Dynamically generated electric charge distributions in Abelian projected SU(2) lattice gauge theories.

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We show in the maximal Abelian gauge the dynamical electric charge density generated by the coset fields, gauge fixing and ghosts shows antiscreening as in the case of the non-Abelian charge. We verify that with the completion of the ghost term all contributions to flux are accounted for in an exact lattice Ehrenfest relation.

Lattice studies based on Abelian projection have had considerable success identifying the dynamical variables relevant to the physics of quark confinement. There is no definitive way as yet of choosing the optimum variables, but in the maximal Abelian gauge [1,2] the U(1) fields remaining after Abelian projection produce a heavy quark potential that continues to rise linearly [3]. Further the string tension is almost, but not exactly, equal to the full SU(2) quantity; 92% in a recent study at $\beta = 2.5115$ [4].

All elements of a dual superconducting vacuum appear to be present [5,1]; in the maximal Abelian gauge magnetic monopoles reproduce nearly all of the U(1) string tension [6,4]. The spontaneous breaking to the U(1) gauge symmetry is signalled by the non-zero vacuum expectation value of monopole operator [7,8]. The profile of the electric field and the persistent magnetic monopole currents in the vortex between quark and antiquark are well described by an effective theory, the Ginzburg–Landau, or equivalently a Higgs theory giving a London penetration depth and Ginzburg–Landau coherence length [9,10].

Central to finding the effective theory is the definition of the field strength operator in the Abelian projected theory, entering not only in the vortex profiles but also in the formula for the monopole operator. All definitions should be equivalent in the continuum limit, but use of the appropriate lattice expression should lead to a minimization of discretization errors.

In Ref. [11] we exploit lattice symmetries to derive such an operator that satisfies Ehrenfest relations; Maxwell’s equations for ensemble averages irrespective of lattice artifacts.

The charged coset fields are normally discarded in Abelian projection, as are the ghost fields arising from the gauge fixing procedure. Since the remainder of the SU(2) infrared physics must arise from these, an understanding of their rôles is central to completing the picture of full SU(2) confinement. In the maximal Abelian gauge a localised cloud of like polarity charge is induced in the vacuum in the vicinity of a source, producing an effect reminiscent of the antiscreening of charge in QCD. In other gauges studied, the analogous current acts to screen the source [14]. (This is a tentative result, however, without the benefit of the refined definition of flux.)

Consider the effect of a ‘right shift’ of a particular link, $U_\mu(x_0) \to U_\mu(x_0)U^\ast_{\mu}(x_0)$:

$$Z_W(\{U^\ast\}) = \int [d(UU^\ast)] W_3(U) \Delta_{FP} \delta[F] e^{\beta S(U)},$$

where we have introduced

$$1 = \Delta_{FP} \int \prod_j dg_j(y) \prod_i \delta[F^g(U(y_i);x)],$$

and integrated out the $g$ variables in the standard way. So $\Delta_{FP} = \det M$ where

$M_{ix;jy} = \left. \frac{\partial F^g(x)}{\partial g_j(y)} \right|_{g=0}$
We take as the source term an SU(2) plaquette with a σ₃ insertion to check the theorem.

\[ W_3 = \frac{1}{2} Tr(U^1 U^1 U U i \sigma_3). \]

\( Z_W \) is not invariant under \( U^* \). The shift is inconsistent with the gauge condition. It is invariant, however, under an infinitesimal shift together with an infinitesimal ‘corrective’ gauge transformation that restores the gauge fixing

\[ U^*(x_0) = 1 - \frac{i}{2} \epsilon_3(x_0) \sigma_3; \ G(x) = 1 - \frac{i}{2} \eta(x) \cdot \sigma. \]

Using the invariance of the measure under combination of a shift and a ‘corrective’ gauge transformation we obtain

\[ \left[ \frac{\partial}{\partial \epsilon_\mu(z_0)} + \sum_{k,z} \frac{\partial \eta_k(z)}{\partial \epsilon_\mu(z_0)} \frac{\partial}{\partial \eta_k(z)} \right] Z_W = 0. \]

The Ehrenfest relation reads (where \((\cdots)_\mu \) indicates a derivative [13])

\[
\left\langle (W_3)_\mu \right\rangle_s + \left\langle (W_3)_\mu \right\rangle_g + W_3 \times \\
\left( \left\langle \frac{\Delta_{FP}}{\Delta_{FP}} \right\rangle_s + \left\langle \frac{\Delta_{FP}}{\Delta_{FP}} \right\rangle_g + \beta(S)_\mu \right) \right\rangle = 0 (1) 
\]

Assuming the source involved the shifted link,

- \( (W_3)_\mu \) comes from the corrective gauge transformation acting on the source which is U(1) invariant but not SU(2) invariant.
- \( \left\langle \frac{\Delta_{FP}}{\Delta_{FP}} \right\rangle_s \) is the effect of the shift on the Faddeev-Popov determinant.
- \( \left\langle \frac{\Delta_{FP}}{\Delta_{FP}} \right\rangle_g \) is due to the corrective gauge transformation of the Faddeev-Popov determinant.
- \( \beta(S)_\mu \) is a shift term of the (gauge invariant) action.

<table>
<thead>
<tr>
<th>Source: ( W_3 )</th>
<th>( W_3 )</th>
<th>( W_3(U \rightarrow D) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left\langle (W_3)_\mu \right\rangle_s )</td>
<td>0.65468(10)</td>
<td>0.63069(20)</td>
</tr>
<tr>
<td>( \left\langle (W_3)_\mu \right\rangle_g )</td>
<td>0.06095(7)</td>
<td>0.04463(4)</td>
</tr>
<tr>
<td>( \left\langle \frac{\Delta_{FP}}{\Delta_{FP}} \right\rangle_s )</td>
<td>0.00127(21)</td>
<td>0.00132(50)</td>
</tr>
<tr>
<td>( \left\langle \frac{\Delta_{FP}}{\Delta_{FP}} \right\rangle_g )</td>
<td>0.00529(3)</td>
<td>0.00564(3)</td>
</tr>
<tr>
<td>( \left\langle \beta(S)_\mu \right\rangle_s )</td>
<td>-0.72246(68)</td>
<td>-0.68275(50)</td>
</tr>
<tr>
<td>Zero</td>
<td>-0.00026(77)</td>
<td>-0.00045(64)</td>
</tr>
</tbody>
</table>

Table 1

Terms in the Ehrenfest relation, Eqn.(1) on a 4^4 lattice at \( \beta = 2.5 \). The column labeled \( W_3 \) corresponds to the source described in the text. In the second column the source links are replaced by their diagonal parts of the links to test a second source. The theorem gives zero for the sum.

Imposing the gauge constraint up to first order quantifies \( \eta \)

\[ F_i(x) + \frac{\partial F_i(x)}{\partial \epsilon_\mu(z_0)} \epsilon_\mu(z_0) + \sum_{k,z} \frac{\partial F_i(x)}{\partial \eta_k(z)} \eta_k(z) = 0, \]

and we define the shifted Faddeev-Popov matrix as a derivative with respect to a general gauge transformation of the corrected constraint.

\[ M_{ix:ijy} + \delta M_{ix:ijy} = \frac{\partial}{\partial g_j(y)} \times \\
\left\{ \frac{F_i^g(x) + \frac{\partial F_i^g(x)}{\partial \epsilon_\mu(z_0)} \epsilon_\mu(z_0) + \sum_{k,z} \frac{\partial F_i^g(x)}{\partial \eta_k(z)} \eta_k(z)}{\Delta} \right\}. \]

Finally we evaluate the derivative using

\[ \frac{(\Delta)}{\Delta} = Tr[M^{-1}(M)_\mu]. \]

A check of this Ehrenfest theorem is given in Table 1. Some of the terms require a 2N×2N matrix.
$\beta$ & $\frac{1}{\beta}$ & $\text{div}E$ (on source) & total flux \\
\hline
10.0 & 0.1 & 0.1042(1) & 0.0910(8) (mid) \\
& & & 0.0148(8) (back) \\
& & & 0.1092(8) (total) \\
2.4 & 0.4166 & 0.5385(19) & 0.7455(70) (mid) \\
& & & 0.0359(72) (back) \\
& & & 0.7815(95) (total) \\
\hline

Table 2

div$E \equiv \langle \Delta_{\nu} F_{\nu\mu} \rangle$, normalized to $\frac{1}{\beta}$ for a ‘classical’ point charge, measured on a $3 \times 3$ Wilson loop source on an $8^4$ lattice. Integrated electric flux is measured on the midplane centered on the Wilson loop and on a plane on the far side of the torus, and the sum being the total flux.

inversion, where $N$ is the lattice volume. Hence we chose a $4^4$ lattice for the numerical test of what is an exact relation on all lattice sizes.

We separate the links $U_\mu$ into diagonal $D_\mu$ and off-diagonal $O_\mu$ parts. Grouping all $O_\mu$ terms on the right as a set of conserved currents we get the final form of the Ehrenfest-Maxwell relation:

$$
\langle \Delta_{\mu} F_{\mu\nu} \rangle = \left. \langle J^{\text{dyn.}}_{\nu} \rangle \right|_s + \left. J^{\text{static}}_{\nu} \right|_s + \left. \left( \langle J^{FP}_{\nu} \rangle \right|_s + \langle J^{FP}_{\nu} \rangle \right|_g.
$$

The first term in the current comes from the excitation of the charged coset fields, the static term has an extra non-local contribution coming from the corrective gauge transformation, and the last two contributions are from the ghost fields. These terms give a non vanishing charge density cloud around a static source. The left hand side can be used as a lattice operator to measure the total charge density and does not require the matrix inversions needed to measure the individual terms separately which limited the numerical tests to small lattices.

Table 2 gives an application showing:
(i) a ‘classical’ point charge is dressed with like charge,
(ii) the total integrated flux is larger than div$E$ on the source, both indicating anti-screening.

In summary the coset fields renormalise the charge of the Wilson loop as measured by $\langle \Delta_{\nu} F_{\nu\mu} \rangle$ and charge is also induced in the surrounding vacuum. Full SU(2) has antiscreening/asymptotic freedom of color charge, and in the maximal Abelian gauge alone we have seen analogous behaviour, in that the source charge is increased and induces charge of like polarity in the neighboring vacuum. The improved field strength expression defined by the Ehrenfest identity does not coincide with the lattice version [14] of the ‘t Hooft field strength operator [15].

REFERENCES