A Simple Derivation of the
Hard Thermal Loop Effective Action

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ABSTRACT

We use the background field method along with a special gauge condition, to derive the hard thermal loop effective action in a simple manner. The new point in the paper is to relate the effective action explicitly to the $S$-matrix from the onset.

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1 INTRODUCTION

It is by now well established that the resummation of the so-called hard thermal loops (HTLs) is a necessary part of any perturbative scheme to finite temperature QCD. Since the original work by Pisarski[1], Braaten and Pisarski[2] and Frenkel and Taylor[3], there have been several papers deriving the HTL amplitudes using various techniques, including Chern-Simons eikonal[4], color transport theory[5], and Wong equations[6]. It has also been shown that all the HTL amplitudes can be derived from a simple gauge-invariant effective action that incorporates the Ward-identities originally derived by Braaten and Pisarski[2] and Frenkel and Taylor[3]. This effective action has been given in several forms first by Taylor and Wong[7] who gave an expression involving string operators, and by Braaten and Pisarski, who found the following particularly elegant expression (SU(N) Yang-Mills theory)

\[ \Gamma_{\text{HTL}}[A_\mu] = \frac{g^2 N T^2}{6} \int d^4 x \int d^2 \hat{q} \frac{Q_\sigma Q_\lambda}{4 \pi} \frac{\hat{Q} \cdot D(A)}{[\hat{Q} \cdot D(A)]^2} F_{\mu}^\lambda, \]  

where \( D(A)_\mu = \partial_\mu + g A_\mu \) with \( A \) in the adjoint representation, and \( \hat{Q} \) a 4-vector of the form \( \hat{Q} = (1, \hat{q}) \), so \( \hat{Q}^2 = 0 \).\(^3\) This form was guessed by Braaten and Pisarski based on general properties of perturbation theory, and indeed was shown to reproduce the HTL Ward identities in the latter. At the same time Frenkel and Taylor derived essentially the same result by proving that the action satisfies certain conditions that are restrictive enough to have a unique solution[8]. Again their analysis was based on an analysis of the explicit (one-loop) \( n \)-gluon amplitudes in the HTL approximation.

The purpose of this paper is to give a simple derivation of Eq. (1), using the background field method and the specific gauge choice

\[ \hat{Q}_\mu A_\mu = 0. \]  

It is crucial in the derivation to show that it is possible to use a \( \hat{Q} \) dependent gauge choice. Using this gauge we can solve for the gauge potential,

\[ A_\mu(\hat{Q}) = \frac{1}{Q \cdot D(A)} \hat{Q}_\nu F_\nu^\mu, \]  

\(^3\)We shall always use Minkowski metric, so an Euclidean 4-vector is of the form \((i q_0, \vec{q})\).
where the parametric dependence on the gauge condition is shown explicitly. Substituting
in Eq. (1), we obtain
\[ \Gamma_{\text{HTL}}[A_{\mu}] = \frac{g^2 N T^2}{6} \int d^4x \int \frac{d^2\hat{q}}{4\pi} \text{tr} A(\hat{Q}; x)^2, \]
(4)
as was first shown by Frenkel and Taylor[8], who also stressed that nothing is gained in
simplicity by rewriting \( \Gamma_{\text{HTL}} \) in this gauge, since the full non-local and non-linear structure
is hidden in the complicated parametric \( \hat{Q} \)-dependence of \( A_{\mu}(\hat{Q}) \).

It is also not clear that the gauge in Eq. (2) would be useful in trying to derive the
effective action, in spite of the simple form of Eq. (4), since it involves an integral over
the parameter \( \hat{Q} \) that enters in the gauge condition. Although, using the background
field method, the effective action by construction is invariant under background gauge
transformations, it is by no means obvious that it can be expressed as an integral over \( \hat{Q} \)
of a gauge invariant object, as in Eq. (1). The key ingredient in the derivation to follow in
this paper is to find such a formulation by relating the effective action to a gauge invariant
quantity, namely the \( S \)-matrix. The \( \hat{q} \) integral is naturally interpreted as an integral over
the (on-shell) momenta \( Q \) of the particles in the heat bath.

Many of the results used and/or derived in this paper have already appeared in the
literature. For example, the connection between \( \Gamma_{\text{HTL}} \) and forward scattering amplitudes
was already stressed by Frenkel and Taylor, and the gauge condition Eq. (2) was discussed
in [8] and used to simplify the derivation of the effective action in [9]. We believe, however,
that our approach, where the starting point is a direct relation between the effective action
and the \( S \)-matrix, is novel and provides a very simple and physical way to derive Eq. (1).

When facing the problem of how to express the HTL effective action in terms of a gauge
invariant object, it is important to understand its physical significance beyond the formal
definition of being the generating functional of the \( O(T^2) \) parts of the proper \( n \)-point
functions. For static fields, \( \Gamma_{\text{HTL}} \) is nothing but the \( O(T^2) \) contribution to the (negative)
pressure of thermal particles interacting with the field. For time-dependent configurations
the current is the natural object to relate to the \( S \)-matrix, as was stressed in this context
by Jackiw and Nair [10]. The action \( \Gamma_{\text{HTL}} \) is then obtained by integration with respect
to the gauge field. In the next section we shall first consider the static case and then the
time-dependent one. It is worth remembering that although the final formulae derived in
these cases are the same, the physical interpretations are rather different.

The detailed derivations in the next section will be given for adjoint scalars rather
than gauge particles. From the presentation it will be quite obvious that the only thing
that matters for the HTL effective action is the number of physical degrees of freedom, and their charge and statistics. For those readers who are convinced that this is the case, the result for a complex adjoint scalar can immediately be taken over to the case of YM theory since the gluon also has two physical degrees of freedom. For those who want a formal proof, this is provided in Appendix A using HTL power counting arguments. There, we also prove that the HTL effective action, as calculated using background field technique, is independent of the quantum gauge fixing parameter. The extension of our method to include fermions is straightforward and can be found in Appendix B.

### 2 The Free Energy and the Effective Action

As mentioned above, we shall consider a heat bath of charged scalars in the background of a non-abelian field $A_\nu$. By construction, the one-loop background field effective action for static field configurations is nothing but the free energy of a gas of scalar particles interacting with the background. The free energy may in turn be directly related to the $S$-matrix [11]. Hence,

$$
\Gamma_{\text{HTL}}^{\text{stat}} = \frac{1}{\beta} \text{Tr} \beta \ln[-\Box] = F = F_0 - \frac{1}{\beta} \int_0^\infty dE \, e^{-\beta E} \, \frac{1}{4\pi i} \text{Tr} \left( S^\dagger \frac{\partial S}{\partial E} - S \frac{\partial S^\dagger}{\partial E} \right)_C,
$$

where the trace is over all connected diagrams in the notation of [11]. For particles that do not interact mutually, but only with the external field, the sum over multiparticle states can be performed and the free energy can be related to the one-particle density of states

$$
F = \frac{1}{\beta} \int_0^\infty dE \, \ln(1 - e^{-\beta E})(\rho_0 + \Delta\rho(E)),
$$

where the shift of the density of states is related to the one-particle $S$-matrix by

$$
\Delta\rho(E) = \frac{1}{4\pi i} \text{Tr} \left( S^\dagger \frac{\partial S}{\partial E} - S \frac{\partial S^\dagger}{\partial E} \right)_{1\text{-part}}.
$$

Since the $S$-matrix is gauge invariant for each physical momentum state we can use different gauge choices for different momenta and thus the choice in Eq. (2) is allowed. Then, only the first two terms in

$$
S = \ldots + \ldots + \ldots + \ldots + \ldots + \ldots
$$
contribute. All other diagrams are either zero because of the gauge choice or suppressed at high temperature. The counting here is very much the same as in Appendix A. A direct expansion of \( S = 1 + iT \) in a static background gives

\[
\langle q | S(E) | q \rangle = 1 + \frac{i g^2 N}{E_q V} 2\pi \delta (E_q - E) \int d^3 x \left( A^a_\mu(Q; x) \right)^2 ,
\]

(9)

using the normalization \( \langle q | q' \rangle = (2\pi)^3 \delta^{(3)}(q - q') \) and \( \text{tr} = \frac{V}{(2\pi)^3} \int d^3 q \). From Eq. (7) we obtain

\[
\Delta \rho(E) = \frac{g^2 N}{2\pi^2} \int d^3 x \int \frac{d^2 q}{4\pi} \left( A^a_\mu(Q; x) \right)^2 .
\]

(10)

The energy integral in Eq. (6) is then trivial and we arrive at

\[
F = F_0 - \frac{g^2 N T^2}{12} \int d^3 x \int \frac{d^2 q}{4\pi} \left( A^a_\mu(Q; x) \right)^2 ,
\]

(11)

which agrees with Eq. (4) when we use the relation \( \Gamma = -\int_0^\infty dx_0 F \).

The calculation above can be generalized to time dependent background fields but, as already mentioned, the physical interpretation is different. The free energy is an equilibrium concept and we shall instead start from the expectation value of the current in a background field to derive an effective action. The current is given by [10, 12]

\[
j(t, x) = \frac{1}{Z} \text{tr} \left[ e^{-\beta H} U^\dagger(t, -\infty) j(-\infty, x) U(t, -\infty) \right] = \frac{1}{Z} \text{tr} \left[ e^{-\beta H S} \frac{i \delta}{\delta A(t, x)} S \right] ,
\]

(12)

with the \( S \)-matrix in the interaction picture. There are two pieces in the current when written in terms of the \( T \)-matrix \( S = 1 + iT \)

\[
j(t, x) = \frac{1}{Z} \text{tr} e^{-\beta H} \left[ \frac{\delta T}{\delta A(t, x)} - iT^\dagger \frac{\delta T}{\delta A(t, x)} \right] ,
\]

(13)

The first term is a total derivative of an action (which we call the effective action for time-ordered \( n \)-point functions). To one-loop order the second term is imaginary and has support only when the external field is on the light-cone. This is the term that makes up for the difference between time-ordered and retarded \( n \)-point functions.

Looking only for the real part we can integrate the first piece in Eq. (13) with respect to \( A \). We obtain

\[
e^{\Gamma[A]} = \frac{1}{Z} \text{tr} \left[ e^{-\beta H S} \right] = \frac{1}{Z} \exp \left[ V \int \frac{d^3 q}{(2\pi)^3} \ln \left( \sum_{n_q} e^{-\beta E_q n_q} \langle n_q | S | n_q \rangle \right) \right] .
\]

(14)
where $|n_q\rangle$ is a state with $n_q$ particles of momentum $q$. Since the particles do not interact with each other the expectation value of the S-matrix factorizes

\[\langle n_q|S|n_q\rangle = (\langle q|S|q\rangle)^{n_q} = (1 + i\langle q|T|q\rangle)^{n_q}.\]

The sum over $n_q$ and the internal group indices can be performed easily and we find

\[i\Gamma[A] = -\ln Z + VN \int \frac{d^3q}{(2\pi)^3} \ln \left( \frac{1}{1 - e^{-\beta E_q}(1 + i\langle q|T|q\rangle)} \right).\]  

(15)

Since the expectation value of the $T$-matrix is gauge invariant we can again choose the gauge $Q_{\mu}A^\mu = 0$. Only the $g^2A^2$ piece in the interaction enters and in fact only a single such insertion since multiple insertions are again suppressed by powers of $1/T$. An expansion in $\langle q|T|q\rangle$ gives

\[i\Gamma[A] = iVN \int \frac{d^3q}{(2\pi)^3} \frac{1}{e^{\beta E_q} - 1} \langle q|T|q\rangle.\]  

(16)

Using Eq. (9) for non-static background fields one finds

\[\langle q|T|q\rangle = \frac{1}{E_qV} \int d^4x A^2(\hat{Q}; x),\]  

(17)

After substituting into Eq. (16) and performing the $q$-integration, we have

\[\Gamma_{HTL}[A] = \frac{g^2N\tau^2}{12} \int d^4x (A^\mu_{\tau})^2(\hat{Q}; x),\]  

(18)

which again agrees with Eq. (4). This completes our derivation of the HTL effective action for general field configurations.\(^4\)

Finally, we want to stress the simplicity of the arguments leading from Eq. (5) to Eq. (11) and from Eq. (12) to Eq. (18). Basically all steps are written out, and the only thing that requires some care is to get the various normalizations of the S-matrix elements right. This should be contrasted with the rather involved chain of arguments that have appeared in previous derivations of the HTL effective action. This paper was concerned with a new derivation of known results, but one could also try to use our methods to calculate subleading terms by including interactions between the fast thermal particles. This would amount to include contributions from the 2-body, 3-body etc S-matrix. Whether or not this would lead to physically interesting approximations remains to be seen.

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\(^4\)As usual, the form Eq. (1) is appropriate only off the light cone.
A  Appendix

We shall now formally show that the leading high $T$ contribution to the one-loop effective action for gluons is the same as for a single complex scalar field in the adjoint representation. Again we stress that this is rather straightforward from the calculation in Section 2. We start from the Euclidean formulation of finite temperature field theory, and, in the notation of [13], write the one-loop finite $T$, gauge invariant background field effective action as

$$i\Gamma[A, \alpha] = -\text{Tr}_\beta \ln [-\square] + \frac{1}{2} \text{Tr}_\beta \ln [-\square g_{\mu\nu} - 2gF_{\mu\nu} - \left(\frac{1}{\alpha} - 1\right) D_\mu D_\nu],$$

(19)

where $\square = D_\mu D^\mu$, $gF_{\mu\nu} = [D_\mu, D_\nu]$, $\alpha$ is the (covariant background field) gauge-fixing parameter, and the trace is over color and Lorentz indices as well as over spacetime. The background field effective action is by construction gauge invariant, with respect to the background field $A^\mu$, but to prove physical gauge invariance one must also establish $\alpha$-independence. We first consider the background field Feynman gauge, $\alpha = 1$, expand the gluon trace in powers of $F^{\mu\nu}/\square$, and combine the leading term with that from the ghosts to get,

$$i\Gamma[A, 1] = \text{Tr}_\beta \ln [-\square] + \text{Tr}_\beta \left[\frac{1}{\square} 2F^{\mu\nu}\right] - \frac{1}{2} \text{Tr}_\beta \left[\frac{1}{\square} 2F^{\mu\nu} \frac{1}{\square} 2F^{\mu\nu}\right] + \ldots,$$

(20)

where $\square = \partial^2 + g\{\partial_\mu, A_\mu\} + g^2A^2$. Now recall the rules for power counting in HTL. Each gluon propagator contributes a term $\sim 1/q^2$, so naively it would be expected to give a suppression $\sim T^{-2}$ to the amplitude, but as stressed by Braaten and Pisarski, this is not correct. The leading contribution arises when the loop momentum is large but the propagators also almost on shell. Performing the $q_0$ integration by closing the contour puts one propagator on shell, i.e. $q_\mu = q\hat{Q}_\mu$, contributing a factor $1/2q$ to the $dq$ integration, while all the other propagators contribute with denominators $(\hat{Q} + p_i)^2 = 2q\hat{Q} \cdot p_i + p_i^2 \approx 2q\hat{Q} \cdot p_i$. In this approximation the $dq$ integration factorizes and immediately gives the $T$ behavior of the graph by power counting. The leading contribution to a diagram with $m$ 3-gluon vertices, $n$ 4-gluon vertices, and $l$ insertions of $F^{\mu\nu}$ is $\sim T^{3}T^{m}T^{-2-(m+n+l-1)} = T^{3-n-l}$ where the contributions are from the integration measure, the momentum dependence of the 3-gluon vertices and the propagators respectively. Note that the terms corresponding to $l = 1$ is $\sim F^{\mu\nu}$ and thus zero because of the Lorentz trace, and since the $l = 2$ term is already at most $\sim T$, the HTL action comes entirely from the graphs with no $F^{\mu\nu}$.
insertions. This conclusion is independent of the gauge choice for the background field. Thus, the YM effective action $\text{Tr}_B \ln[-\Box]$ can be calculated using charged scalar inside the loop and multiplying with the appropriate group factors.

Finally we show that the $\alpha$-dependence in Eq. (19) is suppressed by powers of $1/T$. Following [14, 13] we write

$$\frac{\partial \Gamma}{\partial \alpha} = \frac{1}{2} \text{Tr}_\beta \left( \frac{1}{\Box} D_\mu \mathcal{E}_\mu \frac{1}{\Box} \right) \left( \frac{1}{\Box} \mathcal{E}_\mu G_{\mu\nu} \mathcal{E}_\nu \frac{1}{\Box} \right), \quad (21)$$

where $\mathcal{E}_\mu = [D_\nu, F_{\mu\nu}]$ and $G$ the full covariant gluon propagator satisfying

$$\left( \square g_{\mu\lambda} + 2F_{\mu\lambda} + \frac{1-\alpha}{\alpha} D_\mu D_\lambda \right) G_{\lambda\nu}(x,y) = g_{\mu\nu} \delta^4(x-y). \quad (22)$$

It is easy to see that, since $\mathcal{E}_\mu$ is independent of the loop momentum and the presence of a double pole in the first term in Eq. (21) gives an extra power of $T$ compared to diagrams with only single poles, there is no $T^2$ contribution. In the second term we can expand $G$ like

$$G = \sum_n \left( \frac{1}{\Box} \left( 2F_{\mu\nu} + \frac{1-\alpha}{\alpha} D_\mu D_\nu \right) \right)^n \frac{1}{\Box}. \quad (23)$$

Now we have potentially dangerous terms with powers of $\Box D_\mu D_\nu$ which naively go like $T^n$. However, the $D$ factors can always be commuted around so that a contraction is possible. For example

$$\frac{1}{\Box} D_\alpha D_\beta \frac{1}{\Box} D_\beta D_\gamma = \frac{1}{\Box} D_\alpha D_\gamma + \frac{1}{\Box} D_\alpha D_\beta \frac{1}{\Box} [D_\beta, \Box] \frac{1}{\Box} D_\gamma, \quad (24)$$

where the second term goes like $T$ since $[D_\beta, \Box] = g(D_\beta F_{\mu\alpha} + F_{\beta\mu} D_\mu) \sim T$. Each contraction lowers the naive power by one factor of $T$ and therefore the dangerous powers can be eliminated. Terms with factors of $F$ are of course subleading according to the same power counting.

**B Appendix**

It is not hard to generalize the above argument to include fermions by adding a background of fermionic sources. In the Feynman gauge the total effective action is

$$i\Gamma[A, \bar{\Psi}, \Psi] = -\text{Tr}_B \ln[-\Box] + \frac{1}{2} \text{STr}_\beta \ln \left[ \frac{-\Box g_{\mu\nu} - 2gF_{\mu\nu} + ig\bar{\Psi}\gamma_\mu\Psi}{i\partial} \right], \quad (25)$$
where the second trace now is a supertrace over both gauge bosons and fermions. This is related to the $S$-matrix in the same way as Eq. (5) and therefore we can use the same $Q$-dependent gauge as in Eq. (2). It can be rewritten in terms of an ordinary trace as [15]

$$i\Gamma[A, \overline{\Psi}, \Psi] = \frac{1}{2} \text{Tr}_B \ln \left[ -\Box g_{\mu\nu} - 2gF_{\mu\nu} \right] - \frac{1}{2} \text{Tr}_B \ln \left[ i\slashed{\partial} \right] \left[ 1 - \frac{g^2}{i\slashed{\partial} \gamma_\mu \Psi \left( \frac{1}{\Box} + 2gF \right)_{\mu\nu} \overline{\Psi} \gamma^\nu \right].$$

The second term is the contribution to the gauge boson effective action from dynamical fermions which is simply equal to $-\frac{1}{4} \text{Tr}_B \ln[-\Box]$. The only difference with the term evaluated earlier is the statistics of the hard particles and that the group trace is in the fundamental representation and gives a factor $N_f/2$. The last term in Eq. (26) gives the effective action for the fermionic background fields. It can also be analyzed with the methods described above. After expanding in powers of $\overline{\Psi}\Psi$ and in powers of $F$ it is only the zeroth order term

$$N \text{tr}_B \left[ g^2 C_f T^2 \overline{\Psi} i\slashed{\partial} \Psi \right]$$

that can go like $T^2$. The remaining factors of $1/\Box$ are expanded in powers of $g$ and with the gauge in Eq. (2) only the leading $1/\partial^2$ remain in the high $T$ limit. The two poles correspond to forward scattering of gauge bosons and fermions, respectively. After performing the thermal trace over Lorentz indices we obtain

$$i\Gamma[\overline{\Psi}, \Psi] = \frac{g^2 C_f T^2}{8} \int \frac{d^3q}{4\pi} \overline{\Psi} \gamma_\mu \hat{Q}^\mu \Psi.$$

Equation (28), just like Eq. (4), is gauge dependent and valid only in the gauge Eq. (2). It is, however, straightforward to write them in a explicitly gauge invariant by expressing $A$ and $\hat{Q} \cdot \partial$ in terms of $F$ and $\hat{Q} \cdot D$ and thereby recovering the standard HTL effective action.

**References**


