp; \lambda_1', \lambda_2').
\]
The cutoff enters into the problem via the definition of the integration domain. It is restricted by the Lepage-Brodsky cutoff on the kinetic energies
\[
\frac{m_f^2 + \vec{k}_\perp^2}{x(1 - x)} \leq \Lambda^2 + 4m_f^2.
\]
Figure 2: Compiled spectra of positronium ($\alpha = 0.3, \Lambda = 1.0 \, m_f, N_1 = N_2 = 21$). The mass squared eigenvalues $M_n^2$ in units of the electron mass $m_e^2$ are shown in different sectors of the $z$-component of the total angular momentum $J_z$.

To simplify the numerical calculations we want to integrate out the azimuthal angle in the problem ($\vec{k}_f = k_\perp e^{i \varphi}$) by substituting it by the discrete quantum number $J_z$. This transformation is described by

$$\langle x, k_\perp; J_z, \lambda_1, \lambda_2 | \tilde{V}_{\text{eff}} | x', k_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle := \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \int_0^{2\pi} d\varphi \, e^{-i(L_\varphi - L_\varphi')} \langle x, k_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', k_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle.$$ \hspace{1cm} (27)

To be able to perform this integral analytically, we have to choose a special cutoff function

$$f_\Lambda(\Delta) = \theta(\Lambda^2 - |\Delta|).$$ \hspace{1cm} (28)

This sharp cutoff leads to difficulties in the calculations, namely the collinear singularity is not canceled exactly anymore [11]. With this cutoff function, however, the scale integrals become very simple in the limit $\Lambda \to 0$

$$\int_{\Lambda \to 0}^{\Lambda} \frac{df_\Lambda(\Delta_i)}{d\lambda} f_\Lambda(\Delta_j) d\lambda = \theta(\Delta_i - \Delta_j).$$ \hspace{1cm} (29)

Summarizing this section, we perform our calculations using the Wegner scheme within the similarity flow framework, together with a special choice of the cutoff function, Eq. (28). The actual matrix elements, integrated analytically over the azimuthal angle are listed in the appendix. Note that they differ only by a re-definition of the spin-dependent function $\operatorname{Int}(n)$ from the matrix elements of the Pauli ansatz, cf. e.g. Ref. [6].
This makes it possible to use the numerical techniques and the computer code of Ref. [7] to calculate the positronium spectrum. The matrix elements used in the present paper have an additional singular spin-independent part. We argue that this is an artifact of the choice of the cutoff function and will omit it in the numerical calculations. We comment on the justification of this step below.

4 Numerical results

The Hamiltonian matrix elements were derived by applying the flow equation scheme to the positronium problem in front form dynamics. Contrary to preceding work, at this point of the calculations we use a non-perturbative method to extract the spectrum of this Hamiltonian rather than light-cone bound state perturbation theory. We have to solve the eigenvalue problem

\[ H_{LC}|\phi_i\rangle = M_n^2|\phi_i\rangle, \]

or equivalently the integral equation, Eq. (25). We use the algorithm set up in Ref. [7]. For details of the calculations and numerical methods applied, see there. To be able to trace possible violations of rotational invariance (a non-trivial issue in front form dynamics), we chose to work with an unphysically large coupling constant \( \alpha = 0.3 \). This is possible, because the integral equation is an algebraic function of the coupling.

The results of the computations are handily compiled in Fig. 2. We get the expected Bohr spectrum, the hyperfine splitting and even the correct multiplet structure.
The multiplets are (almost) degenerate. We will investigate the associated question of rotational invariance further below. We suppressed plots showing the numerical convergence of the eigenvalues with the discretization parameter $N$. They look similar to those of Ref. [6]. In particular, we again find exponential convergence of the eigenvalues when approaching the continuum limit ($N \rightarrow \infty$).

We emphasize the weak dependence of the eigenvalues on the cutoff $\Lambda$, cf. Fig. 3(a). From the theory, we expect the eigenvalues to diverge logarithmically because we used flow equations derived up second order only. However, as we shall see, the coefficient of the hyperfine splitting is very well described at moderate values of the cutoff. We therefore compare our eigenvalues to results of equal time perturbation theory at $\Lambda = 50 m_f$. The eigenvalues are listed in Table 1 and agree with the known results. Note however that perturbation theory might be not very reliable at this large coupling.

An excellent test for the theory is the value of the coefficient of the hyperfine splitting, defined as

$$ C_{hf} = \frac{M_{\text{triplet}} - M_{\text{singlet}}}{\alpha^4}. $$

The corresponding Fig. 3(b) is impressive and encouraging: we obtain a smooth curve, converging for large cutoffs to a value [14]

$$ C_{hf} \xrightarrow{\Lambda \rightarrow \infty} 0.2825. $$

<table>
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<tr>
<th>No.</th>
<th>$M^2_{\text{theor}}(J_z=0)$</th>
<th>$M^2_{\text{ETPT}}$</th>
<th>$\Delta M^2$</th>
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<tr>
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Table 1: The positronium spectrum for $\alpha = 0.3, \Lambda = 50 m_f, N_1 = N_2 = 21$. The first row compiles our results. The second row of eigenvalues ($M^2_{\text{ETPT}}$) are equal time perturbation theory calculations up to order $O(\alpha^4)$. Also shown is the difference $\Delta M^2 := M^2_{\text{theor}} - M^2_{\text{ETPT}}$. 

The corresponding Fig. 3(b) is impressive and encouraging: we obtain a smooth curve, converging for large cutoffs to a value [14]
From equal time perturbation theory we would expect a value between $1/3$ (up to order $\alpha^4$) and 0.2379 (up to order $\alpha^6 \log \alpha$). This is exactly what we obtain.

We also had a look on violations of rotational invariance. This was a main issue in Ref. [6]. We plotted the deviation of corresponding eigenvalues as a function of the discretization parameter $N$ and as a function of the cutoff in Fig. 4. The first plot shows that the violation of rotational symmetry is not explained by a finite discretization: the curve is approximately given by

$$\Delta M^2(N) = a - [a - \Delta M^2(5)] \exp\{(N - 5)/b\}$$

($a = -1.0262 \times 10^{-4}, b = 4.0$) and clearly doesn’t go to zero in the continuum limit $N \to \infty$. The cutoff dependent curve of Fig. 4 is excellently fitted by the function

$$M^2(\Lambda) = \hat{a} + \hat{b} \log \Lambda + \hat{c} \log^2 \Lambda,$$

where $\hat{a} = -1.054 \times 10^{-4}$, $\hat{b} = -6.105 \times 10^{-4}$, and $\hat{c} = -1.0 \times 10^{-4}$. We find indeed a very small violation of rotational invariance. This suggests that to obtain full rotational invariance, one has to go to higher orders in the derivation of the flow equations. In detail, we find a discrepancy between the triplet levels of 1% of the relevant (hyperfine splitting) scale at a cutoff of $1m_f^2$, rising to roughly 10% when $\Lambda = 18m_f^2$.

The original idea of the flow equations implies the use of a smooth cutoff function to cancel the collinear singularity completely. We had to omit this spin independent singular part by hand to be able to perform the integration of a numerical counterterm.
We argue that this part is an artifact of the choice of the cutoff function. Indeed, it was shown in Ref. [11] that the singular part of the matrix elements vanishes for the standard flow equation cutoff function, Eq. (7). Wegner [15] raised some doubts that this would change the eigenfunctions significantly. To show that at least on this level our model works fine, we plot the ground state wavefunctions in Fig. 5. They are almost identical to those displayed in Ref. [6], with a slight change in the peak amplitudes.

A remark is in order concerning the numerical integrations. The fine-tuning of the parameters of the algorithm for the numerical integrations performed with the NAG library was unexpectedly delicate. Additionally, for large values of the cutoff $\Lambda$, we had to ignore some warnings of the routines to proceed. However, errors in this part of the calculations are due to artifacts of the discretization and should vanish in the continuum limit. Moreover, the resulting curves all are very smooth. This gives us further confidence in our results.

5 Discussion

Summarizing our results, we presented the positronium spectrum and wavefunctions in all sectors of the angular momentum $J_z$. The results are encouraging especially when looking at their cutoff dependence. Quite in general we find a weak, logarithmic smooth dependence on the cutoff. The comparison to results of equal time perturbation theory gives good agreement. With respect to its cutoff related properties, the flow equation scheme seems to work better than the model of Ref. [6], inspired by the method of iterated resolvents. This is expected by construction of the methods and new ideas.
have been proposed to deal with renormalization issues in the latter method [5].

The inclusion of the annihilation channel would be a straightforward calculation. The aim of this note was to extract results non-perturbatively from the flow equation scheme in general. We found that even rotational invariance is obeyed to a large extent on the numerical level and withstood therefore from implementing the annihilation channel here. An interesting investigation would be to use different cutoff functions and to compare them. From the numerical point of view this implies the use of an additional numerical integration, which would heavily increase the used CPU time.

It remains to be investigated how the singular terms influence the spectrum. We argued that they are relics of the chosen cutoff function and omitted them. The numerical results seem to support this claim. In general, it would be interesting to calculate the effective Hamiltonian to a higher order in the bare coupling constant. This would be useful to find out about the structure of the generated (irrelevant) operators and also to test the improved cutoff dependence of the results. It seems however quite tedious to go beyond the order of calculations presented here. In particular, the attempt to solve for the spectrum of this Hamiltonian would pose a new class of problems, both in perturbative approaches and in numerical calculations.

Acknowledgments

This work was supported by the MINERVA foundation. I am indebted to E. Gubanova for enlightening discussions about her analytic calculations and the method of flow equations in general.

A Helicity Tables

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<td>-\tilde{G}_3(1, 2)</td>
<td>\tilde{G}_1(1, 2)</td>
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</table>

Table 2: Helicity table of the effective interaction for \( J_z = \pm n, x > x' \).

The functions \( G_i(1, 2) := G_i(x, k_\perp; x', k'_\perp) \) in Table 2 are given by

\[
G_i(x, k_\perp; x', k'_\perp) = m_n^2 \left( \frac{1}{x x'} + \frac{1}{(1 - x)(1 - x')} \right) \text{Int}(|1 - n|)
\]
In volved in this expression are the de/n0cnitions

\[ G_2(x, k_\perp; x', k_\perp') = \left[ m_f^2 \left( \frac{k_\perp k_{1\perp}}{x'x(1-x)(1-x')} \text{Int}(|n|) + \frac{|a_1 - a_2|}{x(x-x')^2} \text{Int}(|1-n|)\delta_{n,1} \right) + k_\perp k_{1\perp} \left[ \frac{\text{Int}(|1-n|)}{x'} + \frac{\text{Int}(|1+n|)}{(1-x)(1-x')} \right] \text{Int}(|n|) \right. \]

\[ \left. + k_\perp k_{1\perp} \left[ \frac{\text{Int}(|1-n|)}{x'x} + \frac{\text{Int}(|1+n|)}{(1-x)(1-x')x'} \right] + \frac{|a_1 - a_2|}{x(x-x')^2} \text{Int}(|n|)\delta_{n,0} \right] \text{Int}(|n|) \]

\[ G_3(x, k_\perp; x', k_\perp') = -m_f \frac{1}{x'x} \left[ k_{1\perp} \text{Int}(|1-n|) - k_\perp \frac{1-x'}{1-x} \text{Int}(|n|) \right] \]

\[ G_4(x, k_\perp; x', k_\perp') = -m_f^2 \frac{(x-x')^2}{x'x(1-x)(1-x)} \text{Int}(|n|) \]

An asterisk denotes the permutation of particle and anti-particle

\[ G_\ast(x, k_\perp; x', k_\perp') := G_3(1-x, -k_\perp; 1-x', -k_\perp'). \]

and it is \( \tilde{G}_i(n) := G_i(-n) \). The function \( \text{Int}(n) \) is defined as

\[ \text{Int}(n) = \theta(a_1 - a_2)\text{Int}_1(n) + \theta(a_2 - a_1)\text{Int}_2(n), \]

with

\[ \text{Int}_i(n) := \frac{\alpha}{\pi} (-\mathcal{A}_i)^{-n+1} \left( \frac{B_i}{k_\perp k_{1\perp}} \right)^n. \]

Involved in this expression are the definitions

\[ \mathcal{A}_i = \frac{1}{\sqrt{a_i^2 - 4k_i^2 k_{1\perp}^2}}, \quad B_i = \frac{1}{2} (1 - a_i \mathcal{A}_i), \]

and

\[ a_1 = m_f^2 \left( \frac{x-x'}{x'x} \right) + k_\perp + k_{1\perp} - \left( x-x' \right) \left[ \frac{k_{1\perp}^2}{x} - \frac{k'_{1\perp}^2}{x'} \right], \]

\[ a_2 = m_f^2 \left( \frac{(x-x')^2}{(1-x)(1-x')} + k_\perp + k_{1\perp}^2 + \left( x-x' \right) \left[ \frac{k_{1\perp}^2}{1-x} - \frac{k'_{1\perp}^2}{1-x'} \right]. \]

Note that the analogue of the latter expressions in the calculations with the Pauli ansatz [6], is their average \( a = (a_1 + a_2)/2 \).

References


[2] We are a bit sloppy here with the nomenclature of what is finite and infinite.


[14] We obtained a good fit to the function $(a = 0.2825, b = 1.54)$

$$C_{hf}(\Lambda) \simeq a - [a - C_{hf}(1)] e^{-b \log \Lambda}.$$