VISCOS STABILITY OF RELATIVISTIC KEPLERIAN ACCRETION DISKS

Pranab Ghosh\textsuperscript{1,2}

Laboratory for High Energy astrophysics
NAS Goddard Space Flight Center, Greenbelt, MD 20771

\textsuperscript{1} Senior NAS/NRC Resident Research Associate

\textsuperscript{2} On sabbatical leave from Tata Institute of Fundamental Research, Bombay 400 005, India

Received July 20, 1998; accepted August 19, 1998

To appear in the Astrophysical Journal Letters
Received ____________; accepted ________________
We investigate the viscous stability of thin, Keplerian accretion disks in regions where general relativistic (GR) effects are essential. For gas pressure dominated (GPD) disks, we show that the Newtonian conclusion that such disks are viscously stable is reversed by GR modifications in the behaviors of viscous stress and surface density over a significantly large annular region not far from the innermost stable orbit at $r = r_{\text{ms}}$. For slowly-rotating central objects, this region spans a range of radii $14 \lesssim r \lesssim 19$ in units of the central object’s mass $M$. For radiation pressure dominated (RPD) disks, the Newtonian conclusion that they are viscously unstable remains valid after including the above GR modifications, except in a very small annulus around $r \approx 14M$, which has a negligible influence. Inclusion of the stabilizing effect of the mass-inflow through the disk’s inner edge via a GR analogue of Roche-lobe overflow adds a small, stable region around $r_{\text{ms}}$ for RPD disks, but leaves GPD disks unchanged.

We mention possible astrophysical relevance of these results, particularly to the high-frequency X-ray variabilities observed by the Rossi X–ray Timing Explorer.

Subject headings: accretion, accretion disks – relativity – gravitation – black hole physics – X-rays:stars
1. Introduction

Extensive studies of the stability of geometrically thin, optically thick, Keplerian accretion disks in the Newtonian regime (Lightman & Eardley 1974, henceforth LE; Lightman 1974a,b, henceforth L74a,b; Shakura & Sunyaev 1976; Piran 1978, henceforth P78) have established that such disks are unstable to both viscous and thermal modes when they are radiation pressure dominated (RPD), but not when they are gas pressure dominated (GPD). In this Letter, we show that GPD disks are viscously unstable in an essentially relativistic region not far from the innermost stable orbit at \( r = r_{ms} \), due to general relativistic (GR) modifications in the behaviors of the disk’s viscous stress and surface density. For a slowly-rotating central accreting object, this region spans a range of radii \( 14 \lesssim r \lesssim 19 \) in units of the central object’s mass \( M \) (we use the convention \( G = c = 1 \) throughout this work). We demonstrate that, after including these GR modifications, RPD disks remain viscously unstable in this region, except in an extremely small annulus around \( r \approx 14M \), which has a negligible effect. Inclusion of the stabilizing effect of the mass-inflow through the disk’s inner edge via a GR analogue of Roche-lobe overflow (Abramowicz 1981, 1985, henceforth A81, A85) adds a further, small, stable annulus around \( r_{ms} \) for RPD disks, but leaves GPD disks unchanged. The inner parts of GPD disks (around black holes or sufficiently compact, weakly-magnetized neutron stars) extending to radii \( \gtrsim r_{ms} \) are thus expected to have blobs of matter: we indicate the relevance of this result for high-frequency X-ray variability observed with the Rossi X – ray Timing Explorer (RXTE).

2. The Viscous Instability

Viscous stability in thin, Keplerian disks in GR is described by the evolution of the disk’s surface density \( \Sigma(r, t) \), which is obtained by combining the conditions of conservation
of mass and angular momentum. In Kerr metric, these are

\[
\mathcal{D}^{\frac{1}{2}} \frac{\partial}{\partial r} \left( \mathcal{D}^{-\frac{1}{2}} \dot{M} \right) = 2\pi r \mathcal{A}^{\frac{1}{2}} \mathcal{D}^{-\frac{1}{2}} \frac{\partial \Sigma}{\partial t},
\]

\[
\dot{M} \frac{\partial \ell}{\partial r} = \frac{\partial}{\partial r} \left( 2\pi r^2 \mathcal{B} \mathcal{C} \mathcal{D}^{-1} \dot{W} \right).
\]

Here, \(\dot{M}(r,t)\) is the local, instantaneous mass flux, \(W\) is the vertically-integrated viscous stress, \(\ell\) is the specific angular momentum, and \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\) are relativistic factors (Novikov & Thorne 1973, henceforth NT73) whose values reduce to unity in the Newtonian regime:

\[
\mathcal{A} \equiv 1 + \frac{a^2}{r^2} + \frac{2a^2}{r^3}, \quad \mathcal{B} \equiv 1 + \frac{a}{r^{3/2}}, \quad \mathcal{C} \equiv 1 - \frac{3}{r} + \frac{2a}{r^{3/2}}, \quad \mathcal{D} \equiv 1 - \frac{2}{r} + \frac{a^2}{r^2}.
\]

Here \(a\) is the specific angular momentum of the central accreting object, and we express lengths and times in units of \(M\) in all relativistic factors only. Equations (1) and (2) yield

\[
\mathcal{A}^{\frac{1}{2}} \mathcal{D}^{-\frac{1}{2}} \frac{\partial \Sigma}{\partial t} = \mathcal{D}^{\frac{1}{2}} \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{\mathcal{D}^{-\frac{1}{2}}}{(\partial \ell/\partial r)} \frac{\partial}{\partial r} \left( \mathcal{B} \mathcal{C}^{-1} r^2 \dot{W} \right) \right],
\]

showing how the viscous stress \(W\) determines the evolution of \(\Sigma\).

The Newtonian limits of equations (1)-(4) were used by LE and L74b to argue that viscous instability sets in when viscous stress is a decreasing function of surface density, \(\partial W/\partial \Sigma < 0\), while the disk is stable when \(\partial W/\partial \Sigma > 0\) (also see Pringle 1981). This is transparent upon expressing the radial derivative of \(W\) in equation (4) in terms of \(\partial W/\partial \Sigma\) (L74b), and comparing with the standard form for diffusion equations: the effective diffusion coefficient is negative in the former case, indicating the formation of density clumps, while it is positive in the latter case, indicating normal viscous diffusion. The physical reason is equally transparent: in the former case, a local upward fluctuation in \(\Sigma\) causes a decrease in the viscous stress, making the viscous smoothing of the density clump more difficult, and the situation runs away (albeit on the slow, viscous timescale). The latter case is stable, as any tendency towards clumping is nullified by the stronger viscous smoothing it generates. Since for “standard” Newtonian Keplerian accretion disks (Shakura
& Sunyaev 1973, henceforth SS73) $W \propto \Sigma^{-1}$ in RPD regions (LE) and $W \propto \Sigma^{5/2}$ in GPD regions (see §3.2), LE and L74b derived the well-known Newtonian result that the former are viscously unstable, while the latter are stable.

The crucial point about relativistic disks is that the basic criterion for viscous stability in terms of $\partial W/\partial \Sigma$ remains the same as above, as an inspection of equation (4) readily shows. This is as expected, since the physical picture given above remains unaltered in the GR regime. Therefore, the stability of a disk model in the GR regime is determined by the relation between $W$ and $\Sigma$ in that regime. In §3, we summarize this relation for thin, Keplerian disks in the GR regime, and its effects on viscous stability. In so doing, we use the GR formulation of geometrically thin, optically thick, Keplerian accretion disks of NT73 throughout. The formulation of SS73 was basically Newtonian, with an appropriate GR boundary condition applied ad hoc to the viscous stress at $r_{ms}$. While the consistent GR formulation of NT73 is more suitable for our purposes, we indicate later that the effects described here can be seen qualitatively even in the SS73 formulation, and in the pseudo-Newtonian generalization (Abramowicz et al. 1988, henceforth ACLS) thereof.

3. Stability in the Relativistic Regime

The vertically-integrated viscous stress $W$ is obtained in the GR formulation (NT73) by combining the conservation laws for angular momentum and energy, subject to the boundary condition that $W$ vanishes at $r = r_{ms}$, where the gas “falls out” of the disk and spirals down to the central object:

$$W = \frac{M}{2\pi} \left(\frac{M}{r^3}\right)^{1/2} \frac{C_i^2 Q}{B^D}. \quad (5)$$

Here, $Q$ is a relativistic factor (NT73) which describes the effects of the boundary conditions on $W$: at $r = r_{ms}$, $Q$ vanishes, while in the Newtonian regime, $Q$ reduces to unity. $Q$ has
been calculated for the Kerr metric in Page & Thorne (1974). Explicit computations given in this Letter will be confined to the Schwarzschild metric\(^1\), in which limit \(W\) is given by\(^2\)

\[
W = \frac{\dot{M}}{2\pi} \left( \frac{M}{r^3} \right)^{\frac{1}{2}} (1 - \frac{2}{r})^{-1} \left[ 1 - \sqrt{\frac{6}{r}} + \frac{1}{2} \sqrt{\frac{3}{r}} \ln \left( \frac{1 + \sqrt{\frac{3}{r}}}{1 - \sqrt{\frac{3}{r}} \sqrt{\frac{3}{r} - 1}} \right) \right],
\]

and shown in Figure 1. From the Newtonian limit \(W \propto r^{-3/2}\) at large distances, \(W\) passes through a maximum at \(r \approx 14M\), vanishing at \(r = r_{ms} = 6M\).

### 3.1. Vertical Structure

The disk’s surface density \(\Sigma\) is determined by its vertical structure, the physics of which is threefold: vertical hydrostatic equilibrium, vertical radiative transport of energy, and equation of state (SS73; NT73). In GR, the first is given by

\[
\frac{\partial p}{\partial z} = -\frac{\rho \dot{M} z}{r^3} \mathcal{K}^{-1},
\]

where \(p\) and \(\rho\) are the disk pressure and density, respectively, and the relativistic factor \(\mathcal{K}\) is given by

\[
\mathcal{K} = 1 - \frac{4a}{r^{3/2}} + \frac{3a^2}{r^2}.
\]

Note that this is the improved version of the original NT73 formulation as given by Riffert & Herold (1995); for more detailed GR formulation and discussion, see Abramowicz, Lanza & Percival (1997) and references therein. The second is given by

\[
F = \frac{3}{4} \left( \frac{M}{r^3} \right)^{\frac{1}{2}} W \mathcal{D}^{-1} \mathcal{C}^{-1} = \frac{2bT^4}{3\kappa_{es}\Sigma},
\]

\(^1\)Extension of our results to non-zero values of the angular momentum parameter \(a\) presents no difficulties or surprises, and will be described elsewhere.

\(^2\)In both the SS73 formulation and its ACLS generalization, the logarithmic term within the square brackets in eq.[6] is missing; in addition, the relativistic factor \((1 - \frac{2}{r})^{-1}\) is missing as well in the SS73 formulation. However, \(W\) still has the same qualitative shape.
where $F$ is flux of radiant energy from each face of the disk, $T$ is the disk temperature, $b$ is the radiation constant, and $\kappa_{es}$ is the electron-scattering opacity (appropriate for the inner disk). The equation of state is given by

$$p = \begin{cases} \frac{\Sigma kT}{h m_p} & \text{GPD}; \\ \frac{1}{3} b T^4 & \text{RPD}. \end{cases}$$

(10)

Here, $h$ is the semi-thickness of the disk, $m_p$ is the proton mass, and $k$ is Boltzmann’s constant.

### 3.2. $W - \Sigma$ Relation and Stability

Together with the relation $W = 2\alpha h p$ for the viscous stress, $\alpha$ being the standard disk viscosity parameter (SS73; NT73), equations (7)-(10) yield the following results for the surface densities of GPD and RPD disks:

$$\Sigma_{GPD} = \left( \frac{m_p}{k\alpha} \right)^{4/5} \left( \frac{b}{18\kappa_{es}} \right)^{1/5} M^{-1/10}(W\sqrt{r})^{3/5}C^{1/5}D^{-1/5}.$$  

(11)

$$\Sigma_{RPD} = \left( \frac{16}{9\alpha \kappa^2_{es}} \right) W^{-1} CKD^{-2}.$$  

(12)

Radial variations of $\Sigma_{GPD}$ and $\Sigma_{RPD}$ are shown in Figure 1 in the Schwarzschild limit. $W - \Sigma$ relations for GPD and RPD disks are shown explicitly in Figures 2 and 3, respectively.

---

3The apparent rise in Fig.1 of $\Sigma_{RPD}$ to large values at large $r$, and also as $r \to r_{ms}$, are unphysical, of course. In reality, the former rise is terminated where the RPD region joins smoothly to a GPD region at sufficiently large radii, and the latter is terminated close to $r_{ms}$, where there is a transition from Keplerian flow to free-fall via a GR analogue of Roche-lobe overflow (see text).
For GPD disks, the Newtonian property $\partial W/\partial \Sigma > 0$ is valid for all radii larger than that at which $\Sigma_{GPD}$ attains its maximum, i.e., $r \approx 19M$. However, between the maxima of $\Sigma_{GPD}$ and $W$, i.e., in the region $14M \lesssim r \lesssim 19M$, $\partial W/\partial \Sigma < 0$, since $W$ decreases with increasing $r$ there, but $\Sigma_{GPD}$ increases. At $r \lesssim 14M$, $\partial W/\partial \Sigma > 0$ again, as both $W$ and $\Sigma_{GPD}$ increase with $r$. Thus, between the lower, Newtonian branch of the $W - \Sigma$ relation shown in Figure 2 (on which $W \propto r^{-3/2}$, and $\Sigma_{GPD} \propto r^{-3/5}$ as eq.[11] shows, so that $W \propto \Sigma^{5/2}$), and the upper, GR branch, both of which are viscously stable, there is a considerable transition region ($14M \lesssim r \lesssim 19M$ in the Schwarzschild limit) which is unstable to the viscous mode. Hence, blobs of matter are expected to form in this annular region of GPD disks. The ultimate fate of these blobs merit further study: it is clear that, as they drift radially into the stable region at $6M \lesssim r \lesssim 14M$, viscous stresses will tend to smoothen them out, but a significant fraction of blobs may survive to the disk’s inner edge, since the drift timescale through this stable region exceeds the fastest stabilization timescale only by a factor $\lesssim 3$.

For RPD disks, the lower, Newtonian branch of the $W - \Sigma$ relation (on which $W \propto \Sigma^{-1}$ and $\partial W/\partial \Sigma < 0$) is almost identical to the upper, GR branch, with no transition region visible in the main panel of Figure 3. A closer inspection of the curve’s tip (inset) does reveal an extremely small transition region ($13.89 \lesssim r \lesssim 14.26$) over which $\partial W/\partial \Sigma > 0$, corresponding to the region between the maximum of $W$ and the minimum of $\Sigma_{RPD}$ in Figure 1. Although this tiny region is viscously stable in principle, it has no significant effect on the stability properties of RPD disks, which would be expected to form blobs over their entire extent from the arguments given so far (but see below).

Finally, the stabilizing effect (A81, A85) of the mass-inflow through the inner edge of the disk to the central object via a GR analogue of Roche-lobe overflow (Abramowicz, Calvani & Nobili 1980, henceforth ACN, and references therein) is likely to have minor
consequences for most of the results found here, particularly for GPD disks. Found originally
(A81) by applying P78’s Newtonian stability analysis to the above GR mass-inflow through
inner edges of thick, RPD, non-Keplerian disks with super-Eddington accretion rates
(ACN), this effect is now understood to stabilize the *transonic* part (A85) of the accretion
flow thermally (and possibly also viscously), as well as the supersonic part of the flow
interior to $r_{ms}$. However, the radial extent $\delta r_{r_{ms}}$ of the transonic regions at the inner edges of
the thin, Keplerian disks under consideration here is very small, typically $\lesssim 10^{-2}$ (Miller,
Lamb & Psaltis 1998, henceforth MLP, and references therein), comparable to the small
vertical thicknesses (A85) of the transonic flows. For RPD disks, this will lead to the
formation of an additional, small, stable annulus around $r_{ms}$, but there is virtually no
change for GPD disks, since this region is stable anyway. Similar arguments apply to those
Keplerian disks around weakly-magnetized neutron stars which are terminated at radii
$\gtrsim r_{ms}$ by non-gravitational forces, *e.g.*, radiation drag (MLP).

4. Discussion

Underlying the phenomenon described here is a basic difference between the scalings
of $\Sigma_{GPD}$ and $\Sigma_{RPD}$ with $W$ in $\alpha$-disks which does not seem to have been fully appreciated
so far. Apart from the (minor) effects of the factors $C$, $D$ and $K$, the $W - \Sigma$ relation for
RPD disks, $\Sigma_{RPD} \propto W^{-1}$, is independent of $r$, but that for GPD disks, $\Sigma_{GPD} \propto (W \sqrt{r})^{3/5}$,
is not. Hence, when $W$ develops a maximum at $r \sim 14M$ as a result of the GR boundary
condition, and runs through the same set of values (see Figure 1) for $r > 14M$ (Newtonian
branch) and $R < 14M$ (GR branch), corresponding values of $\Sigma_{RPD}$ are essentially the same
on the two branches, but those of $\Sigma_{GPD}$ are quite different (see Figures 2 and 3). The
maximum of $\Sigma_{GPD}$ occurs at that of $W \sqrt{r}$, which is quite different from that of $W$. In the
region between the two maxima, the viscous stress is clearly always a decreasing function
of $\Sigma_{GPD}$, and the disk viscously unstable. This behavior is robust (but see below) in the sense that it depends not on the details of how GR is introduced, but only on the basic scaling $\Sigma_{GPD} \sim (W\sqrt{r})^{3/5}$, which is valid even in the Newtonian limit: this is why the SS73 formulation and its ACLS generalization give results which are qualitatively the same as that given here.

All $\alpha$-disks show the above scalings for RPD and GPD regions because $\Sigma_{RPD}$ scales as $W^{-1}$ in such disks, but $\Sigma_{GPD}$ as $(W\Omega^{-1/3})^{3/5}$, $\Omega$ being the Keplerian angular velocity: the reason for this is clear from the vertical structure relations. These relations involve $\Omega$ twice: in that of vertical equilibrium (eq.[7]), and in that of viscous dissipation rate (eq.[9]). There is a cancellation between the two $\Omega$s for RPD disks, but not for GPD disks, because pressure is independent of $\Sigma$ for the former, but not for the latter (LE; L74b). Of course, this reasoning applies only to $\alpha$-disks, as did that of LE. Investigations of the corresponding properties of GR non-Keplerian disks, particularly advection-dominated disks (Gammie & Popham 1998 and references therein) appear worthwhile.

Astrophysical applications of our results will be given in detail elsewhere. While the focus of the stability analyses of inner disks in the 1970s was on bright, galactic black-hole candidates, where accretion rates are normally high enough to ensure RPD inner regions, it is now clear that accretion rates in many X-ray binaries containing black holes and weakly-magnetized neutron stars can be low enough ($\dot{M} \lesssim 10^{-2}$ of the Eddington rate) that inner disks are GPD all the way to their inner edges at $\sim r_{ms}$. Low-mass X-ray binaries (LMXB), and, in particular, “atoll” type LMXBs with very low neutron-star magnetic fields and accretion rates of the above order, are prime examples of the latter case. Signatures of the phenomena described here may be observable in timing studies of LMXBs by RXTE, particularly in the kilohertz quasiperiodic oscillations (henceforth kHzQPO; see, e.g., van der Klis 1998, henceforth K98). We note that the Keplerian frequencies
corresponding to the unstable region in GPD disks around slowly-rotating central objects are \( \sim 210 - 350(M/2M_\odot)^{-1} \) Hz, similar to the frequency difference between the twin kHzQPOs observed in LMXBs. Furthermore, kHzQPO diagnostics of any residual blobs present in the region between the disk’s inner edge and that of the viscously unstable region (see §3.2) does not appear impossible (Angelini 1998, private communication).

It is a pleasure to thank L. Angelini, J. K. Cannizzo, J. H. Swank, and N. E. White for stimulating discussions, and the referee, M. C. Miller, for helpful comments.
REFERENCES


Abramowicz, M. A. 1985, PASJ, 37, 727 (A85)


FIGURE CAPTIONS

Fig. 1.— Radial variation of the integrated viscous stress $W$ (solid line) in a thin, Keplerian disk in the Schwarzschild limit, as well as that of the surface density $\Sigma_{GPD}$ of a GPD disk (dashed line), and that of the surface density $\Sigma_{RPD}$ of a RPD disk (dotted line), also in the same limit. Radii shown in units of the central object’s mass $M$. Since only the shapes of the curves matter for this work, vertical units have been adjusted for convenient simultaneous display of the profiles. Also shown at the top of the panel is the range of radii over which GPD disks are viscously unstable, and, at the bottom of the panel, the (extremely small) region over which RPD disks are viscously stable, because of GR effects (see text).

Fig. 2.— Integrated viscous stress $W$ vs. surface density $\Sigma_{GPD}$ of a Keplerian GPD disk in the Schwarzschild limit. The lower branch of the curve is the Newtonian one, and the upper branch the ultrarelativistic one, both of which are viscously stable (see text). Between these, there is a transition region (marked by X’s) which is viscously unstable (see text), corresponding to the range of radii similarly marked at the top of Fig. 1. Note that the radial coordinate decreases monotonically along the curve in going from the lower to the upper branch.

Fig. 3.— Similar to Fig. 2, but for RPD disks. The main panel shows that the lower, Newtonian branch is almost identical to the upper, relativistic branch, both being viscously unstable (see text), and that there is almost no transition region. The inset, a magnification of the tip of the main curve, reveals an extremely small transition region which is viscously stable, corresponding to the marks at the bottom of Fig. 1, but this has little effect on the overall stability of RPD disks.