T-duality for boundary-non-critical point-particle and string quantum mechanics\(^1\)

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Abstract

It is observed that some structures recently uncovered in the study of Calogero-Sutherland models and anyons are close analogs of well-known structures of boundary conformal field theory. These examples of “boundary conformal quantum mechanics”, in spite of their apparent simplicity, have a rather reach structure, including some sort of T-duality, and could provide useful frameworks for testing general properties of boundary conformal theories. Of particular interest are the duality properties of anyons and Calogero-Sutherland particles in presence of boundary-violations of conformal invariance; these are here briefly analyzed leading to the conjecture of a general interconnection between (deformed) boundary conformal quantum mechanics, T-type duality, and (“exchange” or “exclusion”) exotic statistics. These results on the point-particle quantum-mechanics side are compared with recent results on the action of T-duality on open strings that satisfy conformal-invariance-violating boundary conditions. Moreover, it is observed that some of the special properties of anyon and Calogero-Sutherland quantum mechanics are also enjoyed by the M(atrix) quantum mechanics which has recently attracted considerable attention.

1 Introduction

Over the last few years a large number of studies has been devoted to anyons [1] and Calogero-Sutherland particles [2, 3], with emphasis on the fact that the former provide the canonical laboratory for the study of anomalous exchange statistics [1] while the latter exhibit anomalous exclusion statistics [4]. Interestingly, these point-particle non-relativistic quantum-mechanical systems enjoy scale invariance if (as customary) the domain of the relevant Hamiltonians is specified by the requirement that the wave functions be regular everywhere. Some recent studies have examined the implications of a certain class of “deformations” of these systems, in which one consistently introduces violations of scale invariance via self-adjoint extensions prescribing that the wave functions have isolated and square-integrable singularities at the boundary of the fundamental domain (the points of configuration space where the positions of two of the particles coincide). As I shall observe in the following, in spite of its apparent simplicity this type of boundary deformation of a scale-invariant quantum mechanics leads to a rather reach structure, including many of the properties of boundary deformations of more complicated boundary-conformal field theories.

These “boundary-conformal quantum mechanical systems” (which I shall qualify as “boundary-non-critical quantum mechanical systems” once deformed by non-conformally-invariant boundary physics) could be useful as simple settings in which to test ideas concerning more complicated theories with nontrivial boundary physics. I shall illustrate this by observing that in the Calogero-Sutherland and anyon problems one can find dualities that share some of the properties of dualities holding in open-string theories. This analogy still holds when non-conformally-invariant boundary physics is introduced, in which case the relevant duality properties on the open-string side are the ones here briefly reviewed in Sec. 3 (and originally derived in Ref. [5]).

2 Boundary-non-critical quantum mechanics and T-duality

The possibility of self-adjoint extensions mentioned in the Introduction can be illustrated very simply by looking at the Hamiltonian

\[ H = H^\nu_{\text{any}}(r) = -\frac{1}{r} \partial_r (r \partial_r) + \frac{\nu^2}{r^2}. \]

This Hamiltonian not only describes the s-wave relative motion of a 2-anyon system with exchange-statistics parameter \( \nu \), but its eigensolutions are also simply related to the ones of a corresponding Calogero-Sutherland model. The 2-body (relative-motion) Calogero-Sutherland model with exclusion-statistics parameter \( \beta \) is described
in the “Calogero limit” (i.e. the infinite-radius limit for the circle on which Calogero-Sutherland particles are constrained) by the Hamiltonian

$$H^\beta_{CaSu}(x) = -\frac{d^2}{dx^2} + \frac{\beta(\beta - 1)}{x^2},$$

which is related to $H^{\nu}_{any}$ by

$$H^\beta_{CaSu}(x) = x^{-1/2} H^{\beta+1/2}_{any}(x) x^{-1/2}$$

It is therefore sufficient to discuss the properties of the Hamiltonian $\mathcal{H}$ defined in (1) in order to obtain insight in both the s-wave sector of the (2+1-dimensional) 2-body anyon problem and the full (1+1-dimensional) 2-body Calogero-Sutherland problem. (Generalizations to the N-body problems with $N > 2$ have been discussed elsewhere, see e.g. Refs. [6, 7], but it will be sufficient to consider the 2-body problems for the illustrative purposes of the present discussion.)

A first observation is that the $\mathcal{H}$-eigenproblem is scale-invariant if the domain of $\mathcal{H}$ only includes wave functions that are regular everywhere, which is the preferred choice in most of the related literature. However, it is well known [8, 9] that “meaningful” (self-adjoint $\mathcal{H}$) $\mathcal{H}$-eigenproblems are obtained also from the more general class of boundary conditions

$$\left[r^{|\nu|}\psi(r) - w^{2|\nu|} \frac{d}{d(r^{2|\nu|})} \right] r=0 = 0,$$

which in general is scale-dependent. The scale-independent limits of (4) are obtained at the special values $0, \infty$ of the dimensionless parameter $w$, which characterizes the self-adjoint extension once the reference scale $\rho$ is assigned.\(^3\) In particular, the popular case of wave functions regular everywhere corresponds to the scale-invariant limit $w=0$, and it is conventional to consider the other elements of the one-parameter family of domains described by (4) as “deformations” of the ordinary regular-wave-function case.

Interestingly, these deformations that I have until now discussed as encoded in boundary conditions at $r=0$ allow for a “dual” description in which the wave functions are all along taken to be regular everywhere (i.e. satisfy (4) for $w=0$) and the boundary deformation is introduced via a $\delta(r)/r$ contact interaction (i.e. a boundary interaction, since particles only collide at the boundary of the domain) with running coupling $g(\mu)$.

\(^2\)Eq. (4) and some of the following equations do not naively apply to the special limit $\nu = 0$. Although very simple limiting prescriptions can be introduced to make these formulas hold even in the $\nu = 0$ limit, for the present paper it suffices to focus on $\nu \neq 0$. The interested reader can find useful insight in the study of the special case $\nu = 0$ in Ref. [10]. Concerning the parameter $w$ the reader should notice that, as in Ref. [9], in the following only $w \geq 0$ are considered.

\(^3\)In order to emphasize this $\rho \leftrightarrow w$ interdependence (which has sometimes been misinterpreted in the related literature) in the following I also use the compact notation $w_\rho$. 
Evidence of the equivalence of these two dual descriptions has emerged in several studies [6, 7, 9]. By looking at the first few terms in an appropriate perturbative expansion it has been shown how \( g(\mu) \) captures the physics of \( \rho \) and \( w \) (which is actually the physics of the combination \( w\rho^{2|\nu|} \), since (4) does not depend on \( w \) and \( \rho \) independently). In particular, in the \( w, \rho \) parametrization here adopted one finds [9] a simple formula that relates \( w \) to \( g \) once the reference scales \( \rho \) and \( \mu \) are fixed.

The realization that the class of deformations (4) could be described using a contact interaction also led [9] to a formulation of the anyon problem in the powerful language of (non-relativistic) field theory, in which of course the contact interaction is of the form \( g\Phi^4 \). (Here \( \Phi \) is understood as the bosonic field used in the “bosonic gauge” description of anyons.) Interestingly, although the Calogero-Sutherland problem is completely solved in the quantum-mechanical formulation, we are yet unable to reformulate it as a local field theory. In practice, while we have indentified the Chern-Simons field as the mediator of the exchange-statistical “interaction”, we (still ?) are unable to describe exclusion-statistical interactions as mediated by a field.

Since it is by default set up as perturbative, the Chern-Simons/anyon field-theoretical approach is directly connected with the corresponding perturbative approach in the quantum-mechanical formulation of anyons, and in fact these two perturbative approaches are essentially the same thing, although one might be more convenient than the other from the point of view of computations, depending on the quantities of interest. Of course, less direct is the correspondence between these perturbative (contact-interaction-based) approaches and the “dual” formulation in which the deformation is enforced via the boundary condition (4). This “duality” has been tested by comparing the first few terms in the perturbative (contact-interaction based) expansion of some quantities of interest to the corresponding approximation that can be obtained in the boundary-condition-deformation formulation, which in the 2-body case can be solved exactly. In light of the positive outcome of these tests there is growing confidence in the “duality” between contact-interaction formulation and boundary-condition formulation, but, as also emphasized in [11], one should probably keep open to the possibility that unexpected nonperturbative effects might spoil the “duality”.\(^4\)

One more aspect of the contact-interaction formulation that deserves mention is the role of renormalization. Interestingly, this type of contact interactions in non-

\(^4\)It is probably worth emphasizing that, while one should perhaps keep open to the possibility that nonperturbative effects might spoil the “duality”, one should not misinterpret the above-mentioned “delicate mathematics” required to describe the special case \( \nu = 0 \) (within the \( w, \rho \) parametrization) as a signal of possible perturbative-level failures of the “duality” between contact-interaction formare of the formulation and boundary-condition formulation. For example, in Ref. [11] a “parametrization singularity” that emerges in the \( \nu \to 0 \) limit has been erroneously interpreted as a problem for this “duality”, while that parametrization singularity has no more physical content than (and is somewhat related to) the well-known singularity that emerges in trying to obtain the \( \mathcal{H} \)-eigenfunctions having \( \ln r \) behavior at small \( r \) for \( \nu = 0 \) as a naive \( \nu \to 0 \) limit of the \( \mathcal{H} \)-eigenfunctions with \( r^{\nu} \) or \( r^{-\nu} \) behavior at small \( r \).
relativistic quantum mechanics (or, equivalently, non-relativistic field theory) requires [6, 7, 9, 10] the full machinery of renormalization that we are most accustomed to encounter in the context of relativistic field theory. The above-mentioned running coupling $g(\mu)$ emerges from the regularization/renormalization procedure. In particular, one finds that the one-loop renormalization-group equation satisfied by $g(\mu)$ is

$$\frac{dg}{d\ln \mu} = g^2 - \nu^2 .$$

Of course the fixed points $g = \pm \nu$ correspond to the scale-invariant limits of the boundary condition (4) ($g = \nu \leftrightarrow w = 0$ whereas $g = -\nu \leftrightarrow w = \infty$).

The formalism and language of renormalization might be used also to reinterpret certain other aspects of this subject. In particular, for the $\nu = 0$ case one often finds in the literature statements to the extent that the contact interactions discussed above can only be attractive. A similar statement is encountered in the $\nu \neq 0$ case, but there one allows for a small repulsive contact interaction, small enough that the associated repulsion does not overcome the "centrifugal" barrier associated with the anyon anomalous spin. These observations which arise in the mathematical setting of studies of self-adjoint extensions are probably related to the "triviality issues" that one encounters in a renormalization group analysis. For any given ultraviolet (short-distance) cut-off scale these models appear to be well defined for any value (however positive or negative) of the contact coupling $g$. As the cut-off is removed one finds that the repulsive models are trivial, i.e. they are not really repulsive since their renormalized coupling vanishes in the limit in which the cut-off is removed. This is probably the renormalization-group counterpart of the self-adjoint extension consistency arguments finding that repulsive contact interactions are not allowed. In physical applications the renormalization-group viewpoint might be most relevant, since in most cases the models here of interest only make physical sense with a finite cut-off. For example, if anyons are seen as collective modes of a physical system they should also be well-defined only up to a maximum energy-scale where the description in terms of the fundamental degrees of freedom becomes necessary.

I close this section by providing some evidence that another "duality" might be present in this framework, a duality with some elements of the T-duality of strings and perhaps even more closely related to the Kramers-Wannier duality of the Ising model in two dimensions. This evidence is found by examining the structure of the two-particle relative partition function $Z_2$ that is associated to the two-particle relative Hamiltonian $\mathcal{H} + \omega^2 r^2$. (Of course, $Z_2$ is the only nontrivial ingredient of the 2nd virial coefficient, so one could rephrase what follows by emphasizing the implications for the 2nd virial coefficient.)

Since the eigenvalues of $\mathcal{H} + \omega^2 r^2$ are of the form

$$E_n = 2 (2n + 1 + \Delta) \omega ,$$

(6)
where $\Delta$ depends on the boundary conditions (i.e. $w, \rho$) and $\nu, Z^2$ can be written as

$$Z^2 = \sum_{n=0}^{\infty} \exp[-2(2n + 1 + \Delta)\beta \omega],$$

(7)

Interestingly, all the dependence on $\nu$ and $w, \rho$ is in the factor $\exp(-2\Delta \beta \omega)$, and it is therefore plausible that the same $Z^2$ be obtained for different combinations of $w, \rho, \nu, \beta$, suggesting a duality in which different boundary conditions be connected by various coupling and temperature combinations. These dualities would characterize the full two-particle relative-motion sector of the Calogero-Sutherland problem, but in the anyon problem $Z^2$ only takes into account the $L=0$ projection of the two-particle relative motion, and for the full result one would have to add

$$Z^{L\neq0}_2 = \cosh[2(1 - \nu)\beta \omega] - \frac{\exp(-2\nu \beta \omega)}{2 \sinh[2\beta \omega]},$$

(8)

which depends independently on $\nu$ and $\beta$. The resulting properties of $Z^2 + Z^{L\neq0}_2$ have been discussed (however implicitly) in the recent Ref. [12], and will be discussed in detail in Ref. [13].

3 T-duality and boundary-non-critical strings

Open strings provide another class of theories that admits interesting boundary-non-critical deformations. As a first example in which to analyze the applicability to boundary-non-critical strings of T duality, Ref. [5] discussed the case of a linear-dilaton boundary background. Let me here review how that works in a flat 26-dimensional target spacetime. Upon introduction of the linear dilaton boundary background in a non-critical string theory with central charge deficit $Q^2$ one is confronted by the action

$$S = \frac{1}{4\pi} \int d^2 \sigma \partial X^i \partial \tilde{X}^i + \frac{1}{4\pi} \int d^2 \sigma \partial Y^j \partial \tilde{Y}^j - \int_{\partial \Sigma} \hat{k} \eta^i X^i$$

(9)

where $\eta$ is an n-dimensional constant number-valued vector, and (for later convenience) I have divided the 26 fields into $n$ fields of type $X$ (i.e. $i = 1 \ldots n$) and $26-n$ fields of type $Y$ (i.e. $j = 1 \ldots 26 - n$). I have also used conventional notations $\hat{k}$ for the extrinsic curvature of the fiducial metric, $\Sigma$ for the world-sheet manifold, and $\partial \Sigma$ for the boundary of the world-sheet manifold. The fields $X$ and $Y$ are assumed to satisfy Neumann boundary conditions:

$$\partial_\sigma X^i = \partial_\sigma \tilde{Y}^j = 0 \quad \text{on} \quad \partial \Sigma$$

(10)

where I use the notation $\partial_\sigma$ (or $\partial_\tau$) for normal (tangential) derivatives on $\partial \Sigma$ and I also use the notation $\hat{\Phi}$ to emphasize that a field $\Phi$ is being evaluated on $\partial \Sigma$.  

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The path-integral formulation of this $\sigma$-model is:

$$Z = \int DX\,DY\, \delta(\partial_\sigma \hat{X}^i)\,\delta(\partial_\sigma \hat{Y}^j)\, \exp \left[ -\frac{1}{4\pi} \int_\Sigma \left[ \partial X^i \partial X^i + \partial Y^j \partial Y^j \right] + \int_{\partial\Sigma} \hat{k} Q \eta^k X^i \right],$$

where I indicated explicitly via boundary delta-functionals that the $X^i$ and the $Y^j$ are Neumann fields.

As observed in Ref. [5] a functional T-duality transformation on the fields $X^i$ in the path integral (11) can be implemented by a straightforward generalization of the formulation of T duality in the path-integral formalism for critical (Dirichlet or Neumann) open-string theories (see e.g. Ref. [14]). In particular, also in the boundary-non-critical case the T-duality transformation has as crucial element the introduction of a vectorial field variable corresponding to the partial derivative of the fields $X^i$ that are being dualized:

$$W^i_\alpha \equiv \partial_\alpha X^i. \quad (12)$$

The fields $W$ are introduced in the path integral via the identity:

$$\int DW^i_\alpha \delta(W^i_\alpha - \partial_\alpha X) \delta(\epsilon_{\alpha\beta}\partial_\alpha W^j_\beta) = 1, \quad (13)$$

which takes of course into account the ‘Bianchi identity’:

$$\epsilon_{\alpha\beta}\partial_\alpha W^j_\beta = 0 \quad (14)$$

The path integral (11) is therefore rewritten as:

$$Z = Z_Y \int DX^i\,DW^i_\alpha\,D\chi^i\,D\lambda^i_\alpha \delta(\partial_\sigma \hat{X}^i) \, \exp \left[ \int_{\partial\Sigma} \hat{k} Q \eta^k X^i - i \int_\Sigma \hat{\lambda}^i_\sigma \hat{W}^i_\sigma - i \int_{\partial\Sigma} \hat{\lambda}^i_{\tau} (W^i_\tau - \partial_\tau \hat{X}^i) \right]$$

$$\exp \left[ -\frac{1}{4\pi} \int_\Sigma (W^i_\alpha)^2 - i \int_\Sigma \chi^i(\epsilon_{\alpha\beta}\partial_\alpha W^j_\beta) - i \int_\Sigma \lambda^i_\alpha (W^i_\alpha - \partial_\alpha X) \right], \quad (15)$$

where (consistently with the notation already introduced for the normal and tangential derivatives) I denoted the normal (tangential) components of world-sheet vectors with a lower index $\sigma$ ($\tau$). Summation is of course understood on all repeated indices apart from $\sigma$ and $\tau$ which are fixed labels for boundary fields. I am also using the short-hand notation

$$Z_Y \equiv \int DY \, \delta(\partial_\sigma \hat{Y}^j) \, \exp \left[ -\frac{1}{4\pi} \int_\Sigma \partial Y^j \partial Y^j \right], \quad (16)$$

for the portion of the partition function that concerns the $Y^j$ fields, which are "spectators" of the T-duality transformation being performed on the $X^i$ degrees of freedom. Moreover I adopt the convention $\epsilon_{\sigma\tau} = 1$ and the following functional representation of a $\delta(\phi)$ constraint

$$\delta(\phi) = \int D\lambda e^{-i \int_{k^i} \phi \lambda} \quad (17)$$
with \( M \) an appropriate manifold, indicating the range of definition of the arguments of the fields \( \phi \). (In the cases here of interest \( M = \Sigma \) or \( \partial \Sigma \).)

The Lagrange multipliers fields \( \chi^i \) and \( \lambda^i_\alpha \) play a highly non-trivial role in the T-duality transformation; in particular, the fields \( \chi^i \), which implement the Bianchi identity (14), turn out to be directly related to the fields that are T-dual to the fields \( X^i \), just as expected from the analysis reported in Ref. [14].

As best seen by rearranging terms using integration by parts [5], the functional integration over \( X \) and \( W \) can be done easily, and one obtains (up to an irrelevant overall factor coming from the gaussian integration over \( W \))

\[
Z = Z_Y \int D\chi D\lambda \delta(\partial_\alpha \lambda_\sigma) \delta(\hat{\lambda}_\sigma) \delta(\hat{\chi} - \hat{\lambda}_\tau) \delta(\partial_\tau \hat{\lambda}_\tau + iQk\eta^i + \hat{\lambda}_\sigma) \exp \left[ -\pi \int_{\Sigma} (\lambda^i_\alpha - \epsilon_{\alpha\beta} \partial_\beta \chi^i)^2 \right]
\]

(18)

The fields \( X^i_D \) that are T-dual to the fields \( X^i \) are easily identified as the ones satisfying the relation

\[
\frac{\epsilon_{\alpha\beta} \partial_\beta X^i_D}{2\pi} = \epsilon_{\alpha\beta} \partial_\beta \chi^i - \lambda^i_\alpha
\]

(19)

whose consistency follows from the constraint \( \partial_\alpha \lambda^i_\alpha = 0 \) (see Eq. (18)).

Upon the change of variables \( \chi^i \to X^i_D \), and disposing of the then trivial functional integration over the \( \lambda \) fields, one can easily rewrite the partition function of (18) as (up to another irrelevant overall factor)

\[
Z = Z_Y \int DX_D \delta(\partial_\tau \hat{X}^i_D + 2\pi Qk\eta^i) \exp \left[ -\int_{\Sigma}(\partial_\alpha X^i_D)^2 \right]
\]

(20)

where I also used the fact that (19), when combined with the constraint \( \hat{\lambda}^i_\sigma = 0 \) (see Eq. (18)), implies \( \partial_\tau \hat{X}^i_D = -i2\pi \partial_\tau \hat{\chi}^i \).

The T duality transformation implemented via the path-integral manipulations that take from (11) to (20) evidently maps a Neumann open string with boundary interactions corresponding to the linear dilaton boundary background present in (11) into a free open string satisfying nonconformal boundary conditions

\[
\partial_\tau \hat{X}^i_D = -2\pi Qk\eta^i
\]

(21)

This boundary condition reduces to the Dirichlet boundary condition in the limit \( Q \to 0 \). For every \( Q \neq 0 \) it encodes a “conformal anomaly” for the free T-dual theory that reflects the conformal anomaly of the corresponding boundary interactions of the original Neumann theory.

4 Outlook

Some of the topics here discussed might evolve very quickly in the coming years. It would be important to establish in more general contexts (more general structures
allowed on the boundary) the doings of T-duality transformations in boundary-non-critical open-string theories. It would also be interesting to find additional evidence of the general interconnection between boundary non-critical quantum mechanics, T-type duality, and (exchange or exclusion) exotic statistics that appears to be suggested by the line of argument advocated in the present paper.

It is also tempting to consider the possibility that some of the ideas of “renormalization in quantum mechanics” and “boundary-non-critical quantum mechanics” might turn out to have applications in the framework of M(atrix) quantum mechanics [15]. In particular, in the formulation adopted in Ref. [16] one ends up considering the effects of a $r^{-2}$ perturbation in a 9-dimensional space and one is of course confronted with infrared problems (and possibly ultraviolet problems if wave functions with appropriate structure can be considered). While the analysis reported in Ref. [16] avoided the regularization/renormalization machinery by advocating some (rather formal) manipulations based on supersymmetry, a rigorous treatment would require regularization/renormalization (e.g. of the type of the Hulthen-potential infrared regularization discussed in Ref. [17]).

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