THE TOPOLOGICAL STRUCTURE OF THE VORTICES
IN THE $O(n)$ SYMMETRIC TDGL MODEL*

YISHI DUAN, YING JIANG† and TAO XU
Institute of Theoretical Physics, Lanzhou University, Lanzhou, 730000, P.R.China

In the light of $\phi$–mapping method and topological current theory, the topological structure of the vortex state in TDGL model and the topological quantization of the vortex topological charges are investigated. It is pointed out that the topological charges of the vortices in TDGL model are described by the Winding numbers of $\phi$–mapping which are determined in terms of the Hopf indices and the Brouwer degrees of $\phi$–mapping.

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The world of defects is amazingly rich and have been the focus of much attention in many areas of contemporary physics$^{[1]}$–$^{[3]}$. The importance of the role of defects in understanding a variety of problems in physics is clear$^{[4]}$–$^{[7]}$. As is well known, topology now becomes much more important and necessary in physics, hence it is necessary for us to investigate the topological properties of the defects meticulously. In our previous work, by the use of the $\phi$–mapping topological current theory$^{[8]}$, we have investigated the topological invariants$^{[9]}$–$^{[11]}$ and the topological structures of physical systems$^{[12]}$–$^{[14]}$ successfully. Now, in the light of this useful method, we will study the topological properties of the vortices in the context of an $O(n)$ symmetric time–dependent Ginzburg–Landau (TDGL) model for the case of point defects$^{[15]}$ where $n = k$ and $k$ is the spatial dimensionality.

We consider a time-dependent Ginzburg–Landau model for an $n$–component order parameter $\vec{\phi}(\vec{r}, t) = (\phi^1(\vec{r}, t), \ldots, \phi^n(\vec{r}, t))$ governed by the Langevin equation

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{K} = -\Gamma \frac{\delta F}{\delta \vec{\phi}} + \vec{\eta}$$

(1)

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† Corresponding author; E-mail: itp3@lzu.edu.cn
where $\Gamma$ is a kinetic coefficient and $\tilde{\eta}$ is a thermal noise which is related to $\Gamma$ by a fluctuation–dissipation theorem. $F$ is a Ginzburg–Landau effective free energy assumed to be of the form

$$F = \int d^k r \left[ \frac{c}{2}(\nabla \phi)^2 + V(\phi) \right]$$

(2)

where $c < 0$ and the potential $V$ is assumed to be of the degenerate double–well form.

In recent years, many works have been done on the system of TDGL model. Liu and Mazenko have discussed the growth kinetics of the systems with continuous symmetry\cite{16}, the defect–defect correlation in the dynamics of first–order phase transitions\cite{17} and the vortex velocities in the TDGL model\cite{18}. Ryusuke Ikeda has presented the hydrodynamical description for vortex states in type II superconductors on the TDGL equation\cite{19}. By the use of the TDGL model, Schönborn and Desai have investigated the intra–surface kinetics of phase ordering on toroidal and corrugated surfaces\cite{20}. Two–dimensional XY models with resistively–shunted junction dynamics have also been discussed by Kim et.al\cite{21}. However, most of them concentrated on the dynamical properties of the TDGL model. In this letter, we will focus on the intrinsic topological structure of the vortex topological current and give the topological quantization of the topological charges of the vortices in TDGL model.

It is well known that the $n$–dimensional order parameter field $\tilde{\phi}(\vec{r}, t)$, which is governed by the Langevin equation, determines the defect properties of the system, and it can be looked upon as a smooth mapping between the $(n + 1)$–dimensional space–time $X$ and the $n$–dimensional Euclidean space $R^n$ as $\phi : X \rightarrow R^n$. By analogy with the discussion in our previous work\cite{14;22}, from this $\phi$–mapping, a topological current can be deduced as

$$j^\mu = \frac{1}{A(S^{n-1})(n-1)!} e^{\mu \mu_1 \cdots \mu_n} \epsilon_{a_1 \cdots a_n} \partial_\mu n^{a_1} \cdots \partial_\mu n^{a_n}$$

(3)

$$\mu, \mu_1, \cdots \mu_n = 0, 1, \cdots n; \quad a_1, \cdots a_n = 1, \cdots, n$$

to describe the vortex state of the system and its zeroth component is defined as the density of the total vortex charge $\rho = j^0$. In the expression, $\partial_\mu$ stands for $\partial/\partial x^\mu$, $A(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$ the area of $(n – 1)$–dimensional unit sphere $S^{n-1}$ and $n^a(x)$ is the direction
field of the $n$–component order parameter field $\vec{\phi}(\vec{r}, t)$

$$n^a(\vec{r}, t) = \frac{\phi^a(\vec{r}, t)}{||\phi(\vec{r}, t)||}, \quad ||\phi(x)|| = \sqrt{\phi^a(\vec{r}, t)\phi^a(\vec{r}, t)}$$  \hspace{1cm} (4)

with

$$n^a(\vec{r}, t)n^a(\vec{r}, t) = 1.$$  \hspace{1cm} (5)

It is obvious that $n^a(\vec{r}, t)$ is a section of the sphere bundle $S(X)$ and it can be looked upon as a map of $X$ onto $(n - 1)$–dimensional unit sphere $S^{n-1}$ in order parameter space. Clearly, the zero points of the order parameter field $\vec{\phi}(\vec{r}, t)$ are just the singular points of $n^a(\vec{r}, t)$.

From the formulas above, we conclude that in the TDGL model, there exists a conservative equation of the topological current in (3)

$$\partial_\mu j^\mu = 0, \quad \mu = 0, 1, \cdots n$$

or

$$\partial_\mu + \nabla \cdot \vec{j} = 0.$$  

In the following, we will investigate the intrinsic structure of this vortex topological current (3) by the use of the $\phi$–mapping method. From (5) and (4), we have

$$\partial_\mu n^a = \frac{1}{||\phi||} \partial_\mu \phi^a + \phi^a \partial_\mu \left( \frac{1}{||\phi||} \right)$$

$$\frac{\partial}{\partial \phi^a} \left( \frac{1}{||\phi||} \right) = - \frac{\phi^a}{||\phi||^3}$$

which should be looked upon as generalized function$^{[23]}$. Using these expressions the topological current (3) can be rewritten as

$$j^\mu = C_n \varepsilon^{\mu\nu_1 \cdots \nu_n} \epsilon_{a_1 \cdots a_n} \partial_{\mu_1} \phi^a \partial_{\mu_2} \phi^{a_2} \cdots \partial_{\mu_n} \phi^{a_n} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^{a_1}} (G_n(||\phi||))$$

where $C_n$ is a constant

$$C_n = \left\{ \begin{array}{ll} -\frac{1}{2\pi} \frac{1}{(n-2)(n-1)!} & \text{for } n > 2 \\ \frac{1}{2\pi} \frac{1}{2} & \text{for } n = 2 \end{array} \right.$$  

3
and \( G_n(||\phi||) \) is a generalized function

\[
G_n(||\phi||) = \begin{cases} 
\frac{1}{||\phi||^{n-1}} & \text{for } n > 2 \\
\ln||\phi|| & \text{for } n = 2.
\end{cases}
\]

If we define \( n + 1 \) Jacobians \( J^\mu (\frac{\phi}{x}) \) as

\[
\epsilon^{a_1 \ldots a_n} J^\mu (\frac{\phi}{x}) = \epsilon^{\mu \nu_1 \ldots \nu_n} \partial_{\nu_1} \phi^{a_1} \ldots \partial_{\nu_n} \phi^{a_n},
\]

in which \( J^0 (\frac{\phi}{x}) \) is just the usual \( n \)-dimensional Jacobian determinant

\[
J^0 (\frac{\phi}{x}) = D (\frac{\phi}{x}) = \frac{D (\phi^1, \ldots, \phi^n)}{D (x^1, \ldots, x^n)},
\]

and make use of the \( n \)-dimensional Laplacian Green’s function relation in \( \phi \)-space\cite{22}

\[
\Delta_\phi (G_n (||\phi||)) = -\frac{4\pi^{n/2}}{\Gamma(n/2 - 1)} \delta(\vec{\phi})
\]

where \( \Delta_\phi = (\frac{\partial^2}{\partial \phi \partial \phi}) \) is the \( n \)-dimensional Laplacian operator in \( \phi \)-space, we do obtain the \( \delta \)-function structure of the vortex topological current rigorously

\[
j^\mu = \delta (\vec{\phi}) J^\mu (\frac{\phi}{x}).
\]

This expression involves the total defect information of the TDGL system and it indicates that all of the vortices are located at the zero points of the order parameter field \( \phi (\vec{r}, t) \).

From this expression, the density of \( j^\mu \) is also changed into a compact form

\[
\rho = j^0 = \delta (\vec{\phi}) D (\frac{\phi}{x}).
\]

We find that \( j^\mu \neq 0, \rho \neq 0 \) only when \( \vec{\phi} = 0 \), which is the singular point of \( j^\mu \). In detail, the Kernel of \( \phi \)-mapping is the singularities of the topological current \( j^\mu \) in \( X \), i.e. the inner structure of topological current is labelled by the zeroes of \( \phi \)-mapping. We think that this is the core of topological current theory and \( \phi \)-mapping is the key to study it. The essential and elegance of \( \phi \)-mapping theory with singularities are that the long and complex form of the topological current \( j^\mu \) in (3) can be simply expressed in the form of a generalized function \( \delta (\vec{\phi}) \).
From the above discussions, we see that the Kernel of $\phi$–mapping plays an important role in topological current theory. So we will search for the solutions of the equations

$$\phi^a(\vec{r}, t) = 0, \quad a = 1, \cdots, n$$

by means of the implicit function theorem, and further give the dynamic form of the topological current $j^\mu$. Suppose the function $\phi^a(\vec{r}, t)$ possesses $l$ isolated zeroes. The implicit function theorem\textsuperscript{[24]} says that when these zeroes are regular points of $\phi$–mapping and require the Jacobian $D(\phi/x) \neq 0$, there is one and only one system of continuous functions of $x^0 = t$

$$\vec{x} = \vec{z}_i(t), \quad i = 1, \cdots, l,$$

(9)

which is trajectory of the $i$–th zero and is called the $i$–th one–dimensional singular line $L_i$ in the space–time $X$. On the other hand, putting the solutions (9) back into $\phi^a(x)$, we have

$$\phi^a(t, \vec{z}_i(t)) \equiv 0, \quad a = 1, \cdots, n,$$

from which one can prove that the velocity of the $i$–th vortex is determined by\textsuperscript{[24]}

$$\nu^\mu = \frac{d\vec{x}_i^\mu}{dt} = \frac{J^\mu(\phi/x)}{D(\phi/x)}|_{\vec{z}_i(t)},$$

(10)

taking account of (15), the topological current (14) can be rewritten in a simple and compact form

$$j^\mu = \rho v^\mu.$$

It is surprising that the topological current (3) just can take the same form as the current density in classical electrodynamics or hydrodynamics. The expression given by (10) for the velocity is very useful because it avoids the problem of having to specify the positions of the vortices explicitly. The positions are implicitly determined by the zeros of the order parameter field. The general expression with $J^\mu(\frac{\phi}{x})$ should be useful in looking at the motion of vortices in the presence of external fields beyond a growth kinetics context.

Following, we will investigate the topological charges of the vortices and their quantization. Let $M$ be a spatial hypersurface in $X$ with variables $x^1, \cdots, x^n$ for a given $t$, and $M_i$
a neighborhood of \( z_i(t) \) on \( M \) with boundary \( \partial M_i \) satisfying \( z_i \notin \partial M_i \), \( M_i \cap M_j = \emptyset \). Then, the generalized Winding Number \( W_i \) of \( n^o(\vec{r}, t) \) at \( z_i(t) \) can be defined by the Gauss map\(^{25}\) 
\[
W_i = \frac{1}{A(S^{n-1})(n-1)!} \int_{\partial M_i} n^*(\epsilon_{a_1\cdots a_n} n^{a_1} dn^{a_2} \wedge \cdots \wedge dn^{a_n})
\]
where \( n^* \) is the pull back of map \( n \). The generalized Winding Number is a topological invariant and is also called the degree of Gauss map\(^{26}\). It is well known that \( W_i \) is corresponding to the first homotopy group \( \pi[S^{n-1}] = \mathbb{Z} \) (the set of integers)\(^{28}\) Using the Stokes’ theorem in exterior differential form, one can deduce that
\[
W_i = \int_{M_i} \rho d^nx.
\]
Using the result in (8), we get the compact form of \( W_i \)
\[
W_i = \int_{M_i} \delta(\vec{\phi}) D\left(\frac{\vec{\phi}}{x}\right) d^nx. \tag{11}
\]
Following, by analogy with the procedure of deducing \( \delta(f(x)) \), we can expand the \( \delta \)-function \( \delta(\vec{\phi}) \) as
\[
\delta(\vec{\phi}) = \sum_{i=1}^{l} c_i \delta(\vec{x} - \vec{z}_i(t)) \tag{12}
\]
where the coefficients \( c_i \ (i = 1, \cdots, l) \) must be positive, i.e. \( c_i = |c_i| \). Substituting (12) into (11) and calculating the integral, we get
\[
W_i = \int_{M_i} \sum_{i=1}^{l} c_i \delta(\vec{x} - \vec{z}_i(t)) D\left(\frac{\vec{\phi}}{x}\right) d^nx = c_i D\left(\frac{\vec{\phi}}{x}\right)|_{\vec{x} = \vec{z}_i(t)}
\]
which gives
\[
c_i = \frac{|W_i|}{|D\left(\frac{\vec{\phi}}{x}\right)|_{\vec{x} = \vec{z}_i(t)}}.
\]
Let \( |W_i| = \beta_i \), the \( \delta \)-function \( \delta(\vec{\phi}) \) can be expressed by the zeroes of \( \phi^o(x) \) as
\[
\delta(\vec{\phi}) = \sum_{i=1}^{l} \frac{\beta_i \eta_i}{D\left(\frac{\vec{\phi}}{x}\right)|_{\vec{x} = \vec{z}_i(t)}} \delta(\vec{x} - \vec{z}_i(t)) \tag{13}
\]
where the positive integer \( \beta_i \) is called the Hopf index\(^{27}\) of \( \phi \)-mapping on \( M_i \), the obvious meaning of \( \beta_i \) is that when the point \( \vec{x} \) covers the neighborhood of the zero point \( \vec{z}_i(t) \) on \( M \)
once, the function $\tilde{\phi}(x)$ covers the corresponding region $\beta_i$ times. $\eta_i = \text{sign}D(\phi/x)z_i = \pm 1$ is the Brouwer degrees of $\phi$–mapping. The formula (13) has the topological information $\beta_i\eta_i$ and then is the generalization of ordinary $\delta$–function theory. Using this expansion of $\delta(\tilde{\phi})$ in (13), it is evidently that the topological current $j^\mu$ in (7) can be further expressed in the form

$$ j^\mu = \sum_{i=1}^{l} \beta_i \eta_i \delta(x - z_i(t)) \frac{J^\mu(\phi/x)}{D(\phi/x)}, $$

and the density of topological current $\rho$ is

$$ \rho = j^0 = \sum_{i=1}^{l} \beta_i \eta_i \delta(x - z_i(t)). $$

which are exactly the current and density of a system of $l$ vortices with topological charges $g_i = \beta_i \eta_i$ moving in the $n + 1$ dimensional space-time. The $l$ one–dimensional singular manifolds $L_i(i = 1, \cdots l)$ in the space-time $X$, which are locus of the zero points $z_i(t)$, are just the trajectory of these vortices in the space–time. The total topological charge of this system is

$$ G = \int_M \rho d^n x = \sum_{i=1}^{l} \beta_i \eta_i = \sum_{i=1}^{l} g_i. $$

In summary, from our theory, for the first time, we obtain the topological charges of the vortices $g_i = \beta_i \eta_i$ in the context of TDGL model and these charges of vortices are topological quantized in terms of the Hopf indices and the Brouwer degrees of the $\phi$–mapping. Here we see that these vortices are located at the zeros of $\tilde{\phi}(\tilde{r}, t)$, i.e. the singularities of the unit vector $\tilde{n}(\tilde{r}, t)$ and, the Hopf indices $\beta_i$ and Brouwer degree $\eta_i$ classify these vortices. In detail, the Hopf indices $\beta_i$ characterize the absolute values of the topological charges and the Brouwer degrees $\eta_i = +1$ correspond to vortices while $\eta_i = -1$ to antivortices. From (8) and (15) the total topological charge of these vortices system in TDGL model can also be expressed as

$$ G = \int_M \rho d^n x = \deg \phi \int_{\phi(M)} \delta(\tilde{\phi}) d^n \phi = \deg \phi. $$

We see that the total topological charge $G$ of these system is equal to the degree of $\phi$–mapping $\deg \phi$. And from (16) we have the obvious result that $\deg \phi = \sum_{i=1}^{l} \beta_i \eta_i$, i.e. the
degree of $\phi$–mapping is equal to the sum of the indices of $n$–component order parameter field $\tilde{\phi}$ at its zeros or the topological charge of the vortices. With the discussion mentioned above, we know that the results in this letter are obtained straightly from the topological viewpoint under the condition $D(\phi/x) \neq 0$. When this condition failed, i.e. $D(\phi/x) = 0$, there should exist some kinds of branch processes in the topological current and we will discuss this problem in other papers.

References


