A Reexamination of Proton Decay in Supersymmetric Grand Unified Theories

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Abstract

We reconsider dimension–five proton decay operators, making semi-quantitative remarks which apply to a large class of supersymmetric GUTs in which the short-distance operators are correlated with the fermion Yukawa couplings. In these GUTs, which include minimal \(SU(5)\), the operators \((u^c d^c_i)(d_j \nu_{\tau})\), induced by charged Higgsino dressing, completely dominate for moderate to large \(\tan \beta\). The rate for \(p \to (K^+, \pi^+)\nu_{\tau}\) grows rapidly, as \(\tan^4 \beta\), and the \(K^+\) to \(\pi^+\) branching ratio can often be precisely predicted. At small \(\tan \beta\) the operators \((ud_i)(d_j \nu)\) are dominant, while the operators \((ud_i)(u\ell^-)\), with left-handed charged leptons, are comparable to the neutrino operators in the generic GUT and suppressed in minimal GUTs. Charged-lepton branching fractions are always small at large \(\tan \beta\). The electron to muon ratio is small in minimal GUTs but can be larger, even of order one, in other models. All other operators are very small. At small \(\tan \beta\) in non-minimal GUTs, gluino and neutralino dressing effects on neutrino rates are not negligible.

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Unification of the three gauge forces of the standard model into a single symmetry group is an attractive idea with a long history [?]. Grand Unified Theories (GUTs) provide a beautiful explanation of the multiplet structure of the standard model fermions, and predict the value of one of the gauge couplings at the weak scale in terms of the other two. The success of this prediction in the context of the minimal supersymmetric extension of the standard model (MSSM) has given new impetus to the study of SUSY GUTs over the past decade; for a review see [?].

GUTs place quarks, leptons and their antiparticles in the same multiplet, thereby making the nucleon unstable. The momentum scale of grand unification, as inferred from the extrapolation of the standard model gauge couplings, is $M_{\text{GUT}} \sim 2 \times 10^{16}$ GeV, so large that nucleon decay rates are compatible with present experimental limits. In supersymmetric GUTs there are two sources of baryon number violation. Dimension–six (four–fermion) operators are generated by the exchange of superheavy gauge bosons of the GUT symmetry group. With $M_{\text{GUT}} \sim 2 \times 10^{16}$ GeV, the expected proton lifetime from these operators is $\tau(p \rightarrow \pi^0 e^+) \sim 10^{34 \pm 2}$ yr. [?]; the present experimental lower limit is $\tau(p \rightarrow \pi^0 e^+) \geq 2 \times 10^{33}$ yr. [?]. The second source of proton decay — the subject of this paper — is the dimension–five (two–fermion/two–sfermion) operators induced by the exchange of color–triplet Higgsinos, the GUT partners of the MSSM Higgs(ino) doublets [?,?]. The associated amplitudes scale as $M_{\text{GUT}}^{-1}$, but are suppressed by light fermion Yukawa couplings. They are also suppressed by a loop factor, which arises from the “dressing” of the operator by a gaugino or Higgsino that converts the two sfermions into light fermions. The size of the loop integral is highly uncertain, due to the unknown masses of the supersymmetric particles. Additional uncertainty stems from the unknown values of $\tan \beta \equiv \langle H_u \rangle / \langle H_d \rangle$ and the triplet Higgsino mass. (In branching ratios, some of the unknown parameters cancel, so they can be more reliably predicted than the overall rate.) Since the strength of these operators is proportional to Yukawa couplings, the dominant modes are those with strange quarks — the heaviest fermion into which the proton can decay kinematically. In minimal SUSY $SU(5)$, including all the uncertainties, the lifetime may be estimated as $\tau(p \rightarrow K^+ \nu) = (10^{28} - 10^{34})$ yr. This range is nearly eliminated by the experimental limit $\tau(p \rightarrow K^+ \nu) \geq 5 \times 10^{32}$ yr. [?], suggesting that proton decay must be discovered imminently if the idea of grand unification is on the right track.

These issues have been studied extensively in minimal $SU(5)$ [?,?—— see [?,?] for more references to the early literature — as well as in other GUTs [?,?,?,?]. Most calculations have been specific to particular models. In this letter we attempt to make statements which, while more qualitative than quantitative, apply to a wide class of theories. Our purpose in this paper is to summarize and clarify the expectations concerning branching ratios in a large ensemble of supersymmetric GUTs, in which the dimension-five operators are roughly correlated with the fermion Yukawa couplings. We will make this more definite below. Our approach is somewhat similar in spirit to that of [?].

General statements of this type require a thorough analysis. To this end a systematic method has been developed in which rough upper bounds on baryon-number-violating operators may be established; this will be presented elsewhere [?]. These bounds apply under the following conditions. The dimension-five operators take the form

$$\mathcal{L}_5 = \int d^2 \theta \left\{ \lambda_{ijkr} Q_i Q_j Q_k L_r + \tilde{\lambda}_{ijkr} U_i^c D_r^c U_j^c E_r^c \right\} + h.c.$$  \hspace{1cm} (1)

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(Capital letters denote superfields; \(U_i\) contains the squark \(\tilde{u}_i\) and the quark \(u_i\), etc.) We take as our ansatz the relations \(\lambda_{ijkr} \sim \lambda_{ijkr} \sim \frac{1}{\pi} \epsilon_1 \epsilon_2 \epsilon_k\), independent of the index \(r\), where \(M\) is an overall mass scale independent of \(i, j, k, r\) and \(\epsilon_3 \sim 1, \epsilon_2 \equiv \epsilon \sim 1/25\) and \(\epsilon_1 \sim \epsilon_2^2\).

(The motivation [?,[?] for the ansatz is associated with the fact that, near the GUT scale, most of the flavor structure of the theory can be roughly encoded in powers of \(\epsilon\) [?].) In (1) the \(\lambda_{ijkr}\) are expressed in the gauge basis, but it will be more convenient to work in the “supersymmetric basis”, in which the matter fermions are expressed as mass eigenstates and the sfermions are expressed as the supersymmetric partners of those eigenstates. To convert from the gauge \((g)\) to the supersymmetric \((m)\) basis, we must rotate the superfields as \(U_i^{(g)} = R_i^g U_i^{(m)}, \ D_j^{(g)} = R_j^d D_j^{(m)}, \) etc. so that the fermion mass matrices are diagonal. In the supersymmetric basis,

\[
W_{\text{dim}=5} = (\hat{\lambda}_{ijkl} U_i^a D_j^b U_k^c E_r - \hat{\lambda}_{ijkl} U_i^a D_j^b U_k^c N_r + \hat{\lambda}_{ijkl} U_k^c D_j^b U_i^a N_r) \epsilon_{\alpha \beta \gamma}.
\]

where \(\alpha, \beta, \gamma\) are color indices, we have dropped the superscript \((m)\), and we have defined

\[
\hat{\lambda}_{ijkl} = \sum_{i', j', k', l'=1}^3 \lambda_{i'j'k'l'} R_{ii'}^a R_{jj'}^b R_{k'k}^c R_{ll'}^d;
\]

the coefficients \(\hat{\lambda}_{ijkl}\) and \(\hat{\lambda}_{ijkl}\) are defined analogously. As part of our ansatz, we assume the matrices \(R_i^a, R_i^d, R_i^e, R_i^c\) — those associated with the \(10\) representations of \(SU(5)\) — have the same texture as \(V_{CKM}\); we do not assume this for the other \(R\) matrices. If \(V_{CKM}\) satisfied \(V_{ij} \sim \min\{\epsilon_i/\epsilon_j, \epsilon_j/\epsilon_i\}\), then we would have \(\hat{\lambda}_{ijkl} \sim \lambda_{ijkl} \sim \lambda_{ijkl} \sim \epsilon_i \epsilon_j \epsilon_k\).

However, the Cabibbo angle \(\theta_c\) is of order \(5 \epsilon_1/\epsilon_2\) and can enter into the \(\lambda\). An enhancement by a factor of \(\theta_c/(\epsilon_2) \sim 5\) may occur for each index \(i, j, k\) taking value 1 [?]; thus

\[
\hat{\lambda}_{ijkl} \sim \lambda_{ijkl} \sim \hat{\lambda}_{ijkl} \sim \epsilon_i \epsilon_j \epsilon_k \left[\theta_c/(\epsilon_2)\right]^3 (\delta_{11} + \delta_{j1} + \delta_{k1}).
\]

Note \(\delta_{11} + \delta_{j1} + \delta_{k1} \leq 2\) because of the antisymmetry properties of the \(\hat{\lambda}\) coefficients. More details of this ansatz are given in [? and [?].

If the coefficients \(\hat{\lambda}\) in a given GUT are all of order or less than those appearing in Eq. (4), then we can apply the results of [?] to the model. Although we will not classify them here, many GUTs (including minimal \(SU(5)\) and \(SO(10)\) and their more realistic variants, as well as the ten-centered models of [?,?,[?] are compatible with the ansatz (4). We will now demonstrate that the \(\hat{\lambda}\) coefficients in the minimal \(SU(5)\) GUTs are consistent with (4). In the supersymmetric basis, the superpotential induced by the color–triplet Higgsinos in minimal \(SU(5)\) is

\[
W_C = (U_i D_j e^{i \sigma_i} f_i V_{ij} + U_i^a E_j^a f_j V_{ij}) H_C + (D_i N_i h_i - U_i E_j V_{ij}^* h_j + U_i^a D_j^e e^{-i \sigma_i} V_j^* h_j) \tilde{H}_C.
\]

Here \(H_C\) and \(\tilde{H}_C\) are the color triplet and anti-triplet from the \(5_H\) and \(\bar{5}_H\) of the Higgs multiplet, \(V\) is the CKM matrix, \(f_i\) \((h_i)\) is the diagonal Yukawa coupling matrix of the up–quarks (down-quarks and leptons), and the \(\sigma_i\) are phase factors with \(\sum_{i=1}^3 \sigma_i = 0\). Exchange of the Higgs color triplets, of mass \(M\), leads to the dimension-five operators

\[
W_{d=5} = M^{-1}(U_i D_j D_k N_i e^{i \sigma_i} f_i V_{ij} \delta_{kr} h_r - U_i D_j D_k E_r e^{i \sigma_i} f_i V_{ij} V_{kr}^* h_r + U_i^a D_j^e D_k^e e^{-i \sigma_i} f_j V_{jk} V_{kr}^* h_r)
\]
along with other terms irrelevant for proton decay.

If it were the case that $\theta_c \sim \epsilon_1/\epsilon_2$, so that $V_{ij} \sim \min\{\epsilon_i/\epsilon_j, \epsilon_j/\epsilon_i\}$, along with $f_i \sim \epsilon_i^2$, $h_1 \sim \epsilon_1 \zeta$, where $\zeta = \tan \beta/60$, we would have

$$f_i V_{ij} \delta_{kr} h_r \sim \epsilon_i^2 \epsilon_r \delta_{kr} \min\{\epsilon_i/\epsilon_j, \epsilon_j/\epsilon_i\} \zeta \lesssim \epsilon_i \epsilon_j \epsilon_k \zeta$$

$$f_i V_{ij} V_{kr} h_r \sim \epsilon_i^2 \epsilon_r \min\{\epsilon_i/\epsilon_j, \epsilon_j/\epsilon_i\} \min\{\epsilon_k/\epsilon_r, \epsilon_r/\epsilon_k\} \zeta \lesssim \epsilon_i \epsilon_j \epsilon_k \zeta$$

$$f_j V_{jk} V_{kr} h_r \sim \epsilon_j^2 \epsilon_r \min\{\epsilon_j/\epsilon_k, \epsilon_k/\epsilon_j\} \min\{\epsilon_i/\epsilon_r, \epsilon_r/\epsilon_i\} \zeta \lesssim \epsilon_i \epsilon_j \epsilon_k \zeta$$

All coefficients in (7) would then be equal to or smaller than those in Eq. (4); note the overall factor of $\zeta$ can be absorbed into the overall mass scale which affects all $\lambda$ equally. Now let us account for the fact that $\theta_c \sim 5\epsilon_1/\epsilon_2$. Inspection of Eq. (7) shows that factors of the Cabibbo angle can only enhance a coupling above $\epsilon_i \epsilon_j \epsilon_k$ if one of the indices $i,j,k$ takes value 1; $r = 1$ cannot cause such an enhancement because of the factor $h_1$, which reduces $\lambda^{(e, e)}_{ijk1}, \lambda^{(e, e)}_{ijk1}$ below its ansatz value. If two such indices take value 1, then one may get a double enhancement. The resulting coefficients are the same as or less than those appearing in Eq. (4), and thus the upper bounds of [?] are applicable.

Rough upper bounds from [?] on various four-fermion baryon-number-violating operators are given in the Table. All operators not shown are negligibly small. The Table shows the different contributions from various gauginos and Higgsinos. Wino-Higgsino mixing is not listed; in each case the upper bound on such contributions lies between or below the pure Wino and the pure Higgsino bounds. Explicitly indicated are factors of $\epsilon$, $\theta_c$, gauge coupling constants $g_t$ and third-generation Yukawa couplings $y_b, y_{\tau}$ which stem from left-right mixing or Higgsino couplings; note $y_b$ and $y_{\tau}$ are roughly of order $\zeta$. Also appearing are parameters $\gamma$ and $\delta$. The first measures the extent to which the $\hat{b}, \hat{\ell}$ are split in mass from and mixed with the other left-handed squarks; if the messenger scale of supersymmetry breaking is high, then $\gamma \sim 1$, while if it is near the weak scale $\gamma$ may be small. The factor $\delta = (A \cos \beta + \mu \sin \beta)(A \sin \beta + \mu \cos \beta)v/\tilde{m}^2$ parameterizes the size of left-right squark mixing; here $v, \mu, \tilde{m}, A$ are 246 GeV, the Higgsino mass, the universal squark mass and the coefficient of the trilinear scalar terms. An overall factor of $M^{-1} \zeta$ is omitted from every entry. Neutrino flavors have not been distinguished, and the flavor of a charged lepton is only indicated when the bounds depend on the lepton flavor. For additional details see [?].

<table>
<thead>
<tr>
<th>udsν, uddν</th>
<th>W±</th>
<th>H±</th>
<th>$\tilde{g}$</th>
<th>$W^0, B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>uusℓ, uudd</td>
<td>$g_2^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$\delta y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$g_2^2 \gamma y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$g_2^2 \gamma y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
</tr>
<tr>
<td>${u \delta } , \tilde{\nu}$</td>
<td>$g_2^2 y_c \epsilon^2 \theta_c$</td>
<td>$y_c \epsilon^2 \theta_c$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${u \delta } , \tilde{\nu}$</td>
<td>$g_2^2 y_c \epsilon^2 \theta_c$</td>
<td>$y_c \epsilon^2 \theta_c$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${u \mu } , \tilde{\nu}$</td>
<td>$g_2^2 \delta y_b(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$y_b(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$g_2^2 \gamma \delta y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$g_2^2 \gamma \delta y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
</tr>
<tr>
<td>${u \mu } , \tilde{\nu}$</td>
<td>$g_2^2 \delta y_b(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$y_b(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$g_2^2 \gamma \delta y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
<td>$g_2^2 \gamma \delta y_b^2(\epsilon^2 \theta_c^2, \epsilon^2 \theta_c^2)$</td>
</tr>
</tbody>
</table>

Rule-of-thumb upper bounds on coefficients of four-fermion baryon-number-violating operators, as found in [?] using the ansatz in Eq. (4). Important loop factors, matrix elements and overall coefficients are omitted and must be accounted for when using this Table. See the text for further explanation.

At small $\tan \beta$ the operators $uds\nu$, $uus\mu$ and $uuse$ have the largest bounds, while at
large $\tan \beta$ the operator $(u^c d^c)\dagger (s v)$ potentially exceeds all others. The replacement of the $s$ quark by a $d$ quark engenders a small suppression [? , ?]. As we will see, it is always true that many of these operators saturate or nearly saturate their bounds. It follows from the Table that operators with right-handed leptons, even $u^c \mu^c s u$, are negligible at all values of $\tan \beta$, as are operators with a left-handed charged lepton and two right-handed quarks. We will therefore restrict our attention to the first four rows in the Table.

At small $\tan \beta$ the largest four-fermion operator is $u d s v$, induced through dressing of the $C D S N_\mu$, $T D B N_\mu$ term in the superpotential by the charged Wino, as in Fig. 1 [? , ?]. In minimal GUTs the second-generation amplitude is proportional to $y_t y_b \theta_c^2$ ($y_c$ is the charm quark Yukawa coupling, etc.), while the third-generation effect is roughly of the same order. Since $y_s y_t \theta_c^2 \sim (e_2 \zeta)(e_2^2) \theta_c^2 \sim \epsilon \theta_c^2 \zeta$, the coefficient of $u d s v$ saturates its upper bound in the Table. Similar statements apply to $u d d \nu$, with an additional factor of $\theta_c$. Interference effects between second- and third-generation diagrams [which unfortunately depend on the otherwise-unmeasurable phases $\sigma_i$ in Eq. (6)] might suppress either $p \to K^+ \nu$ or $p \to \pi^+ \nu$ [? ] but not both [? ]. Accounting for this and for the different hadronic matrix elements and phase space factors, one finds the branching ratio $\Gamma(p \to K^+ \nu)/\Gamma(p \to \pi^+ \nu)$ tends to lie between 0.1 and 1. (Throughout this letter we assume that sfermions of the same charge and phase space factors, one finds the branching ratio $\Gamma(K^+ \nu)$ does not saturate its upper bound in the Table.)

The rates for these processes grow as $\tan^2 \beta$. At large $\tan \beta$, the largest operator in these GUTs is $(u^c d^c)\dagger (s v)$, first discussed in [? , ?]. This operator is obtained through Higgsino exchange or through left-right sfermion mixing, which always entails a factor of the associated right-handed fermion mass. The largest effect therefore comes from changing a third-generation $right-handed$ squark or sfermion to a light $left-handed$ quark or lepton. Specifically, begin with the operator $U^c D^c T^c \tau^c$ in the superpotential, and then convert the sfermions $\tilde{t} \tilde{c} \tilde{\tau}$ to the fermions $s v$. The Higgsino and Wino dressing diagrams are given in Fig. 2; there are also diagrams with Higgsino-Wino mixing whose size is intermediate between them. The diagrams are proportional to the coefficient $\lambda_{1331}$. For the Higgsino dressing one has the couplings $y_t V_{ts} \sim y_t e_2$ for the $\tilde{t} - s$ vertex, $y_t$ for the $\tilde{c} - \nu_{\tau}$ vertex. A similar structure emerges for the Wino diagram but suppressed by gauge couplings and mixing factors; the Higgsino diagram usually dominates. The coefficient of this operator in minimal $\text{SU}(5)$ is proportional to $y_d y_t^2 y_t V_{ts} \sim \epsilon^2 \zeta^2$. This lies $\epsilon/\theta_c$ below its upper bound, because, as is also the case in many other GUTs, $\lambda_{1331} = y_d y_t$ is a factor of $\epsilon/\theta_c$ below the ansatz in Eq. (4). This will not be true in non-minimal GUTs where the matrix $R^d$ is non-hierarchical and $R^d_{12} \sim \theta_c$, as in certain ten-centered models [? ] where the bound on this coefficient can be saturated.

The same set of graphs, with only the external down-type quark flavors changed, gives $(u^c d^c)\dagger (s v)$, $(u^c s^c)\dagger (d v)$, and $(u^c d^c)\dagger (d v)$. In minimal $\text{SU}(5)$ the three amplitudes are in the ratio $y_t V_{ts}, y_s V_{ts}, y_t V_{td}, y_d V_{td}$; note these ratios are independent of the phases $\sigma_i$, unlike the $u d i d_{\tau}$ case. The coefficients of $(u^c d^c)\dagger (s v)$ and $(u^c s^c)\dagger (d v)$ are of opposite sign, and may be of the same order. (This is slightly inconsistent with the Table; however, the estimates therein are rough, and, as noted above, the $(u^c d^c)\dagger (s v)$ amplitude does not saturate its bound.) However, the hadronic matrix elements of these operators have opposite signs (to see this use [? , ?]) so interference in $p \to K^+ \nu_{\tau}$ is constructive; also, since $(u^c d^c)\dagger (s v)$ has a significantly larger matrix element, its contribution dominates. The operator $(u^c d^c)\dagger (d v)$
leads to $p \rightarrow \pi^+ \nu_\tau$; since its coefficient is in a known relationship to the other two, a precise prediction for the branching ratio $\Gamma(p \rightarrow \pi^+ \nu_\tau)/\Gamma(p \rightarrow K^+ \nu_\tau)$ is possible, independent of the supersymmetric spectrum. This predictivity is retained in models where $\hat{\lambda}_{1331}$ is determined. Even this information is unnecessary in GUTs where $\hat{\lambda}_{1331}$ is as large as allowed in Eq. (4), since in this case the $(u^c s^c)^{(1)(d_1 \nu)}$ operator is negligible and only $\hat{\lambda}_{1331}$ appears in the amplitudes.

Since these amplitudes go as $\zeta^2$, the rates for $p \rightarrow K^+ \bar{\nu}_\tau$, $\pi^+ \bar{\nu}_\tau$ grow like $\tan^4 \beta$, leading to short lifetimes and corresponding strong constraints at large $\tan \beta$. We find that the amplitudes for $u d s \nu$ and $(u^c d^c)^{(1)}(s \nu)$ become comparable in most GUTs for $\tan \beta$ somewhere between 3 and 15; in a minimal supergravity $SU(5)$ GUT (accounting for the different short-distance renormalizations and hadronic matrix elements of the two operators) we find this number is of order $9(m_W/\mu)$, with large uncertainties from the ratio $y_d/y_s$, the sfermion spectrum, poorly measured CKM angles, and the third-generation contribution to the $u d s \nu$ operator.* For $\tan \beta \sim 60$, the rate from $(u^c d^c)^{(1)}(s \nu)$ dominates that from $u d s \nu$ by $10-1000$. In those non-minimal GUTs where the bound on $(u^c d^c)^{(1)}(s \nu)$ is saturated, the amplitude is enhanced by another factor of $\theta$, and can dominate for even smaller values of $\tan \beta$. The constraints on large $\tan \beta$ models from these effects, although discussed in [?], do not appear to have been fully incorporated in the literature.

The bounds on $u u s \ell$ and $u u d \ell$ are comparable to those on $u d s \nu$ and $u d d \nu$. However, in minimal (and some non-minimal) GUTs, these bounds are not saturated. From the Table, we see that at small $\tan \beta$ we need only consider Wino exchange. The argument that $u u s \ell$ is highly suppressed [?,?] is that all contributions are proportional to $f_1 = y_u \sim \epsilon^4$, which makes the resulting amplitudes of order $f_1 h_2 \sim \epsilon^3$, smaller than the $\epsilon^4 \theta^2$ bound. To see this, consider the contribution of $U_i D_j D_k N_\tau$ in Eq. (6); the sneutrino must couple to the Wino, implying $i = 1$ and giving a factor of $f_1$. If instead we use $U_i D_j U_k E_\tau$, either $i = 1$ or $k = 1$. The former case gives $f_1$ directly, and the latter gives $f_1$ through a unitarity cancellation: the diagram in Fig. 3 is proportional to

$$h_2 V_{12}^* \sum_{i,j=1}^3 V_{ij}^* V_{ij} f_i V_{ip}^* = h_2 f_1 V_{12}^* V_{ip} \ .$$

(8)

However, the unitarity cancellation in Eq. (8) partly fails due to subtle renormalization group effects. The operator $U_i D_j U_k E_\tau$ is proportional to $V_{ij}$ at the GUT scale, but after renormalization to low-energy it is no longer proportional to $V_{ij}$ at the weak scale. This effect is of order $(1 - y_f^2/y_\nu^2)^{1/24}$ [?,?] (here $y_f \sim 1.1$ is the fixed-point value of $y_u$) or about 0.1 for $\tan \beta \sim 1.4 - 3$. Specifically, consider the $U_i D_j H_C$ coupling $\hat{F}_{ij}$ (we use the notation of [?];) naively $\hat{F}_{ij} = f_i V_{ij}$, but in fact $\hat{F}_{31}, \hat{F}_{32}$ differ from this at the weak scale by $\sim 10\%$. The sum over $j$ in (8) becomes $h_2 V_{12}^* \sum_j V_{ij}^* \hat{F}_{ij} V_{ip} \sim (0.1) h_2 V_{12}^* V_{13}^* V_{33} V_{3p} f_3$, which is about a tenth of $h_2 f_1$. The loop factor for this effect can be enhanced if the third-generation squarks are lighter than those of the first two generations; still, even if the CKM angles, $f_1$ and the

*While this letter was in preparation, a preprint appeared which studies the $(u^c d^c)^{(1)(d_1 \nu)}$ operators in the minimal $SU(5)$ GUT [?].
spectrum are at the edge of their ranges, it can never be as large as the leading contribution. A second effect of the same order (if the messenger scale of supersymmetry breaking scale is high) comes from the mixing and mass splittings between the down-type squarks. Both of these effects, while interesting, are most likely lost in the uncertainties surrounding $y_u$.

In many non-minimal GUTs, the unitarity cancellation in (8) simply does not occur. A sufficient condition for this is that $\hat{F}_{ij}$ be hierarchical but not precisely equal to $f_i V_{ij}$. If $\hat{F}_{ij} \sim f_i \min\{\epsilon_i/\epsilon_j, \epsilon_j/\epsilon_i\}$ as occurs in many realistic models of flavor (including those in which higher-dimension operators or non-minimal Higgs bosons contribute to the up-type quark masses), the sum $V_{ij}^* \hat{F}_{ij}$ does not equal $f_1 \delta_{1i}$; instead it gives $\theta_c f_2, e \theta_c f_3$ for $i = 2, 3$. In this case the diagrams in Fig. 3 with $i = 2, 3$ are proportional to $h_2 V_{12}^* (\theta_c f_2) V_{22}^*, h_2 V_{12}^* (e \theta_c f_3) V_{32}^* \sim e^3 \theta_c^2$. This saturates the bound in the Table, and thus in these theories the neutrino to charged-lepton branching ratio is order one at small $\tan \beta$ (although the hadronic matrix elements favor neutrinos.) Enhancements of $F_{ij}$ by factors of $\theta_c^{\delta_{1i} + \delta_{ij}}$ do not change this conclusion.

The charged lepton rates from these operators increase only as $\tan^2 \beta$. As suggested in the Table, at large $\tan \beta$ a bigger contribution to these operators may come from up-type squark mixing in gluino dressing [? ,?]; however the branching fraction to charged leptons remains small due to the large $(u^c d^f)^s u_d^c \tau^r$ operator.

It is clear from the Table that all observed muons should be left-handed, in contrast to proton decays mediated by dimension-six operators [?,?]. Although the Table suggests branching fractions to electrons can be of the same order as those to muons, in minimal and many non-minimal GUTs they are much smaller [?,?]. In such GUTs the coefficients $\hat{\lambda}_{ijkr}$ are not roughly independent of $r$, in contrast to the ansatz (4). However, the suppression factor is model dependent, and there are theories (such as ten-centered models [?,?]) in which electron and muon decays do have comparable rates. The electron-to-muon branching ratios thus are good probes of flavor physics [?].

Finally, as evident in the Table, dressings involving gluinos and neutralinos can be important at small $\tan \beta$ in their contribution to the $uds\nu, udd\nu$ operators. In minimal GUTs these contributions are suppressed by $y_u$, but in non-minimal GUTs (as in ?) they may become important. Gluino dressing is naively subleading due to the symmetry structure of the dimension-five operators [?,?,?], and can only play a role when there is significant flavor violation [?,?,?]. In the supersymmetric basis, flavor violation appears as intergenerational squark mixing. At small values of $\tan \beta$, $\tilde{d}_i - \tilde{d}_j$ mixings are induced proportional to $y_t^2 V_{3k}^* V_{3j}^* \sim \epsilon_i \epsilon_j$ times the factor $\gamma$ in the Table. In minimal $SU(5)$, because neutral gauginos do not change flavor and intergenerational mixing is small in the up-squark sector if $\tan \beta \ll 60$, the $uds\nu$ and $udd\nu$ operators must come from $U_i D_j D_k N_{ir}$ with $i = 1$; from Eq. (6) this implies a factor of $f_1 = y_u \sim \epsilon^4$, giving effects proportional to $y_u y_u \sim \epsilon^5 \ll \epsilon^3 \theta_c^2$. But in a non-minimal GUT, if $\hat{F}_{ij} \neq f_i V_{ij}$, and both $R_{12}^u$ and $R_{21}^d$ are of order $\theta_c$, then it is possible that $\hat{F}_{11} \sim f_2 \theta_c^2$, giving effects of order $\epsilon^3 \theta_c^2$, comparable to the leading Wino dressing contributions [?]. Dressing by neutralinos ($W_3$-ino and $B$-ino) is similar to the gluino dressing; however, squark mixing is not required for neutralinos to contribute to $uds\nu$, so their effects may overshadow the gluino if $\gamma$ is small [?]. Gluino dressing also can contribute without mixing to $uds\nu$ if there is substantial D-term splitting of squark masses [?].

Further details will appear in [?].

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FIGURES

FIG. 1. Wino-dressing diagrams contributing to $uds\nu$; there are also diagrams with $s, d$ exchanged.

FIG. 2. The Higgsino- and Wino-dressing diagrams leading to $u^c d^c s\nu$.

FIG. 3. The important diagrams contributing to $uus\mu$. 