Non-Standard Embedding and Five-Branes in Heterotic M–Theory

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Abstract

We construct vacua of M–theory on $S^1/Z_2$ associated with Calabi–Yau three-folds. These vacua are appropriate for compactification to $\mathcal{N} = 1$ supersymmetry theories in both four and five dimensions. We allow for general $E_8 \times E_8$ gauge bundles and for the presence of five-branes. The five-branes span the four-dimensional uncompactified space and are wrapped on holomorphic curves in the Calabi–Yau manifold. Properties of these vacua, as well as of the resulting low-energy theories, are discussed. We find that the low-energy gauge group is enlarged by gauge fields that originate on the five-brane world-volumes. In addition, the five-branes increase the types of new $E_8 \times E_8$ breaking patterns allowed by the non-standard embedding. Characteristic features of the low-energy theory, such as the threshold corrections to the gauge kinetic functions, are significantly modified due to the presence of the five-branes, as compared to the case of standard or non-standard embeddings without five-branes.

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1 Introduction

To make contact with low-energy physics, one of the central issues in string theory has been to find
vacua leading to chiral four-dimensional theories with $\mathcal{N} = 1$ supersymmetry. In recent years, the
new understanding of the non-perturbative behavior of string theory has broadened the scope for
approaching these issues. First, the M–theory paradigm of strong-weak coupling duality between
the different string theories has led to new descriptions of familiar vacua. Second, the inclusion
of brane states, that is, vacua with non-trivial form-fields, increases the class of possible backgrounds
giving a chiral $\mathcal{N} = 1$ theory in four dimensions, and has raised the possibility of gauge interactions
arising from the brane world-volume theory itself.

In this paper, we will consider a class of eleven-dimensional M–theory vacua based on the
strongly coupled limit of the $E_8 \times E_8$ heterotic string, as described by Hořava and Witten [1, 2].
At low energy, these are compactifications of eleven-dimensional supergravity on an $S^1/Z_2$ orbifold,
with $E_8$ gauge fields at each of the two orbifold fixed planes. Following Witten [3], we can
further compactify on a Calabi–Yau three-fold to give a chiral $\mathcal{N} = 1$ theory in four-dimensions.
Essentially, all the discussion to date of the low-energy properties of compactifications [4]–[42] has
been limited to the standard embedding, where the Calabi–Yau spin connection is embedded in
one of the $E_8$ gauge groups. Here, we will consider the general configuration leading to $\mathcal{N} = 1$
supersymmetry, where, first, we allow for general gauge bundles, and, second, include five-branes,
states which are essentially non-perturbative in heterotic string theory. The possibility of such
generalizations was first put forward by Witten [3]. Recently, gauge threshold corrections of (0, 2)
orbifold models have been computed in ref. [42] and have, in the large radius limit, been compared
to the expressions calculated from Hořava–Witten theory. Gauge thresholds of non-standard em-
beddings in the strongly coupled limit have also been discussed in [36]. A toy model of gauge fields
coming from five-branes close to the orbifold planes has been presented in [41]. Finally, one notes
that other limits of M–theory leading to four-dimensional $\mathcal{N} = 1$ theories have recently received
attention. In particular, there has been renewed interest in the phenomenology of type I vacua,
dual to the $SO(32)$ heterotic string, which also includes the presence of branes [43, 44]. However,
we will not consider such limits here.

The $\mathcal{N} = 1$ vacua we will discuss have the following structure. One starts with the spacetime
$M_{11} = S^1/Z_2 \times X \times M_4$, where $X$ is a Calabi–Yau three-fold and $M_4$ is flat Minkowski space. As
in the weakly coupled limit, to preserve the four supercharges, arbitrary holomorphic $E_8$ gauge
bundles over $X$ (satisfying the Donaldson–Uhlenbeck–Yau condition) are allowed on each plane. In
particular, there is no requirement that the spin-connection of the Calabi–Yau space be embedded
in the gauge connection of one of the $E_8$ bundles. This generalization is what is meant by non-
standard embedding, and has a long history in the phenomenology of weakly coupled strings (for
early discussions see refs. [45, 46, 47]). In addition, one can add five-branes, located at points
throughout the orbifold interval. The five-branes will preserve some supersymmetry, provided the branes are wrapped on holomorphic two-cycles within $X$ and otherwise span the flat Minkowski space $M_4$ [3].

Both the gauge fields and the five-branes are magnetic sources for the four-form field strength $G$ of the bulk supergravity, and so excite a non-zero $G$ within the compact $S^1/Z_2 \times X$ space. This has two effects. First, since the space is compact, there can be no net magnetic charge, for there is nowhere for the flux to “escape”. Thus, there a cohomological condition that the sum of the sources must be zero. Secondly, the non-zero form field enters the Killing spinor equation and so, to preserve supersymmetry, the background geometry must have a compensating distortion [3]. This leads to a perturbative expansion of the supersymmetric background. Such an expansion is familiar in non-standard embeddings in the weakly coupled heterotic string [45, 46, 47]. In the strongly coupled limit, it appears even for the standard embedding. From this point of view, the generalization to include non-standard embedding and five-branes is very natural.

Having found the vacuum as a perturbative solution, one is then interested in the form of the low-energy theory of the massless excitations around this compactification. It is well known that, in the standard embedding, to match the low-energy Newton constant and grand unified parameters, one needs to take a Calabi–Yau manifold of size comparable to the eleven-dimensional Planck length, with the orbifold an order of magnitude or so larger. Thus, it is natural to consider effective actions both in five dimensions, where only $X$ is compactified, and four, which is appropriate to momenta below the orbifold scale. For the standard embedding, the four-dimensional action has been calculated to leading non-trivial order [15, 22]. Although the expansion is completely non-perturbative, it turns out that, to this order, the form of the effective action is identical to the large coupling Calabi–Yau limit of the one-loop effective action calculated in the weak limit. There are threshold corrections in the gauge couplings as well as in the matter field Kähler potential. In five dimensions, because of the non-zero mode of $G$, the theory is a form of gauged supergravity in the bulk, coupled to gauge theories on the fixed planes [33, 39]. There is no homogeneous background solution but, rather, the correct vacuum is a BPS domain wall solution, supported by sources on the fixed planes and a potential in the bulk.

Calculating the modifications to the low-energy effective actions due to non-standard embedding and five-branes will be the main point of this paper. Our results can be summarized as follows. In section two, we discuss the expansion of the background solution, the cohomology condition on the five-brane and orbifold magnetic sources and the constraints on the zeroth-order background to preserve supersymmetry. We then give the solution to first order. Expanding in terms of eigenfunctions on the Calabi–Yau three-fold, we show that the main contribution comes from the massless modes. Sections three and four discuss the low-energy actions in the case of non-standard embedding and inclusion of five-branes respectively. This requires an analysis of the theory on the
five-brane world-volume, which is given in section 4.2. In summary, we find

- For non-standard embeddings, in the absence of five-branes, the five-dimensional action has the same form as in the standard embedding case both in the bulk and on the orbifold planes. However, the values of the gauge coupling parameters depend on the form of the non-standard embedding.

- The non-standard embedding allows many different breaking patterns for the $E_8$ groups. In particular, it is no longer necessary that the visible sector be broken to $E_6$. Rather, more general gauge groups $G^{(1)}, G^{(2)} \subset E_8$ and corresponding gauge matter can occur on the respective orbifold planes.

- In the presence of five branes, the form of the bulk five-dimensional action between any pair of neighboring branes is the same as in the case of standard embedding. The four-dimensional fixed-plane theories also have the same form and couplings to the bulk fields. However, there are additional four-dimensional theories, arrayed throughout the orbifold and again coupling to the bulk fields, which arise from the five-brane world-volume degrees of freedom.

- In the conventional picture, the five-brane worldvolume theories provide new hidden sectors. Generically, the theory for a single five-brane is $\mathcal{N} = 1$ supersymmetric with $g U(1)$ vector multiplets, together with a universal chiral multiplet and a set of chiral fields parameterizing the moduli space of holomorphic genus $g$ two-cycles in $X$. This gauge group can be enhanced when five-branes overlap or when the embedding of a single fivebrane degenerates. In general, the total rank of the gauge group remains unchanged.

- The presence of five-branes also allows for new types of $E_8 \times E_8$ breaking patterns, beyond those associated with non-standard embeddings alone. This is because the presence of five-brane sources leads to a wider range of solutions satisfying the zero cohomology condition.

- Reducing to four dimensions, the effective action is modified with respect to the standard embedding case. For pure non-standard embeddings, both the gauge and Kähler threshold corrections are identical in form to those of the standard embedding. However, the presence of the five-branes significantly modifies these corrections so that, for instance, both $E_8$ sectors can get threshold corrections of the same sign.

The new threshold corrections due to the five-branes have no analog in the weakly coupled limit since, first, the branes are non-perturbative and, second, the corrections depend on the positions of the five-branes across the orbifold, moduli which simply do not exist in the weakly coupled limit. Similarly, the appearance of new gauge groups due to five-branes is a non-perturbative effect. Finally, we note, it appears that there is a constraint on the total rank of the full gauge group
from orbifold fixed planes and five-branes, which arises from positivity constraints in the magnetic charge cohomology condition. We will discuss this issue elsewhere [48].

Let us end by summarizing our conventions. We use coordinates \( x^I \) with indices \( I, J, K, \cdots = 0, \cdots, 9, 11 \) to parameterize the full eleven-dimensional space \( M_{11} \). Throughout this paper, when we refer to orbifolds, we will work in the “upstairs” picture with the orbifold \( S^1/Z_2 \) in the \( x^{11} \)-direction. We choose the range \( x^{11} \in [-\pi \rho, \pi \rho] \) with the endpoints being identified. The \( Z_2 \) orbifold symmetry acts as \( x^{11} \rightarrow -x^{11} \). Then there exist two ten-dimensional hyperplanes fixed under the \( Z_2 \) symmetry which we denote by \( M^{(n)}_{10}, n = 1, 2 \). Locally, they are specified by the conditions \( x^{11} = 0, \pi \rho \). Barred indices \( \bar{I}, \bar{J}, \bar{K}, \cdots = 0, \cdots, 9 \) are used for the ten-dimensional space orthogonal to the orbifold. We use indices \( A, B, C, \cdots = 4, \cdots, 9 \) for the Calabi–Yau space. Holomorphic and anti-holomorphic indices on the Calabi–Yau space are denoted by \( a, b, c, \ldots \) and \( \bar{a}, \bar{b}, \bar{c}, \ldots \), respectively. Indices \( \mu, \nu, \rho, \cdots = 0, 1, 2, 3 \) are used for the usual four space-time coordinates. Fields will be required to have a definite behaviour under the \( Z_2 \) orbifold symmetry in \( D = 11 \). We demand a bosonic field \( \Phi \) to be even or odd; that is, \( \Phi(x^{11}) = \pm \Phi(-x^{11}) \). For an 11–dimensional Majorana spinor \( \Psi \) the condition is \( \Gamma_{11} \Psi(-x^{11}) = \pm \Psi(x^{11}) \) so that the projection to one of the orbifold planes leads to a ten-dimensional Majorana–Weyl spinor with positive chirality. The field content of 11–dimensional supergravity is given by a metric \( g_{IJ} \), a antisymmetric tensor field \( C_{IJK} \) and the gravitino \( \Psi_I \). While \( g_{IJ}, g_{11,11}, C_{IJK} \) and \( \Psi_I \) are \( Z_2 \) even, \( g_{111}, C_{IJK} \) and \( \Psi_{11} \) are odd. Finally, we note that we will usually adopt the convention that the standard model gauge fields live in the bundle on the \( M^{(1)}_{10} \) fixed plane bundle, and so refer to it as the “observable” sector, while the bundle on the \( M^{(2)}_{10} \) plane becomes the “hidden” sector.

2 Vacua with non-standard embedding and five-branes

In this section, we are going to construct generalized heterotic M–theory vacua appropriate for a reduction of the theory to \( \mathcal{N} = 1 \) supergravity theories in both five and four dimensions. To lowest order (in the sense explained below), these vacua have the usual space-time structure \( M_{11} = S^1/Z_2 \times X \times M_4 \) where \( X \) is a Calabi–Yau three-fold and \( M_4 \) is four-dimensional Minkowski space. As compared to the vacua constructed to date, we will allow for two generalizations. First, we will not restrict ourselves to embedding the Calabi–Yau spin connection into a subgroup \( SU(3) \subset E_8 \) but, rather, allow for general (supersymmetry preserving) gauge field sources on the orbifold hyperplanes. Secondly, we will allow for the presence of five-branes that stretch across \( M_4 \) and wrap around a holomorphic curve in \( X \). As we will see, the inclusion of five-branes makes it much easier to satisfy the necessary constraints. Therefore, their inclusion is essential for a complete discussion non-standard embeddings, and leads to a considerable increase in the number of such vacua.
2.1 Expansion parameters

Before we proceed to the actual computation, let us explain the types of corrections to the lowest order background that one expects. For the weakly coupled heterotic string, it is well known that non-standard embeddings lead to corrections to the Calabi–Yau background. They can be computed perturbatively [45, 46, 47] as a series in

\[ \epsilon_W = \frac{\alpha'}{v_{10}^{1/3}} \]  

(2.1)

where \( v_{10} \) is the Calabi–Yau volume measured in terms of the ten-dimensional Einstein frame metric. At larger string coupling, one also gets contributions from string loops. Thus the full solution is a double expansion involving both \( \epsilon_W \) and the string coupling constant.

On the other hand, in the strongly coupled heterotic string, it has been shown that, even in the case of the standard embedding, there are corrections originating from the localization of the gauge fields to the ten-dimensional orbifold planes [3, 22]. Again, these corrections can be organized in a double expansion. However, one now uses a parameterization appropriate to the strongly coupled theory. The 11-dimensional Hořava–Witten effective action has an expansion in \( \kappa \), the 11-dimensional Newton constant. For the compactification on \( S^1/Z_2 \times X \), there are two other scales, the size of the orbifold interval \( \pi \rho \) and the volume \( v \) of the Calabi–Yau threefold, each measured in the 11-dimensional metric. Solving the equations of motion and supersymmetry conditions for the action to order \( \kappa^{2/3} \), one finds the correction to the background is a double expansion, linear, at this order, in the parameter

\[ \epsilon_S = \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{\pi \rho}{v^{2/3}} \]  

(2.2)

but to all orders in

\[ \epsilon_R = \frac{v^{1/6}}{\pi \rho} \]  

(2.3)

It is natural to use the same expansion for the background with non-standard embedding and the inclusion of five-branes. As we will show explicitly, the solution to the order \( \kappa^{2/3} \) can be obtained as an expansion in eigenfunctions of the Calabi–Yau Laplacian. It turns out that the zero-eigenvalue, or “massless”, terms in this expansion are precisely of order \( \epsilon_S \), while the massive terms are of order \( \epsilon_R \epsilon_S \). Therefore, although one could expect corrections to arbitrary order in \( \epsilon_R \), to leading order in \( \epsilon_S \) only the zeroth-order and linear terms in \( \epsilon_R \) contribute.

Clearly, for the above expansion to be valid both \( \epsilon_S \) and \( \epsilon_R \) should be small. Let us briefly discuss the situation at the physical point, that is, at the values of \( \kappa, v \) and \( \rho \) that lead to the appropriate values for the four-dimensional Newton constant and the grand unification coupling parameter and scale. There, both the 11-dimensional Planck length \( \kappa^{2/9} \), as well as the Calabi–Yau
radius $v^{1/6}$, are of the order $10^{-16} \text{ GeV}^{-1}$ while the orbifold radius is an order of magnitude or so larger. Inserting this into eq. (2.2) and (2.3) shows that $\epsilon_S$ is of order one [5] while $\epsilon_R$ is an order of magnitude or so smaller. At the physical point, therefore, we have

$$\epsilon_R \ll \epsilon_S = O(1) .$$  \hspace{1cm} (2.4)

Consequently, neglecting higher-order terms in $\epsilon_S$ might not provide a good approximation at the physical point. It is, however, the best one can do at the moment given that M–theory on $S^1/Z_2$ is only known as an effective theory to order $\kappa^{2/3}$. On the other hand, in fact, higher-order terms in $\epsilon_R$ should be strongly suppressed and can be safely neglected.

It is interesting to note how this strong coupling expansion is related to the weak coupling expansion with non-standard embedding. Writing $\epsilon_W$ in terms of 11-dimensional quantities, one finds

$$\epsilon_W = \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{1}{\pi^2 \rho^{1/3}} \rho v^{1/3} \hspace{1cm} (2.5)$$

and hence

$$\epsilon_W = \frac{1}{\pi} \epsilon_R \epsilon_S . \hspace{1cm} (2.6)$$

Let us try to make this relation plausible. In the weak coupling limit, the orbifold becomes small. Hence, one expects to extract the weak coupling part of the full background by performing an orbifold average. We recall that the massive terms in the full background are of order $\epsilon_R \epsilon_S$. In addition, we will find that those massive modes decay exponentially as one moves away from the orbifold planes, at a rate set by the Calabi–Yau radius $v^{1/6}$. Therefore, when performing the average, one picks up another factor of $\epsilon_R$ leading to $\epsilon_R^2 \epsilon_S$ as the order of the averaged massive terms. This is in perfect agreement with the expectation, (2.6), from the weakly coupled heterotic string\(^1\).

### 2.2 Basic equations and zeroth-order background

The M–theory vacuum is given in the 11-dimensional limit by specifying the metric $g_{IJ}$ and the three-form $C_{IJK}$ with field strength $G_{IJKL} = 24 \partial_I C_{JKL}$. To the order $\kappa^{2/3}$, the set of equations to be solved consists of the Killing spinor equation

$$\delta \Psi_I = D_I \eta + \frac{\sqrt{2}}{288} (\Gamma_{IJKLM} - 8 g_{IJ} \Gamma_{KLM}) G^{JJKLM} \eta = 0 , \hspace{1cm} (2.7)$$

\(^1\)There is no such comparison for the massless modes as they correspond to trivial integration constants on the weakly coupled side which can be absorbed into a redefinition of the moduli. This will be explained in detail later on.
for a Majorana spinor $\eta$, the equation of motion for $G$

$$D_I G^{IJKL} = 0 \quad (2.8)$$

and the Bianchi identity

$$\begin{align*}
(dG)_{11IJKL} &= 2\sqrt{2}\pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \left[ J^{(0)}(x^{11}) + J^{(N+1)}(x^{11} - \pi \rho) + \frac{1}{2} \sum_{n=1}^{N} J^{(n)}(x^{11} - x_n) + \delta(x^{11} + x_n) \right]_{IJKL} \quad (2.9)
\end{align*}$$

Here the sources $J^{(0)}$ and $J^{(N+1)}$ on the orbifold planes are as usual given by

$$\begin{align*}
J^{(0)} &= -\frac{1}{8\pi^2} \left( \text{tr} F^{(1)} \wedge F^{(1)} - \frac{1}{2} \text{tr} R \wedge R \right)_{x^{11}=0}, \\
J^{(N+1)} &= -\frac{1}{8\pi^2} \left( \text{tr} F^{(2)} \wedge F^{(2)} - \frac{1}{2} \text{tr} R \wedge R \right)_{x^{11}=\pi \rho} \quad (2.10)
\end{align*}$$

We have also introduced $N$ additional sources $J^{(n)}$, $n = 1, \ldots, N$. They come from $N$ five-branes located at $x^{11} = x_1, \ldots, x_N$ where $0 \leq x_1 \leq \cdots \leq x_N \leq \pi \rho$ (see fig. 1). Note that each five-brane at $x^{11} = x_n$ has to be paired with a mirror five-brane at $x^{11} = -x_n$ with the same source since the Bianchi identity must be even under the $Z_2$ orbifold symmetry. Our normalization is such that the total source of each pair is $J^{(n)}$. The structure of these five-brane sources will be discussed below.

We are interested in finding solutions of these equations that preserve 3 + 1-dimensional Poincaré invariance and admit a Killing spinor $\eta$ corresponding to four preserved supercharges and, hence, $\mathcal{N} = 1$ supersymmetry in four dimensions.

The usual procedure to find such solutions is to solve the equations perturbatively. To this order, one chooses a space $S^1/Z_2 \times X \times M_4$, where $X$ is a Calabi–Yau three-fold with a Ricci-flat metric $g_{AB}$, admitting a Killing spinor $\eta^{(CY)}$. To lowest order, the solution, denoted in the following by $(0)$, is then given by

$$\begin{align*}
\eta^{(0)} &= \eta^{(CY)} \\
G^{(0)}_{IJKL} &= 0 \quad (2.11)
\end{align*}$$

Note that it is consistent, to this order, to set the antisymmetric tensor field to zero since the sources in the Bianchi identity are proportional to $\kappa^{2/3}$ and, hence, first order in $\epsilon_S$.

One must also ensure that the theories on the orbifold planes preserve supersymmetry. This leads to the familiar constraint, following from the vanishing of the supersymmetry variation of the

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2Here we are using the normalization given in ref. [2]. Conrad [11] has argued that the correct normalization is smaller. In that case, the coefficient of the right-hand side of the Bianchi identity (2.9) and eqns. (2.20) and (2.26) below are all multiplied by $2^{-1/3}$. Furthermore, the definition of $\epsilon_S$ in eqn. (2.2) should also be multiplied by $2^{-1/3}$.
Figure 1: Orbifold interval with boundaries at 0, πρ and \( N \) five-branes at \( x_1, \ldots, x_N \). The mirror interval from 0 to \(-πρ\) is suppressed in this diagram.

As discussed in [47], this implies that each \( E_8 \) gauge field is a holomorphic gauge bundle over the Calabi–Yau three-fold, satisfying the Donaldson–Uhlenbeck–Yau condition. The holomorphicity implies that \( F^{(1)}_{AB} \) and \( F^{(2)}_{AB} \) are \((1,1)\)-forms. It follows that, since \( R_{AB} \) for a Calabi–Yau three-fold is also a \((1,1)\)-form, the orbifold sources \( J^{(0)} \) and \( J^{(N+1)} \), defined by eq. (2.10), are closed \((2,2)\)-forms.

For the five-brane world-volume theory to be supersymmetric, the branes must be embedded in the Calabi–Yau space in a particular way [3]. To preserve Lorentz invariance in \( M_4 \), they must span the \( 3 + 1 \)-dimensional uncompactified space. The remaining spatial dimensions must then be wrapped on a two-cycle in the Calabi–Yau space. The condition of supersymmetry implies that the cycle is a holomorphic curve [3, 49, 50]. As we will show in section 4.2, in such a situation, we preserve four supercharges on the five-brane world-volume corresponding to \( \mathcal{N} = 1 \) supersymmetry in four dimensions. Since the five-branes are magnetic sources for \( G \), they enter the right-hand side of the Bianchi identity (2.9) as source terms, which should be localized on the five-brane world-volumes. The delta function in \( x^{11} \) gives the localization in the orbifold direction, while the four-forms \( J^{(n)} \) must give the localization of the \( n \)-th five-brane on the two-cycle \( C_2^{(n)} \). Explicitly, for any two-cycle \( C_2 \), one can introduce a delta-function four-form \( \delta(C_2) \), defined in the usual way, such that for any two-form \( \chi \),

\[
\int_x \chi \wedge \delta(C_2) = \int_{C_2} \chi ,
\]
so that \( \delta(C_2) \) is localized on \( C_2 \). In general, we would expect that \( J^{(n)} \) is proportional to \( \delta(C_2^{(n)}) \). In fact, the correct normalization of the five-brane magnetic charge \([52, 53]\) implies that the two are equal, that is

\[
J^{(n)} = \delta(C_2^{(n)}) .
\] (2.14)

Since the cycles are holomorphic, \( J^{(n)} \), like the orbifold sources, are closed (2,2)-forms.

There is one further condition which the five-branes and the fields on the orbifold planes must satisfy. This is a cohomology condition on the Bianchi identity \([3]\). Consider integrating the identity (2.9) over a five-cycle which spans the orbifold interval together with an arbitrary four-cycle \( C_4 \) in the Calabi–Yau three-fold. Since \( dG \) is exact, this integral must vanish. Physically this is the statement that there can be no net charge in a compact space, since there is nowhere for the flux to “escape”. Performing the integral over the orbifold, we derive, using (2.9), the condition

\[
-\frac{1}{8\pi^2}\int_{C_4}\text{tr}F^{(1)} \wedge F^{(1)} - \frac{1}{8\pi^2}\int_{C_4}\text{tr}F^{(2)} \wedge F^{(2)} + \frac{1}{8\pi^2}\int_{C_4}\text{tr}R \wedge R + \sum_{n=1}^{N}\int_{C_4}J^{(n)} = 0 .
\] (2.15)

Hence, the net magnetic charge over \( C_4 \) is zero. Equivalently, this implies that the sum of the sources must be cohomologically trivial, that is

\[
\left[ \sum_{n=0}^{N+1} J^{(n)} \right] = 0 .
\] (2.16)

Let us now return to the normalization of the five-brane charges. We note that in equation (2.15) the first three terms are all integers. They are topological invariants, giving the instanton numbers (second Chern numbers) of the two \( E_8 \) bundles and the instanton number (first Pontrjagin number) of the tangent bundle of the Calabi–Yau three-fold. Hence, the above constraint shows that \( n_5(C_4) = \sum_{n=1}^{N}\int_{C_4}J^{(n)} \) must also be an integer. In fact, with the normalization given in eqn. (2.14), each \( \int_{C_4}J^{(n)} \) is an integer. It is also a topological invariant, giving the intersection number \([51]\) of the \( n \)-th brane, on the two-cycle \( C_2^{(n)} \), with the four-cycle \( C_4 \). This can be understood as follows (see fig. 2). The two cycles naturally intersect at points in the Calabi–Yau manifold. Thus in \( C_4 \), the five-brane appears as a set of point-like magnetic charges located at each intersection. The net contribution of the five-brane to the magnetic charge on \( C_4 \) is then the sum of the point charges, which is precisely the intersection number. Given the normalization of (2.14), each intersection contributes one unit of magnetic charge. We also note that, for a holomorphic curve \( C_4 \), since \( C_2^{(n)} \) is holomorphic, it is a theorem \([51]\) that the intersection number is always positive. This is related to the fact that only five-branes and not anti-five-branes are allowed if we are to preserve supersymmetry. In summary, the main point is that the normalization of the five-brane charge is such that each five-brane intersection with \( C_4 \) and each gauge instanton on the orbifold plane carry the same amount of magnetic charge \([52, 53]\).
Figure 2: Intersection of a five-brane wrapped on the holomorphic cycle $C_2^{(n)}$ and a four-cycle $C_4$. In this example the five-brane contributes two units of magnetic charge on $C_4$.

We can then rewrite the cohomology condition (2.15) on a particular holomorphic four-cycle $C_4$ as

\[ n_1(C_4) + n_2(C_4) + n_5(C_4) = n_R(C_4) \] (2.17)

which states that the sum of the number of instantons on the two $E_8$ bundles and the sum of the intersection numbers of each five-brane with the four-cycle $C_4$, must equal the instanton number for the Calabi–Yau tangent bundle, a number which is fixed once the Calabi–Yau geometry is chosen.

In summary, we see that to define the zeroth-order background we must specify the following data

- a Calabi–Yau three-fold $X$,
- two holomorphic vector bundles over $X$, one for each fixed plane, satisfying the Donaldson–Uhlenbeck–Yau condition. In general, there is no constraint that these bundles correspond to the embedding of the spin-connection in the gauge connection,
- a set of five-branes, each spanning the uncompactified $3 + 1$ dimensional space and wrapping a holomorphic two-cycle in the Calabi–Yau space,
- the sum of the five-branes magnetic charges and the instanton numbers from the gauge bundles, must equal the tangent space instanton number of $X$, as in equation (2.17),

We can then proceed to calculate the first-order corrections to the background.
2.3 First-order background

As an expansion in $\epsilon_S$, we write the bulk fields and the Killing spinor as
\begin{align}
g_{IJ} &= g_{IJ}^{(0)} + g_{IJ}^{(1)} \\
C_{IJK} &= C_{IJK}^{(0)} + C_{IJK}^{(1)} \\
\eta &= \eta^{(0)} + \eta^{(1)}.
\end{align}

(2.18)

where the index (0) refers to the uncorrected background, given in (2.11), and the index (1) to the corrections to first order in $\epsilon_S$.

Expanding to this order in $\epsilon_S$, we get for the Killing spinor equation (2.7)
\begin{align}
\delta \Psi_I &= D^{(0)}_I \eta^{(1)} - \frac{1}{8} \left( D^{(0)}_J g^{(1)}_{KJ} - D^{(0)}_K g^{(1)}_{JI} \right) \Gamma_{JK} \eta^{(0)} \\
&\quad + \frac{\sqrt{2}}{288} \left( \Gamma_{IJKLM} - 8g^{(0)}_{IJ} \Gamma_{KLM} \right) G^{(1)JKLM} \eta^{(0)} = 0
\end{align}

(2.19)

and for the equation of motion for $G$ (2.8) and the Bianchi identity (2.9)
\begin{align}
D^{(0)}_I G^{(1)IJKL} &= 0 \\
(dG^{(1)})_{11i11j11k11l} &= 2\sqrt{2}\pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \left[ J^{(0)}(x^{11}) + J^{(N+1)}(x^{11} - \pi \rho) \right. \\
&\quad \left. + \frac{1}{2} \sum_{n=1}^{N} J^{(n)}(x^{11} - x_n) + \delta(x^{11} + x_n) \right] G^{(1)ijkl} \eta^{(0)}.
\end{align}

(2.20)

First, we note that the only nonvanishing components of the antisymmetric tensor $G^{(1)}$ are $G^{(1)abcd}$ and $G^{(1)abc11}$. This follows from the Bianchi identity for $G^{(1)}$ in eq. (2.20) and the fact that all sources $J^{(n)}$ are $(2,2)$ forms. For $G^{(1)}$ of this form, the Killing spinor equation has been analyzed in ref. [3].

It has been shown in that paper that the corrections first order in $\epsilon_S$ to the metric and Killing spinor should have the structure
\begin{align}
g^{(1)}_{\mu\nu} &= b_{\mu\nu}, \quad g^{(1)}_{AB} = h_{AB}, \quad g^{(1)}_{1111} = \gamma, \quad \eta^{(1)} = \psi \eta^{(0)}
\end{align}

(2.21)

with orbifold and Calabi–Yau dependent functions $b$, $h_{AB}$, $\gamma$ and $\psi$. Furthermore, in [3] a consistent set of differential equations has been derived from eq. (2.19) which determines $b$, $h_{AB}$, $\gamma$ and $\psi$ in terms of $G^{(1)}$. An explicit solution for these differential equations in terms of the dual antisymmetric tensor $B$ defined by
\begin{align}
\mathcal{H} = dB = *G^{(1)}
\end{align}

(2.22)

was presented in ref. [22]. In the following, we adopt the harmonic gauge, $d^*B = 0$. Then, since the sources in the Bianchi identity (2.20) are $(2,2)$ forms, the only nonvanishing components of $B$ are
\begin{align}
B_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} B_{ab}
\end{align}

(2.23)
with $B_{ab}$ a $(1,1)$ form on the Calabi–Yau space. Using the results of ref. [22], the Killing spinor equation in (2.19) is solved by

$$h_{ab} = \sqrt{2}i \left( B_{ab} - \frac{1}{3} \omega_{ab} B \right)$$

$$b = \frac{\sqrt{2}}{6} B$$

$$\gamma = -\frac{\sqrt{2}}{3} B$$

$$\psi = -\frac{\sqrt{2}}{24} B$$

where $B = \omega^{AB} B_{AB}$ and $\omega_{ab} = -ig_{ab}$ is the Kähler form. We have, therefore, explicitly expressed the complete background in terms of the $(1,1)$ form $B_{ab}$. All that remains then is to determine this $(1,1)$ form, which can be done following the methods given in ref. [22]. In the harmonic gauge, which implies

$$D_A (0) B_{AB} = 0 \ , \quad (2.25)$$

$B_{AB}$ is determined from eq. (2.20) by solving

$$\left( \Delta_X + D_{11}^2 \right) B_{AB} = 2\sqrt{2}\pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \left[ *_{X} J^{(0)} \delta(x^{11}) + *_{X} J^{(N+1)} \delta(x^{11} - \pi \rho) + \sum_{n=1}^{N} *_{X} J^{(n)} \left( \delta(x^{11} - x^{n}) + \delta(x^{11} + x^{n}) \right) \right]_{AB} \ . \quad (2.26)$$

where $\Delta_X$ is the Laplacian and $*_{X}$ the Hodge star operator on the Calabi–Yau space. Essentially, this is the equation for a potential between a set of charged plates positioned through the orbifold interval at the fixed planes and the five-brane locations. The charge is not uniform over the Calabi–Yau space. To find a solution, following ref. [22] we introduce eigenmodes $\pi_{iab}$ of this Laplacian with eigenvalues $-\lambda^2_i$ so that

$$\Delta_X \pi_{iab} = -\lambda^2_i \pi_{iab} \ . \quad (2.27)$$

Generically, $\lambda_i$ is of order $v^{-1/6}$. The metric on the space of eigenmodes

$$G_{ij} = \frac{1}{2v} \int_X \pi_i \wedge (*\pi_j)$$

is used to raise and lower $i$-type indices. Particularly relevant are the massless modes with $\lambda_i = 0$, which are precisely the $h^{1,1}$ harmonic $(1,1)$ forms of the Calabi–Yau space. We will also denote...
these harmonic (1, 1) forms by $\omega^A_{AB}$. In the following, in order to distinguish between massless and massive modes, we will use indices $i_0, j_0, k_0, \ldots = 1, \ldots, h^{1,1}$ for the former and indices $i, j, k, \ldots$ for the latter, while we continue to use $i, j, k, \ldots$ for all modes. Let us now expand the sources in terms of the eigenfunctions as

$$*X J^{(n)} = \frac{1}{2\nu^{2/3}} \sum_i \beta_i^{(n)} \pi^i$$

(2.29)

where

$$\beta_i^{(n)} = \frac{1}{\nu^{1/3}} \int_X \pi_i \wedge J^{(n)}.$$

(2.30)

If we introduce four-cycles $C_{4i_0}$ dual to the harmonic (1, 1) forms $\omega_{i_0}$, we can write for the massless modes

$$\beta_{i_0}^{(n)} = \int_{C_{4i_0}} J^{(n)}.$$

(2.31)

Specifically, it follows from (2.10) that $\beta_{i_0}^{(0)}$ and $\beta_{i_0}^{(N+1)}$ represent the instanton numbers of the gauge fields on the orbifold planes minus half the instanton number of the tangent bundle and, hence, it would appear, are in general half-integer. However, since $M_{11}$ must be a spin manifold (since it must admit spinors), the tangent bundle instanton number must be divisible by two [53] and so $\beta_{i_0}^{(0)}$ and $\beta_{i_0}^{(N+1)}$ are, in fact, integer. Furthermore, $\beta_{i_0}^{(n)}$, $n = 1, \ldots, N$ are the five-brane charges, given by the intersection number of each five-brane with the cycle $C_{4i_0}$, and are also integers. Let us also expand $B_{AB}$ in terms of eigenfunctions as

$$B_{AB} = \sum_i b_i \pi^A_{AB}$$

(2.32)

Then inserting this expansion, together with the expression (2.29) for the sources, into eq. (2.26), it is straightforward to obtain

$$\left(\partial^2_{11} - \lambda^2_i\right) b_i = \frac{\epsilon_S}{\sqrt{2}\rho} \left[ \beta_i^{(0)} \delta(x^{11}) + \beta_i^{(N+1)} \delta(x^{11} - \pi\rho) + \frac{1}{2} \sum_{n=1}^N \beta_i^{(n)} \left( \delta(x^{11} - x_i) + \delta(x^{11} + x_i) \right) \right]$$

(2.33)

It is then easy to solve these equation to give an explicit solution for the massive and massless modes. We note that the size of the sources is set by $\epsilon_S/\sqrt{2}\rho$ which, from eq. (2.2), is independent of the size of the orbifold. We first solve eq. (2.33) for the massive modes, that is, for $\lambda_i \neq 0$. In terms of the normalized orbifold coordinates

$$z = \frac{x^{11}}{\pi\rho}, \quad z_n = \frac{x_n}{\pi\rho}, \quad n = 1, \ldots, N,$$

(2.34)
$z_0 = 0$ and $z_{N+1} = 1$, we find

$$b_i = \frac{\pi \epsilon_S}{\sqrt{2}} \delta_i \left[ \left( \sum_{m=0}^{n} c_{i,m} \beta_i^m \right) \sinh(\delta_i^{-1}|z|) + \left( \sum_{m=n+1}^{N+1} s_{i,m} \beta_i^m \right) \right]$$

in the interval

$$z_n \leq |z| \leq z_{n+1} ,$$

for fixed $n$, where $n = 0, \ldots, N$. Here we have defined

$$\delta_i = \frac{1}{\pi \rho \lambda_i}, \quad c_{i,n} = \cosh(\delta_i^{-1}z_n), \quad s_{i,n} = \sinh(\delta_i^{-1}z_n).$$

Note that, since the eigenvalues $\lambda_i$ are of order $\nu^{-1/6}$, the quantities $\delta_i$ defined above are of order $\epsilon_R$. Therefore, as already stated, the size of the massive modes is set by $\epsilon_R \epsilon_S$.

We now turn to the massless modes. First note that, in order to have a solution of (2.33), we must have

$$\sum_{n=0}^{N+1} \beta_i^{(n)} = 0 .$$

However, from the definition (2.31), we see that this is, of course, none other than the cohomology condition (2.16) described above, and so is indeed satisfied. Integrating eq. (2.33) for $\lambda_i = 0$ we then find

$$b_i = \frac{\pi \epsilon_S}{\sqrt{2}} \left[ \sum_{m=0}^{n} \beta_i^{(m)} (|z| - z_m) - \frac{1}{2} \sum_{m=0}^{N+1} (z_m^2 - 2z_m) \beta_i^{(m)} \right]$$

in the interval

$$z_n \leq |z| \leq z_{n+1} ,$$

for fixed $n$, where $n = 0, \ldots, N$. As already discussed, the massless modes are of order $\epsilon_S$ and, unlike for the massive modes, no additional factor of $\epsilon_R$ appears.

It is important to note that there could have been an arbitrary constant in the zero-mode solutions. However, such a constant can always be absorbed into a redefinition of the Calabi–Yau zero modes or, correspondingly, the low energy fields. Consequently, in the solution (2.38) we have fixed the constant by taking the orbifold average of the solution to be zero. This will be important later in deriving low-energy effective actions.

Before we discuss the implications of these equations in detail, let us summarize our results. We have constructed heterotic M–theory backgrounds with non-standard embeddings including the
presence of bulk five-branes. We started with a standard Calabi–Yau background with gauge fields and five-branes to lowest order and showed that corrections to it can be computed in a double expansion in $\epsilon_S$ and $\epsilon_R$. Explicitly, we have solved the problem to linear order in $\epsilon_S$ and to all orders in $\epsilon_R$. We found the massive modes to be of order $\epsilon_R \epsilon_S$ while the massless modes are of order $\epsilon_S$. Therefore, although one could have expected corrections of arbitrary power in $\epsilon_R$, we only find zeroth- and first-order contributions at the linear level in $\epsilon_S$. Concentrating on the leading order massless modes, in each interval between two five-branes, $z_n \leq |z| \leq z_{n+1}$, the massless modes vary linearly with a slope proportional to the total charge $\sum_{m=0}^{n} \beta_{i_0}^{(m)}$ to the left of the interval. (Note that the total charge to the right of the interval has the same magnitude but opposite sign due to eq. (2.37).) At the five-brane locations, the linear pieces match continuously but with kinks which lead to the delta-function sources when the second derivative is computed. (A specific example is given in section 4.1, see fig. 3.) Similar kinks appear for the massive modes which, however, vary in a more complicated way between each pair of five-branes.

3 Backgrounds without five-branes

In this section, we will restrict the previous general solutions to the case of pure non-standard embedding without additional five-branes and discuss some properties of such backgrounds and the resulting low-energy effective actions in both four and five dimensions.

3.1 Properties of the background

To specialize to the case without five-branes, we set $N = 0$ and recall that $z_0 = 0$ and $z_1 = 1$. Also, the vanishing cohomology condition (2.37) implies that we have only one independent charge

$$\beta_{i_0} \equiv \beta_{i_0}^{(0)} = -\beta_{i_0}^{(1)}$$

per mode. Using this information to simplify eq. (2.38), we find for the massless modes

$$b_{i_0} = \frac{\pi \epsilon_S}{\sqrt{2}} \beta_{i_0} \left( |z| - \frac{1}{2} \right).$$

(3.2)

In the same way, we obtain from eq. (2.35) for the massive modes

$$b_{i_1} = \frac{\pi \epsilon_S}{\sqrt{2}} \delta_{i_1} \left[ (\beta_{i_1}^{(0)} - \beta_{i_1}^{(1)}) \frac{\sinh(\delta_{i_1}^{-1}(|z| - 1/2))}{2 \cosh(\delta_{i_1}^{-1}/2)} - (\beta_{i_1}^{(0)} + \beta_{i_1}^{(1)}) \frac{\cosh(\delta_{i_1}^{-1}(|z| - 1/2))}{2 \sinh(\delta_{i_1}^{-1}/2)} \right].$$

(3.3)

Note that, unlike for the massless modes, here we have no relation between the coefficients $\beta_{i_1}^{(0)}$ and $\beta_{i_1}^{(1)}$. Let us compare these results to the case of the standard embedding [22]. We see that the massless modes solution is, in fact, completely unchanged in form from the the standard embedding case, though the parameter $\beta_{i_0}$ can be different. This is a direct consequence of the cohomology
condition (2.37) which, for the simple case without five-branes, tells us that the instanton numbers on the two orbifold planes always have to be equal and opposite. There is no similar condition for the massive modes and we therefore expect a difference from the standard embedding case. Indeed, the standard embedding case is obtained from eq. (3.3) by setting $\beta_i^{(0)} + \beta_i^{(1)} = 0$ so that the second term vanishes. As was noticed in ref. [22], the first term in eq. (3.3) vanishes at the middle of the interval $z = 1/2$ for all modes. Hence, for the standard embedding, at this point the space-time background receives no correction and, in particular, the Calabi–Yau space is undeformed. We see that the second term in eq. (3.3) does not share this property. Therefore, for non-standard embeddings, there is generically no point on the orbifold where the space-time remains uncorrected.

Furthermore, we see that the massive modes depend on the combination $\delta^{-1}z$ only. Therefore, in terms of the normalized orbifold coordinate $z$ (the orbifold coordinate $x^{11}$), the massive modes indeed fall off exponentially with a scale set by $\delta$ (by $v^{1/6}$). In fact, as might be expected, we see that this part of the solution is essentially independent of the size of the orbifold. Averaging the above expression for the massive modes over the orbifold, one should pick up the corresponding weak coupling correction. Clearly, as a consequence of the exponential fall-off, the averaging procedure leads to an additional suppression by $\epsilon R$. Given that the order of a heavy mode is $\epsilon R \epsilon S$, we conclude that its average is of the order $\epsilon^2 \epsilon R \epsilon S$. According to eq. (2.6), this is just $\epsilon W$ and, hence, the expected weak coupling expansion parameter.

### 3.2 Low-energy effective actions

What are the implications of the above results for the low-energy effective action? Since the orbifold is expected to be larger than the Calabi–Yau radius, it is natural to first reduce to a five-dimensional effective theory consisting of the usual 3 + 1 space-time dimensions and the orbifold and, subsequently, reduce this theory further down to four dimensions. First, we should explain how a background appropriate for a reduction to $\mathcal{N} = 1$ supersymmetry in four dimensions can be used to derive a sensible $\mathcal{N} = 1$ theory in five dimensions [22]. The point is that, as we have seen, the background can be split into massless and massive eigenmodes. Reducing from eleven to five dimensions on an undeformed Calabi–Yau background, these correspond to massless moduli fields and heavy Kaluza–Klein modes. Working to linear order in $\epsilon S$, the heavy modes completely decouple from the massless modes and so can essentially be dropped. The background then appears as a particular solution to the five-dimensional effective action, where the moduli depend non-trivially on the orbifold direction. Thus, in summary, to derive the correct five-dimensional action, we need only keep the massless modes in a reduction on an undeformed Calabi–Yau space. However, a similar procedure is not possible for the topologically non-trivial components $G^{(1)}_{ABCD}$ of the

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\[3\] By $\mathcal{N} = 1$ in five dimensions we mean a theory with eight supercharges. In four dimensions, $\mathcal{N} = 1$ means a theory with four supercharges.
antisymmetric tensor field strength. Such a configuration of the internal field strength is not a modulus, but rather a non-zero mode. As a consequence, the proper five-dimensional theory is obtained as a reduction on an undeformed Calabi–Yau background but including non-zero modes for $G$. It is these non-zero modes which introduce all the interesting structure into the theory, notably, that in the bulk we have a gauged supergravity and that the theory admits no homogeneous vacuum. In the case in hand, the precise structure of the non-zero mode can be directly read off from the background as presented.

Let us now briefly review the results of such a reduction for the standard embedding as presented in ref. [33, 39]. It was found that the five-dimensional effective action consists of a gauged $\mathcal{N} = 1$ bulk supergravity theory with $h^{1,1} - 1$ vector multiplets and $h^{2,1} + 1$ hypermultiplets coupled to four-dimensional $\mathcal{N} = 1$ boundary theories. The field content of the orbifold plane at $x^{11} = 0$ consists of an $E_6$ gauge multiplet and $h^{1,1}$ and $h^{2,1}$ chiral multiplets, while the plane at $x^{11} = \pi \rho$ carries $E_8$ gauge multiplets only. The gauging of the bulk supergravity is with respect to a $U(1)$ isometry in the universal hypermultiplet coset space with the gauge field being a certain linear combination of the graviphoton and the vector fields in the vector multiplets. The gauging also leads to a bulk potential for the $(1,1)$ moduli. In addition, there are potentials for the $(1,1)$ moduli confined to the orbifold planes which have opposite strength. As we have mentioned, the characteristic features of this theory, such as the gauging and the existence of the potentials, can be traced back to the existence of the non-zero mode. Furthermore, the vacuum solution of this five-dimensional theory, appropriate for a reduction to four dimensions, was found to be a double BPS domain wall with the two worldvolumes stretched across the orbifold planes.

Which of the above features generalize to non-standard embeddings? The spectrum of zero mode fields in the bulk will, of course, be unchanged. Due to the nonstandard embedding, we can have more general gauge multiplets with groups $G^{(1)}, G^{(2)} \subset E_8$ on the orbifold planes and also corresponding observable and hidden sector matter transforming under these groups. We are interested in the effective action up to linear order in $\epsilon_S$. It is clear that, as above, to this order, the massive part of the background completely decouples from the low-energy effective action since the massless and massive eigenfunction on the Calabi–Yau space are orthogonal [22]. Hence, the form of the effective action to linear order in $\epsilon_S$ is completely determined by the massless part of the background. On the other hand, due to the cohomology condition (2.37), the form of the massless part of the background corrections is same as in the standard embedding case, as we have just shown. Hence, in deriving the five-dimensional effective action for non-standard embedding, we use the same non-zero mode in the reduction as for the standard embedding. This will lead to gauging and bulk and boundary potentials exactly as in the standard embedding case.

Let us explain these last facts in some more detail. First, we identify the non-zero mode of $G$ in the case of non-standard embedding. Inserting the mode (3.2) into the expansion for $B_{AB}$,
eq. (2.32), we can use eq. (2.24) to compute the four-form field strength $G^{(1)}$. While the massless part of $G^{(1)}_{ABC11}$ vanishes, we find for the massless part of $G^{(1)}_{ABCD}$

$$G^{(1)} = \frac{1}{2V} \ast \omega_{i0} \alpha_{i0}$$

(3.4)

where $V$ is the Calabi–Yau volume modulus defined by

$$V = \frac{1}{v} \int_X \sqrt{\delta g}$$

(3.5)

and we have introduced the parameter

$$\alpha_{i0} = \frac{\sqrt{2} \epsilon_S}{\rho} \beta_{i0}.$$  

(3.6)

to conform with the notation of [33, 39]. Eq. (3.4) is precisely the non-zero mode we have mentioned above. Note that $V$ measures the orbifold average of the Calabi–Yau volume in units of $v$. In general, the parameters $\alpha_{i0}$ depend on the choice of both the tangent and the gauge bundles. Explicitly, from eqs. (2.10), (2.31) and the cohomology condition (2.37), we have, for general embeddings,

$$\alpha_{i0} = \frac{-\epsilon_S}{4\sqrt{2} \pi^2 \rho} \int_{\mathcal{C}_{4i0}} \left( \text{tr} F^{(1)} \wedge F^{(1)} - \frac{1}{2} \text{tr} R \wedge R \right)$$

(3.7)

In the case of the standard embedding, the tangent bundle and one of the $E_8$ gauge bundles are identified, while the other gauge bundle is taken to be trivial, so that this reduces to

$$\alpha_{i0} = \frac{\epsilon_S}{8\sqrt{2} \pi^2 \rho} \int_{\mathcal{C}_{4i0}} \text{tr} R \wedge R.$$  

(3.8)

This is the relation given in ref. [39]. The point is that the expression for the non-zero mode (3.4) has the same form for both standard and non-standard embeddings. All that changes are the values of the parameters $\alpha_{i0}$.

Now let us demonstrate how the gauging of the bulk supergravity arises in the case of non-standard embedding. Consider the five-dimensional three-form modulus $C_5$, with field strength $G_5$, and the part of the 11–dimensional three-form that leads to the $h^{1,1}$ vector fields $A^{i0}$, namely $C = A^{i0} \wedge \omega_{i0}$. Inserting these two fields, together with the non-zero mode (3.4), into the Chern–Simons term in the eleven-dimensional supergravity action [39] leads to

$$\int_{M_{11}} C \wedge G \wedge G \sim \int_{M_5} \alpha_{i0} A^{i0} \wedge G_5.$$  

(3.9)

The three-form $C_5$ can be dualized to a scalar in five dimensions, which becomes one of the four scalars $q^u$ in the universal hypermultiplet. Then, the above term directly causes the gauging of the isometry in the hypermultiplet coset space that corresponds to the axionic shift in the dual
scalar. The gauging is with respect to the linear combination $\alpha_{i_{0}}A^{i_{0}}$. Explicitly, we find [39] that the universal hypermultiplet kinetic term is of the form

$$\int_{M_{5}} \sqrt{-g} h_{uv} D_{\alpha}q^{u}D^{\alpha}q^{v}$$  \hspace{1cm} (3.10)$$

with the covariant derivative

$$D_{\alpha}q^{u} = \partial_{\alpha}q^{u} + \alpha_{i_{0}}A^{i_{0}}_{\alpha}k^{u}$$  \hspace{1cm} (3.11)$$

where $k^{u}$ is a Killing vector in the hypermultiplet sigma-model manifold, pointing in the direction of the axionic shift. We see that, since the non-zero mode (3.4) had the same form for both standard and non-standard embeddings, the gauging of the supergravity also has the same form. The only difference is in the values of the charges $\alpha_{i_{0}}$.

Similarly, the bulk potential should have the same form in the standard and non-standard embedding cases. Inserting the non-zero mode (3.4) into the kinetic term $G \wedge *G$ of the four-form field strength in the eleven-dimensional supergravity action leads to a bulk potential for the volume modulus $V$ and the other $(1,1)$ moduli. More precisely, one finds

$$\int_{M_{11}} G^{(1)} \wedge *G^{(1)} \sim \int_{M_{5}} \sqrt{-g} V^{-2}\alpha_{i_{0}}\alpha_{j_{0}} \tilde{G}^{i_{0}j_{0}}$$  \hspace{1cm} (3.12)$$

where

$$\tilde{G}^{i_{0}j_{0}} = \frac{V^{2/3}}{3} G^{i_{0}j_{0}}$$  \hspace{1cm} (3.13)$$

is a renormalized metric that depends on the Calabi–Yau shape moduli (see ref. [39] for details). Note that it follows from supersymmetry that such a potential must arise when an isometry of the universal hypermultiplet sigma-model manifold is gauged.

The potentials on the orbifold planes arise from the ten-dimensional actions on the planes, with the internal gauge fields and curvature inserted. Using identities of the form

$$\int_{X} \omega \wedge \mathrm{tr} R \wedge R \sim \int_{X} \sqrt{-g} \mathrm{tr} R^{2}$$  \hspace{1cm} (3.14)$$

we find

$$\sum_{n=1}^{2} \int_{M_{10}^{(n)}} \sqrt{-g} \left( \mathrm{tr}(F^{(2)})^{2} - \frac{1}{2} \mathrm{tr} R^{2} \right) \sim \int_{M_{4}^{(1)}} \sqrt{-g} V^{-1}\alpha_{i_{0}}b^{i_{0}} - \int_{M_{4}^{(2)}} \sqrt{-g} V^{-1}\alpha_{i_{0}}b^{i_{0}}$$  \hspace{1cm} (3.15)$$

where $b^{i_{0}}$ are the Kähler shape moduli defined by the expansion of the Kähler form $\omega = V^{1/3}b^{i_{0}}\omega_{i_{0}}$. As for the standard embedding case, the potentials come out with opposite strength, again a consequence of the cohomology condition (3.1), $\beta_{i_{0}}^{(0)} = -\beta_{i_{0}}^{(1)}$.

In summary, we conclude that the five-dimensional effective action derived in ref. [33, 39] for the standard embedding is, in fact, much more general and applies, with appropriate adjustment of
the boundary field content and the charges $\alpha_{i_0}$, to any Calabi–Yau-based non-standard embedding without additional five-branes. Furthermore, the double domain wall vacuum solution of the five-dimensional theory is unchanged, since it does not depend on the field content on the orbifold planes.

The four-dimensional theory is obtained as a reduction on this domain wall. Hence, the four-dimensional effective action will be unchanged in the case of non-standard embeddings without five-branes, except for the possibility of more general gauge groups and matter multiplets. One further new feature, in the case of non-standard embedding, is the possibility of gauge matter on the hidden orbifold plane. In this case, the threshold-like correction to the matter part of the Kähler potential will be different for observable and hidden sectors in the same way the gauge kinetic functions of the two sectors differ.

To be more concrete, let us consider the universal case with moduli $S$ and $T$, gauge fields of $G^{(1)} \times G^{(2)} \subset E_8 \times E_8$ and corresponding gauge matter $C^{(1)}$ and $C^{(2)}$, transforming under $G^{(1)}$ and $G^{(2)}$, respectively. Then, we have for the Kähler potential and the gauge kinetic functions

$$
\begin{align*}
K &= -\log(S + \bar{S}) - 3 \log(T + \bar{T}) + Z_1 |C^{(1)}|^2 + Z_2 |C^{(2)}|^2 \\
Z_1 &= \frac{3}{T + T} + \frac{\epsilon_S}{8\pi} \frac{\beta}{S + \bar{S}} \\
Z_2 &= \frac{3}{T + \bar{T}} - \frac{\epsilon_S}{8\pi} \frac{\beta}{S + \bar{S}} \\
f^{(1)} &= S + \frac{\epsilon_S}{8\pi} \beta T \\
f^{(2)} &= S - \frac{\epsilon_S}{8\pi} \beta T.
\end{align*}
$$

(3.16)

where $\beta$ is the single instanton charge, of the type defined in eqn. (3.1), corresponding to the universal Kähler deformation. For vacua based on the standard embedding, it was pointed out in ref. [3] that, if $\beta > 0$ so that the smaller of the two couplings corresponds to the observable sector, then, fitting this to the grand unification coupling, the larger coupling is of order one at the “physical” point. Hence, gaugino condensation in the hidden sector appears to be a likely scenario. We have just shown that, in fact, this statement continues to apply to all Calabi–Yau based non-standard embedding vacua without additional bulk five-branes, provided $\beta > 0$, since the gauge kinetic functions are completely unchanged. Gaugino condensation, therefore, appears to be a generic possibility for such vacua.

4 Backgrounds with five-branes

Let us now turn to the much more interesting case of non-standard embeddings with five-branes in the bulk. We will concentrate on the massless modes, since, as above, it is these modes which will determine the low-energy action.
4.1 Properties of the background

The general solution (2.38) for the massless modes shows a linear behaviour for each interval between two five-branes. The slope, however, varies from interval to interval in a way controlled by the five-brane charges. The same statement applies to the variation of geometrical quantities, like the Calabi–Yau volume, across the orbifold. Let us consider an example for a certain massless mode $b$. Four five-branes with charges $(β^{(1)}, β^{(2)}, β^{(3)}, β^{(4)}) = (1, 1, 1, 1)$ are positioned at $(z_1, z_2, z_3, z_4) = (0.2, 0.6, 0.8, 0.8)$. Note that the third and fourth five-brane are coincident. The instanton numbers on the orbifold planes are chosen to be $(β^{(0)}, β^{(4)}) = (−1, −3)$. Note that the total charge sums up to zero as required by the cohomology constraint (2.37). The orbifold dependence of $(\sqrt{2}/π\epsilon S)b$ is depicted in fig. 3. It is clear that the additional five-brane charges introduce much more freedom as compared to the case without five-branes. For example, while in the latter case one always has $b(0) = −b(1)$ leading to equal, but opposite, gauge threshold corrections, the example in fig. 3 shows that $b(0), b(1) > 0$ is possible. One, therefore, expects the thresholds in the low-energy gauge kinetic functions to change. This will be analyzed in a moment. Another interesting phenomenon in the above example is that the mode is constant between the first and second five-brane. This is a direct consequence of our choice of the charges which sum up to zero both to the left and the right of this interval. If such a property is arranged for all massless modes, the Calabi–Yau volume remains exactly constant throughout this interval.

Figure 3: Orbifold dependence of a massless mode $(\sqrt{2}/π\epsilon S)b$ for four five-branes at $(z_1, z_2, z_3, z_4) = (0.2, 0.6, 0.8, 0.8)$ with charges $(β^{(1)}, β^{(2)}, β^{(3)}, β^{(4)}) = (1, 1, 1, 1)$ and instanton numbers $(β^{(0)}, β^{(4)}) = (−1, −3)$. 
4.2 Five-branes on Calabi–Yau two-cycles

The inclusion of five-branes not only generalizes the types of background one can consider, but also introduces new degrees of freedom into the theory, namely, the dynamical fields on the five-branes themselves. In this section, we will consider what low-energy fields survive on one of the five-branes when it is wrapped around a two-cycle in the Calabi–Yau three-fold.

In general, the fields on a single five-brane are as follows [54, 55]. The simplest are the bosonic coordinates $X^I$ describing the embedding of the brane into 11-dimensional spacetime. The additional bosonic field is a world-volume two-form potential $B$ with field strength $H = dB$ satisfying a generalized self-duality condition. For small fluctuations, the duality condition simplifies to the conventional constraint $H = \ast H$. These degrees of freedom are paired with spacetime fermions $\theta$, leading to a Green–Schwarz type action, with manifest spacetime supersymmetry and local kappa-symmetry [56, 57]. (As usual, including the self-dual field in the action is difficult, but is possible by either including an auxiliary field or abandoning a covariant formulation.) For a five-brane in flat space, one can choose a gauge such that the dynamical fields fall into a six-dimensional massless tensor multiplet with $(0, 2)$ supersymmetry on the brane world-volume [58, 59]. The multiplet has five scalars describing the motion in directions transverse to the five-brane, together with the self-dual tensor $H$.

For a five-brane embedded in $S^1/\mathbb{Z}_2 \times X \times M_4$, to preserve Lorentz invariance in $M_4$, $3 + 1$ dimensions of the five-brane must be left uncompactified. The remaining two spatial dimensions are then wrapped on a two-cycle of the Calabi–Yau three-fold. To preserve supersymmetry, the two-cycle must be a holomorphic curve [3, 49, 50]. Thus, from the point of view of a five-dimensional effective theory on $S^1/\mathbb{Z}_2 \times X \times M_4$, since two of the five-brane directions are compactified, it appears as a flat three-brane (or equivalently domain wall) located at some point $x^{11} = x$ on the orbifold. Thus, at low energy, the degrees of freedom on the brane must fall into four-dimensional supersymmetric multiplets.

An important question is how much supersymmetry is preserved in the low-energy theory. One way to address this problem is directly from the symmetries of the Green–Schwarz action, following the discussion for similar brane configurations in [49]. Locally, the 11-dimensional spacetime $S^1/\mathbb{Z}_2 \times X \times M_4$ admits eight independent Killing spinors $\eta$, so should be described by a theory with eight supercharges. (Globally, only half of the spinors survive the non-local orbifold quotienting condition $\Gamma_{11}\eta(-x^{11}) = \eta(x^{11})$, so that, for instance, the eleven-dimensional bulk fields lead to $\mathcal{N} = 1$, not $\mathcal{N} = 2$, supergravity in four dimensions.) The Green–Schwarz form of the five-brane action is then invariant under supertranslations generated by $\eta$, as well as local kappa-transformations. In general the fermion fields $\theta$ transform as (see for instance ref. [59])

$$\delta \theta = \eta + P_+ \kappa$$

(4.1)
where $P_+$ is a projection operator. If the brane configuration is purely bosonic then $\theta = 0$ and the variation of the bosonic fields is identically zero. Furthermore, if $H = 0$ then the projection operator takes the simple form

$$P = \frac{1}{2} \left( 1 \pm \frac{1}{6! \sqrt{g}} \epsilon^{m_1 \ldots m_6} \partial_{m_1} X^{I_1} \ldots \partial_{m_6} X^{I_6} \Gamma_{I_1 \ldots I_6} \right)$$

(4.2)

where $\sigma^m$, $m = 0, \ldots, 5$ label the coordinates on the five-brane and $g$ is the determinant of the induced metric

$$g_{mn} = \partial_m X^I \partial_n X^J g_{IJ}.$$  

(4.3)

If the brane configuration is invariant for some combination of supertranslation $\eta$ and kappa-transformation, then we say it is supersymmetric. Now $\kappa$ is a local parameter which can be chosen at will. Since the projection operators satisfy $P_+ + P_- = 1$, we see that for a solution of $\delta \theta = 0$, one is required to set $\kappa = -\eta$, together with imposing the condition

$$P_- \eta = 0$$

(4.4)

For a brane wrapped on a two-cycle in the Calabi–Yau space, spanning $M_4$ and located at $x^{11} = x$ in the orbifold interval, we can choose the parameterization

$$X^\mu = \sigma^\mu \quad X^A = X^A(\sigma, \bar{\sigma}) \quad X^{11} = x$$

(4.5)

where $\sigma = \sigma^4 + i\sigma^5$. The condition (4.4) then reads

$$-(i/\sqrt{g}) \partial X^A \bar{\partial} X^B \Gamma^{(4)} \Gamma_{AB} \eta = \eta$$

(4.6)

where we have introduced the four-dimensional chirality operator $\Gamma^{(4)} = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3$. Recalling that on the Calabi–Yau three-fold the Killing spinor satisfies $\Gamma^{\bar{b}} \eta = 0$, it is easy to show that this condition can only be satisfied if the embedding is holomorphic, that is $X^a = X^a(\sigma)$, independent of $\bar{\sigma}$. The condition then further reduces to

$$\Gamma^{(4)} \eta = i\eta$$

(4.7)

which, given that the spinor has definite chirality in eleven dimensions as well as on the Calabi–Yau space, implies that $\Gamma^{11} \eta = \eta$, compatible with the global orbifold quotient condition. Thus, finally, we see that only half of the eight Killing spinors, namely those satisfying (4.7), lead to preserved supersymmetries on the five-brane. Consequently the low-energy four-dimensional theory describing the five-brane dynamics will have $N = 1$ supersymmetry.

The simplest excitations on the five-brane surviving in the low-energy four-dimensional effective theory are the moduli describing the position of the five-brane in eleven dimensions. There is a
single modulus $X^{11}$ giving the position of the brane in the orbifold interval. In addition, there is the moduli space of holomorphic curves $C_2$ in $X$ describing the position of the brane in the Calabi–Yau space. This moduli space is generally complicated, and we will not address its detailed structure here. (As an example, the moduli space of genus one curves in K3 is K3 itself [50].) However, we note that these moduli are scalars in four dimensions, and we expect them to arrange themselves as a set of chiral multiplets, with a complex structure presumably inherited from that of the Calabi–Yau manifold.

Now let us consider the reduction of the self-dual three-form degrees of freedom. (Here we are essentially repeating a discussion given in [62, 63].) The holomorphic curve is a Riemann surface and, so, is characterized by its genus $g$. One recalls that the number of independent harmonic one-forms on a Riemann surface is given by $2g$. In addition, there is the harmonic volume two-form $\Omega$. Thus, if we decompose the five-brane world-volume as $C_2 \times M_4$, we can expand $H$ in zero modes as

$$H = da \wedge \Omega + F^u \wedge \lambda_u + h$$

where $\lambda_u$ are a basis $u = 1, \ldots, 2g$ of harmonic one-forms on $C_2$, while the four-dimensional fields are a scalar $a$, $2g\ U(1)$ vector fields $F^u = dA^u$ and a three-form field strength $h = db$. However, not all these fields are independent because of the self-duality condition $H = *H$. Rather, one easily concludes that

$$h = *da$$

and, hence, that the four-dimensional scalar $a$ and two-form $b$ describe the same degree of freedom. To analyze the vector fields, we introduce the matrix $T^{uv}$ defined by

$$*\lambda_u = T^{uv}\lambda_v$$

If we choose the basis $\lambda_u$ such that the moduli space metric $\int_{C_2} \lambda_u \wedge (*\lambda_v)$ is the unit matrix, $T$ is antisymmetric and, of course, $T^2 = -1$. The self-duality constraint implies for the vector fields that

$$F^u = T^{vu} * F^v.$$ 

If we choose a basis for $F^u$ such that

$$T = \text{diag} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

with $g$ two by two blocks on the diagonal, one easily concludes that only $g$ of the $2g$ vector fields are independent. In conclusion, for a genus $g$ curve $C_2$, we have found one scalar and $g\ U(1)$ vector
fields from the two-form on the five-brane worldvolume. The scalar has to pair with another scalar to form a chiral $\mathcal{N} = 1$ multiplet. The only other universal scalar available is the zero mode of the transverse coordinate $X^{11}$ in the orbifold direction.

Thus, in general, the $\mathcal{N} = 1$ low-energy theory of a single five-brane wrapped on a genus $g$ holomorphic curve $C_2$ has gauge group $U(1)^g$ with $g U(1)$ vector multiplets and a universal chiral multiplet with bosonic fields $(a, X^{11})$. Furthermore, there is some number of additional chiral multiplets describing the moduli space of the curve $C_2$ in the Calabi–Yau three-fold.

It is well known that when two regions of five-brane world-volume in M–theory come into close proximity, new massless states appear [53, 60]. These are associated with membranes stretching between the two nearly overlapping five-brane surfaces. In general, this can lead to enhancement of the gauge symmetry. Let us now consider this possibility, heretofore ignored in our discussion. In general, one can consider two types of brane degeneracy where parts of the five-brane world-volumes are in close proximity. The first, and simplest, is to have $N$ distinct but coincident five-branes, all wrapping the same cycle $C_2$ in the Calabi–Yau space and all located at the same point in the orbifold interval. Here, the new massless states come from membranes stretching between the distinct five-brane world-volumes. The second, and more complicated, situation is where there is a degeneracy of the embedding of a single five-brane, such that parts of the curve $C_2$ become close together in the Calabi–Yau space. In this case, the new states come from membranes stretching between different parts of the same five-brane world-volume. Let us consider these two possibilities separately.

The first case of distinct five-branes is analogous to the M–theory description of $N$ overlapping type IIB D3-branes, which arise as $N$ coincident five-branes wrapping the same cycle in a flat torus. In that case, the $U(1)$ gauge theory on each D3-brane is enhanced to a $U(N)$ theory describing the full collection of branes. Thus, by analogy, in our case we would expect a similar enhancement of each of the $g U(1)$ fields on each five-brane. That is, when wrapped on a holomorphic curve of genus $g$, the full gauge group for the low-energy theory describing $N$ coincident five-branes becomes $U(N)^g$.

The second case is in closer analogy to a system considered by Witten in [62]. There, for example, a system of two type IIA NS five-branes intersecting with $g + 1$ D4-branes in flat space was lifted to an M–theory description in terms of a single five-brane with world volume $\Sigma \times M_4$, where $\Sigma$ is a non-compact Riemann surface of a particular type. This surface could be completed into a compact surface $\bar{\Sigma}$ of genus $g$. In general, $\Sigma$ was such that the type IIA theory was completely “Higgsed”, so that the D4-branes were separated and the gauge symmetry was simply $U(1)^g$. (One might expect $U(1)^{g+1}$ for $g + 1$ D4-branes, but one of the degrees of freedom is “frozen out” [62].) Degenerations, where parts of the Riemann surface came close together in the embedding space, corresponded to overlapping D4-branes and so led to enhanced gauge symmetry. The exact enhancement depended
on the type of the degeneracy; that is, on how many D4-branes were overlapping. However, the enhanced gauge group is always of the form of a product of $U(n)$ and $SU(n)$ groups, such that the total rank of the gauge group, is $g$. The largest allowed group is $SU(g + 1)$. The other allowed groups correspond to various “Higgsings” of $SU(g + 1)$ by fields in the adjoint representation. This situation is very close to our case, except that instead of a single five-brane wrapping a non-compact Riemann surface $\Sigma$ embedded in flat space, we have a single five-brane wrapping a compact Riemann surface $C_2$ embedded in a Calabi–Yau space. We argued above that, generically, we expect a low-energy theory with $U(1)^g$ symmetry. This corresponds to the “fully-Higgsed” case in Witten’s theory. By direct analogy, we would expect similar enhancement to products of $U(n)$ and $SU(n)$ groups for degeneracies of the embedding such that parts of the curve $C_2$ come close together in the Calabi–Yau three-fold. Again, there would be a constraint on the total rank being equal to $g$. We might expect that the largest possible gauge group is similarly $SU(g + 1)$. However, understanding the details of this enhancement requires explicit knowledge of the properties of the moduli space of holomorphic curves. We will return to this subject elsewhere [48].

Summarizing the two case, we see that for $N$ five-branes wrapping the same curve $C_2$ of genus $g$, we expect that the symmetry is enhanced from $N$ copies of $U(1)^g$ to $U(N)^g$. Alternatively in the second case, even for a single brane, we can get enhancement if the embedding degenerates. In general, $U(1)^g$ enhances to a product of unitary groups such that the total rank is equal to $g$. The maximal enhancement is presumably to $SU(g + 1)$, and the other allowed groups correspond to different “Higgsings” of $SU(g + 1)$ by fields in the adjoint representation. For example, if $g = 2$, then $SU(3)$ could be broken to either $SU(2) \times U(1)$ or $U(1) \times U(1)$. In all cases, the total rank of the symmetry group is conserved. Finally, we note that in the case where the Calabi–Yau space itself degenerates to become a singular orbifold, and the five-branes are wrapped at the singularity, we could expect more exotic enhancement, in particular, to gauge groups other than unitary groups. In this paper, however, we will restrict ourselves to the case of smooth Calabi–Yau spaces.

### 4.3 Low energy effective actions

Next, we would like to discuss the five-dimensional effective actions that result from the reduction of Hořava–Witten theory on a background that includes five-branes. It has already been explained in section 3.2 how the vacua without five-branes found in this paper can be used to construct a sensible five-dimensional theory. Essentially the same arguments apply here. We begin with the five-dimensional bulk theory. Clearly, the zero-mode content is unchanged with respect to the case without five-branes. Thus we have $\mathcal{N} = 1$ supergravity coupled to $h^{1,1} - 1$ vector multiplets and $h^{2,1} + 1$ hypermultiplets. What about the gauging of the hypermultiplet coset space? Inserting the
massless modes (2.38) into eq. (2.32) and calculating $G^{(1)}$ via eq. (2.24) one finds

$$G^{(1)} = \frac{1}{2V} \langle \omega_{i0} \rangle \sum_{m=0}^{n} \alpha^{(m),i0}$$

in the interval

$$z_n \leq |z| \leq z_{n+1}$$

for fixed $n$, where $n = 0, \ldots, N$, and as in eqn. (3.6) we have introduced the parameters

$$\alpha^{(m)}_{i0} = \sqrt{2} \frac{\epsilon S}{\rho} \beta^{(m)}_{i0}$$

(4.15)

to conform with the notation of [33, 39]. Hence, we still have a non-zero mode that must be taken into account in the dimensional reduction. Its form, however, depends on the interval one is considering. Consequently, the five-dimensional action again contains a term of the form (3.9), but with $\alpha_{i0}$ being replaced by $\sum_{m=0}^{n} \alpha^{(m)}_{i0}$ for the interval $z_n \leq |z| \leq z_{n+1}$. In other words, we have gauging in the bulk between each two five-branes, but the gauge charge differs from interval to interval. Since the bulk potential (3.12) is directly related to the gauging, it is subject to a similar replacement of charges. In summary, we conclude that the bulk theory between any pair of neighboring five-branes in the interval $z_n \leq |z| \leq z_{n+1}$ is as given in ref. [33, 39], but with $\alpha_{i0}$ replaced by $\sum_{m=0}^{n} \alpha^{(m)}_{i0}$.

We now turn to the orbifold planes. They are described by four-dimensional $\mathcal{N} = 1$ theories at $x^{11} = 0, \pi \rho$ coupled to the bulk. The zero mode spectrum on these planes is, of course, unchanged with respect to the situation without five-branes. It consists of gauge multiplets corresponding to the unbroken gauge groups $G^{(1)}$ and $G^{(2)}$, as dictated by the choice of the internal gauge bundle, and corresponding gauge matter multiplets. The height of the boundary potentials (see eqn. (3.15)) is now set by the charges $\alpha^{(0)}_{i0}$ and $\alpha^{(N+1)}_{i0}$ which, due the presence of additional five-brane charges, are no longer necessarily equal and opposite.

Finally, we should consider the worldvolume theories of the three-branes that originate from wrapping the five-branes around supersymmetric cycles. Applying the results of the previous subsection to each of the $N$ five-branes, we have $N$ additional four-dimensional $\mathcal{N} = 1$ theories at $x^{11} = x_1, \ldots, x_N$ which couple to the five-dimensional bulk. The field content of such a theory at $x^{11} = x_n$ for $n = 1, \ldots, N$ is generically given by $U(1)^{g_n}$ gauge multiplets, where $g_n$ is the genus of the holomorphic curve on which the $n$-th five-brane is wrapped, a universal chiral multiplet and a number of additional chiral multiplets describing the moduli space of the holomorphic curve within the Calabi–Yau manifold. By the mechanisms described at the end of the previous subsection, the $U(1)^{g_n}$ gauge groups can be enhanced to non-Abelian groups. As the simplest example, two five-branes located at $x^{11} = x_n$ and $x^{11} = x_{n+1}$ could be wrapped on the same Calabi–Yau cycle.
with genus $g_n$. As long as two five-branes are separated in the orbifold, that is, $x_{n+1} \neq x_n$, we have two gauge groups $U(1)^{g_n}$, one group on each brane. However, when the two five-branes coincide, that is, for $x_{n+1} = x_n$, these groups are enhanced to $U(2)^{g_n}$. The precise form of the three-brane world-volume theories should be obtained by a reduction of the five-brane world-volume theory on the holomorphic curves, in a target space background of the undeformed Calabi–Yau space together with the non-zero mode for the four-form field strength. We leave this to a future publication [48], but note here that we expect those three-brane theories to have a potential depending on the moduli living on the three-brane and the projection of the bulk moduli to the three-brane world-volume. This expectation is in analogy with the theories on the orbifold planes which, as we have seen, possess such a potential. It has been shown in ref. [33, 39] that those boundary potentials provide the source terms for a BPS double-domain wall solution of the five-dimensional theory in the absence of additional five-branes. This double domain wall is the appropriate background for a further reduction to four dimensions. Again, in analogy, we expect the vacuum of the five-dimensional theory in the presence of five-branes to be a BPS multi-domain wall. More precisely, for $N$ five-branes, we expect $N + 2$ domain walls with two world-volumes stretching across the orbifold planes and the remaining $N$ stretching across the three-brane planes. The role of the potentials on the three-brane world-volume theories is to provide the $N$ additional source terms needed to support such a solution.

Let us finally discuss some consequences for the four-dimensional effective theory. Clearly, there is a sector of the theory which has just the conventional field content of four-dimensional $N = 1$ low-energy supergravities derived from string theory. More precisely, this is $h^{1,1} + h^{2,1}$ chiral matter multiplets containing the moduli, gauge multiplets with gauge group $G^{(1)} \times G^{(2)} \subset E_8 \times E_8$ and corresponding gauge matter. In the presence of five-branes, however, we have additional sectors of the four-dimensional theory leading to additional chiral multiplets containing the five-brane moduli and, even more important, to gauge multiplets with generic gauge group

$$G = \prod_{n=1}^{N} U(1)^{g_n}. \quad \text{(4.16)}$$

At specific points in the five-brane moduli space, one expects enhancement to a non-Abelian group $G = G_1 \times \cdots \times G_M$. As explained above, in typical cases, the factors $G_m$ can be $U(n)$ and $SU(n)$ groups. We expect the enhancement to preserve the rank, that is, we have

$$\text{rank}(G) = \sum_{n=1}^{N} g_n. \quad \text{(4.17)}$$

We recall that $g_n$ is the genus of the curve on which the $n$-th five-brane is wrapped. As it stands, it appears that the rank could be made arbitrarily large. However, for a given Calabi–Yau space, we expect a constraint on the rank which originates from positivity constraints in the the zero-cohomology condition (2.17). This will be further explored in [48]. As is, the five-brane sectors and
the conventional sector of the theory only interact via the bulk supergravity fields. Therefore, at this point, they are most naturally interpreted as hidden sectors.

We should, however, point out that the presence of five-branes provides considerably more flexibility in the choice of \(G^{(1)} \times G^{(2)}\), the “conventional” gauge group that originates from the heterotic \(E_8 \times E_8\). This happens because it is much simpler to satisfy the zero cohomology condition (2.17) in the presence of five-branes. Let us give an an example which is illuminating, although not necessary physically relevant. Consider a Calabi–Yau space \(X\) with topologically nontrivial \(\text{tr} R \wedge R\). In addition, we set both \(E_8\) gauge field backgrounds to zero, which implies that the unbroken gauge group is simply \(E_8 \times E_8\). Without five-branes, such a background is inconsistent since it is in conflict with the zero-cohomology condition (2.17). However, if for each independent four-cycle \(C_{4i_0}\), we can introduce \(N_{i_0}\) five-branes, each having unit intersection number with the cycle \(C_{4i_0}\), such that

\[
N_{i_0} = -\frac{1}{8\pi^2} \int_{C_{4i_0}} \text{tr} R \wedge R
\]

then the zero-cohomology condition is satisfied. Of course, the gauge group will then be enlarged to \(E_8 \times E_8 \times G\) where the gauge group \(G\) originates from the five-branes, as discussed above.

What about the form of the four-dimensional effective action? We have seen that non-standard embedding without five-branes does not change the form of the effective action with respect to the standard embedding case. This could be understood as a direct consequence of the fact that the five-dimensional effective theory remains unchanged. Above we have seen, however, that the five-dimensional effective theory does change in the presence of five-branes. In particular, its vacuum BPS solution is now a multi-domain wall, as opposed to a double-domain wall in the case without five-branes. Hence, we expect the four-dimensional theory obtained as a reduction on this multi-domain wall to change as well. Let us, as an example of this, calculate the gauge kinetic functions in four dimensions to linear order in \(\epsilon_S\). Here, we will not do this using the five-dimensional theory but, equivalently, reduce directly from eleven to four dimensions. We define the modulus \(R\) for the orbifold radius by

\[
R = \frac{1}{2V\pi \rho} \int_{S^1/\mathbb{Z}_2 \times X} \sqrt{\gamma g}.
\]

Note that with this definition, \(R\) measures the averaged orbifold size in units of \(2\pi \rho\). Let us also introduce the \((1,1)\) moduli \(a^{i_0}\) in the usual way as

\[
\omega_{AB} = a^{i_0} \omega_{i_0 AB}.
\]

Then, the real parts of the low energy fields \(S\) and \(T^i\) are given by

\[
\text{Re}(S) = V, \quad \text{Re}(T^{i_0}) = VR^{-1}a^{i_0}.
\]
We stress that with these definitions, $S$ and $T^{i_0}$ have the standard Kähler potential, that is, the order $\epsilon_S$ corrections to the Kähler potential vanish [22]. The gauge kinetic functions can be directly read off from the 10–dimensional Yang–Mills actions (3.15). Using the metric from eq. (2.24) with (2.38), (2.32) inserted and the above definition of the moduli, we find

$$f^{(1)} = S + \frac{\epsilon_S}{8\pi} T^{i_0} \sum_{n=0}^{N+1} (1 - z_n)^2 \beta^{(n)}_{i_0}$$

$$f^{(2)} = S + \frac{\epsilon_S}{8\pi} T^{i_0} \sum_{n=1}^{N+1} z_n^2 \beta^{(n)}_{i_0}$$

where, in addition, we have the cohomology constraint (2.37). Recall from eq. (3.16) that in the case without five-branes, the threshold correction on the two orbifold planes are identical but opposite in sign. Note that here the expressions for these two thresholds are, in fact, different. If, for example, there is only one five-brane with charges $\beta^{(1)}_{i_0}$ at $z = z_1$ on the orbifold, we have

$$f^{(1)} - f^{(2)} = \frac{\epsilon_S}{4\pi} T^{i_0} \left[ \beta^{(0)}_{i_0} + (1 - z_1) \beta^{(1)}_{i_0} \right].$$

We see that the gauge thresholds on the orbifold planes depend on both the position and the charges of the additional five-branes in the bulk. This gives considerably more freedom than in the case without five-branes. In particular, for special choices of the charges and the five-brane position, the difference of the gauge kinetic functions can be small. Thus, for instance, the hidden gauge coupling at the physical point need not be as large as it was in the case without five-branes.

**Note added** When this manuscript was in preparation we received ref. [64] which also discusses non-standard embeddings in heterotic M–theory, however, without considering vacua with five-branes. It also included an interesting discussion of the appearance of anomalous $U(1)$ gauge fields.

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**References**


