Abstract

$AdS_2$ has an $SL(2, R)$ isometry group. It is argued that in the context of quantum gravity on $AdS_2$ this group is enlarged to the full infinite-dimensional 1 + 1 conformal group. Massive scalar fields are coupled to $AdS_2$ gravity and shown to have associated conformal weights $h(m)$ shifted by their mass. For integral values of $h$ primary boundary operators are obtained as $h$ normal derivatives of the scalar field. $AdS_2$ string theories arise in the ‘very-near-horizon’ limit of $S^1$-compactified $AdS_3$ string theories. This limit corresponds to energies far below the compactification scale. The dual conformal field theory has one null dimension and can in certain cases be described as the discrete light cone quantization of a two-dimensional deformed symmetric-product conformal field theory. Evidence is given that the $AdS_2$ Virasoro algebra is related to the right-moving $AdS_3$ Virasoro algebra by a topological twist which shifts the central charge to zero.
1. Introduction

Perhaps the best - understood of the family of \textit{AdS/CFT} dualities [1] is the case \textit{AdS$_3$/2D-CFT} [2,1,3-12], largely because the conformal group is infinite dimensional and greatly constrains the dynamics. In higher dimensions the field theory side is less well-understood, and the duality has largely been used to learn about field theory from gravity. In lower dimensions - namely the \textit{AdS$_2$} case - one confronts the puzzles of black holes undistracted by inessential complications. Here one hopes to learn about gravity from field theory, and new wrinkles are encountered. One such wrinkle is that \textit{AdS$_2$} has two timelike boundaries, one originating from the region inside and one from the region outside the horizon. Hence the dual one-dimensional conformal quantum mechanics does not appear to live on a connected manifold.

At present very little is understood about this \textit{AdS$_2$} case, which is the subject of this paper. Some previous investigations appear in [13]. We first consider general properties of two-dimensional quantum gravity plus matter on \textit{AdS$_2$}. While the subject of two-dimensional quantum gravity has received considerable attention$^1$, we shall see that \textit{AdS$_2$} provides an interesting new context. We then consider properties and dualities of \textit{AdS$_2$} string theories obtained by compactification of the well - understood \textit{AdS$_3$} case.

All theories of quantum gravity in two dimensions are conformal field theories [15-17]. Liouville-like theories with \textit{AdS$_2$} ground states – which necessarily are open-string-like theories on the strip – are no exception. A key distinction of the \textit{AdS$_2$} case discovered in [18,19] is that, unlike on the circle or the plane, the theory on the strip has a ground state which is annihilated by the global $SL(2,R)$ subgroup of the conformal group. This subgroup is identified with the \textit{AdS$_2$} isometry group. We accordingly argue in section 2.1 that the global $SL(2,R)$ conformal symmetry in the \textit{AdS$_2$/CFT$_1$} duality is enlarged to the full local conformal group, much as in the \textit{AdS$_3$/CFT$_2$} case [2]. In section 2.2 we consider a scalar field $\phi$ of mass $m$ coupled to gravity on \textit{AdS$_2$}. It is shown perturbatively that the effect of the mass term is to shift the dimension of the highest weight state created by the scalar field to $h = \frac{1}{2}(1 + \sqrt{1 + m^2\ell^2})$, where $\ell$ is the \textit{AdS$_2$} radius. The corresponding boundary operator for integral $h$ is $(\partial_\sigma)^h \phi$, with $\partial_\sigma$ the normal derivative. This is primary because lower derivatives of $\phi$ vanish on the boundary. Section 2.3 contains a discussion of

$^1$ A review with emphasis relevant to the present context can be found in the first few sections of reference [14].
deficit angles produced by boundary insertions and a consequent upper bound on allowed masses.

In section 3 we turn to string theory. The near-horizon geometry of the five-dimensional black hole of \([20]\) is \(AdS_3\) periodically identified on a circle of asymptotic radius \(R\) \([21]\). In section 2.1 the interpretation of this as a compactification to two dimensions is presented. In section 3.2 it is shown that the geometry of the ‘very-near-horizon’ limit, corresponding to energies much less than \(\frac{1}{R}\), is \(AdS_2\) with a constant Kaluza-Klein \(U(1)\) field strength. It is also found that in this limit the \(AdS_3\) identification lies entirely within the \(SL(2, R)_L\) subgroup of the \(SL(2, R)_L \times SL(2, R)_R\) isometry group. The dual description of the very-near-horizon limit is argued in section 3.3 to be the DLCQ of the deformed symmetric product field theory described in \([20]\). Hence in this context the one dimension of the conformal field theory in the \(AdS_2 \leftrightarrow CFT_1\) duality is null. This is naturally interpreted as half of a 2D conformal field theory rather than as conformal quantum mechanics. In section 3.4 the two-dimensional effective action arising from compactification of \(AdS_3\) is described.

Symmetries are discussed in section 4. The \(AdS_3\) string theory admits the action of two Virasoro algebras with large central charges. Only the right action survives the compactification. \(AdS_2\) on the other hand admits the action of a single Virasoro algebra which has \(c = 0\) since it is a theory of gravity. In section 4.1 we argue, assuming the existence of a suitable Hilbert space on \(AdS_2\), that the \(AdS_2\) Virasoro algebra is related to the \(AdS_3\) algebra by a twisting which shifts the central charge to zero. This twisting is similar or identical to that used in earlier constructions of topological field theories and topological gravity from ordinary supersymmetric conformal field theories \([22]\). In section 4.2 the analysis of neutral scalar fields of section 2.2 is extended to the \(U(1)\) charged scalar fields arising in \(AdS_3\) compactification. It is found that due to cancellations the lowest conformal weight of the states created by such fields is independent of the charge and the Kaluza-Klein mass correction, and hence can be small even for very massive fields.

It should be noted \(AdS_2\) string theories arise as the near-horizon limits of a wide variety of four and five -dimensional black holes without passing through an intermediate \(AdS_3\) region. It is not clear to us which of the features discovered in sections 3 and 4 are generic to all \(AdS_2\) string theories. The discussion of section 2 on the other hand, and in particular the enlargement of the \(SL(2, R)\) to the full conformal group, pertains to all theories of quantum gravity on \(AdS_2\).
2. AdS2 Quantum Gravity as Conformal Field Theory

In this section we consider general properties of two-dimensional quantum gravity coupled to matter with an AdS2 ground state. We take Newton’s constant to be small and the AdS2 radius large so that a semiclassical description is appropriate. Since AdS2 has two timelike boundaries, such theories live on the strip.\(^2\) String theory on AdS\(_2\times M^8\) can be viewed as two-dimensional gravity with an infinite tower of fields arising from massive string and Kaluza-Klein modes. In conformal gauge
\[
\text{}\left(2.1\right)\quad ds^2 = -e^{2\rho}dt^+dt^-,
\]
every two-dimensional quantum theory of gravity is a \(c = 0\) (including ghosts) conformal field theory, with conformal invariance arising as a result of diffeomorphism invariance and background independence. Hence AdS\(_2\) string theory can be viewed as conformal field theory on the strip. This is a lower dimensional analog of the philosophy of [2-3] in which AdS3 string theory is viewed as conformal field theory on the circle. Since the conformal transformations are residual diffeomorphisms, not all of the states in the theory should be regarded as inequivalent physical states. In a BRST formalism physical states will be identified as usual as BRST cohomology classes. In a ghost-free gauge one has a physical state condition of the form \(L_n|\psi> = 0\). These are the quantum versions of the constraints which determine the metric in terms of the matter fields.

2.1. AdS2 Isometries and the Virasoro Algebra

The gravitational degrees of freedom of AdS\(_2\) quantum gravity in conformal gauge are described by a Liouville theory governing the metric conformal factor \(\rho\), possibly generalized by dilaton couplings. The action for a specific case in string theory is given in section 3 below. AdS\(_2\) quantum gravity is like an open string theory on the strip owing to the two boundaries of AdS\(_2\). Hence there is only one copy of the Virasoro algebra. This has an \(SL(2, R)\) subgroup which generates the usual transformations
\[
\text{\left(2.2\right) } L_{-1} = \frac{i}{2}(e^{-2iu^+}\partial_+ + e^{-2iu^-}\partial_-),
\]
\[
L_0 = \frac{i}{2}(\partial_+ + \partial_-),
\]
\[
L_1 = \frac{i}{2}(e^{2iu^+}\partial_+ + e^{2iu^-}\partial_-),
\]
\(^2\) We will mainly consider the covering space of AdS\(_2\), which for brevity will be referred to simply as AdS\(_2\).
$u^\pm = \frac{1}{2}(\tau \pm \sigma)$ and $0 \leq \sigma \leq \pi$. As noted some time ago by D’Hoker and Jackiw [18,19], the Liouville theory with cosmological constant $-\frac{2}{\ell^2}$ has an $SL(2, R)$-invariant vacuum characterized by

$$\ell e^{-\rho} = \sin(u^+ - u^-).$$

(2.3)

This vacuum is of course the $AdS_2$ spacetime. The $SL(2, R)$ generators (2.2), originally introduced without reference to a metric, are the $SL(2, R)$ Killing vectors of $AdS_2$.

This situation contrasts with Liouville theory on the circle or plane, for which there is no vacuum preserving the $SL(2, R)_R \times SL(2, R)_L$ Virasoro subgroup except for the singular configuration $e^\rho = 0$. The absence of such an invariant vacuum has greatly complicated the analysis of the theory on the circle. Hence in this sense Liouville theory lives more naturally on the strip than on the plane.

An important conclusion here is that the global $SL(2, R)$ symmetry of $AdS_2$ string theory is enhanced to (half of) the $1+1$ conformal group, much as the global $SL(2, R)_L \times SL(2, R)_R$ symmetry of $AdS_3$ string theory is enhanced to the full $1+1$ conformal group. This conclusion follows from general properties of $AdS_2$ quantum gravity, and so applies to all $AdS_2$ string theories.

2.2. Coupling to Massive Scalars

Two-dimensional gravity may also contain scalar fields with actions of the form

$$S_\phi = -\frac{1}{4\pi} \int d^2 x \sqrt{-g}((\nabla \phi)^2 + m^2 \phi^2) = \frac{1}{8\pi} \int d^2 u (4 \partial_+ \phi \partial_- \phi - m^2 e^{2\rho} \phi^2)),$$

(2.4)

We wish to study the dynamics of such scalar fields in a perturbation expansion about the $AdS_2$ vacuum (2.3). From the kinetic term we see that $\phi$ has global scaling dimension zero. The mass term is nevertheless a marginal perturbation because $e^{2\rho}$ is dimension $(1, 1)$ operator. The wave equation

$$\nabla^2 \phi = m^2 \phi,$$

(2.5)

reduces in conformal gauge to

$$-4e^{-2\rho} \partial_+ \partial_- \phi = m^2 \phi.$$  

(2.6)

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3 The theory considered in [18,19] differs in the scalar couplings from the string theory case considered below but this is not relevant for the following.

4 An explicit demonstration in a context close to the present one can be found in [23].
It is easily verified the $AdS_2$ laplacian defined by the metric (2.3) is precisely $-\frac{4}{\ell^2}$ times the $SL(2, R)$ Casimir obtained by squaring (2.2). Hence

$$-4(-L_0(L_0 - 1) + L_-L_1)\phi = m^2\ell^2\phi. \quad (2.7)$$

For primary, normalizable solutions $\phi_h$ of (2.7) with $L_1\phi_h = 0$ and $L_0 = h$

$$4h(h - 1)\phi_h = m^2\ell^2\phi_h \quad (2.8)$$

or

$$h = \frac{1}{2}(1 + \sqrt{1 + m^2\ell^2}), \quad (2.9)$$
in agreement with [24,25,4]. $L_1\phi_h = 0$ then implies

$$\phi_h = (e^{-2iu^+} - e^{-2iu^-})^h. \quad (2.10)$$

Further solutions of (2.7) are generated by acting on $\phi_h$ with $L_{-1}$, as this commutes with the $SL(2, R)$ Casimir operator:

$$B_n(\sigma, \tau) = b_n(\sigma)e^{-i(n+h)\tau} = C_n(L_{-1})^n\phi_h, \quad n \geq 0. \quad (2.11)$$

Together with their complex conjugates $B_n^*$ these are a complete set of modes. We choose the normalization constants $C_n$ so that

$$\langle B_m|B_n\rangle = \delta_{mn} \quad (2.12)$$

with respect to the Klein-Gordon inner product

$$\langle A|B\rangle = -i\int_0^\pi d\sigma (A^*\partial_\tau B - \partial_\tau A^*B). \quad (2.13)$$

$\phi$ may then be promoted to a quantum field and expanded in an oscillator basis

$$\phi = \sum_{n=0}^\infty a_n^\dagger B_n^* + a_nB_n. \quad (2.14)$$

The canonical commutation relations for $\phi$ then imply as usual

$$[a_m, a_n^\dagger] = \delta_{mn}. \quad (2.15)$$

5
Using the $\phi$ stress tensor derived from the action (2.4), one finds that the quantum Virasoro generator $L_0^\phi$ gets a contribution

$$L_0^\phi = \sum_{n=0}^{\infty} (h + n)a_n^\dagger a_n.$$  \hspace{1cm} (2.16)

It follows that, as expected, $a_n^\dagger$ creates a state with $L_0^\phi = h + n$. It is also possible to see that $[L_1^\phi, a_n^\dagger] = 0$ so that $a_n^\dagger |0\rangle$, where $|0\rangle$ is defined by $a_n |0\rangle = 0$, is a highest weight state.\footnote{The Virasoro constraints dictate the gravitational dressing of this state, which is higher order in the perturbative regime of small Newton’s constant considered here.}

Therefore for every field of mass $m$ there is a highest weight state of weight (2.9) in the conformal field theory. In conformal field theory on the strip there is a one-to-one correspondence\footnote{Modulo possibly relevant subtleties discussed in [14].} between highest weight states and primary operators at the boundary. For integral $h$ the primary operator corresponding to $a_n^\dagger |0\rangle$ is

$$(\partial_\sigma)^h \phi(0, \tau).$$  \hspace{1cm} (2.17)

Ordinarily an operator such as (2.17) would not be primary due to the appearance of singular terms with lower derivatives of $\phi$ in the OPE with the stress tensor. However the mode expansion (2.14) for $\phi$ implies

$$(\partial_\sigma)^k \phi(0, \tau) = 0, \quad \text{for} \quad k < h,$$  \hspace{1cm} (2.18)

so that (2.17) is indeed primary.

2.3. A Bound on the Mass

The analysis of the previous section was relevant for small dimensionless Newton’s constant $G_N$ and $m^2 \ell^2$ and $h$ of order one. In string theory or Kaluza-Klein theory one encounters arbitrarily large values of $m$ and so one is forced to consider the case $h \sim 1/G_N$. In this case conformal invariance in a semiclassical expansion requires that (2.17) is replaced by the dressed operator\footnote{The coefficient of $\rho$ in the exponent will be altered if there are leading semiclassical corrections to the dimension of $e^{-h\rho}$. This is indeed the case for the pure Liouville theory considered in [26,14].}

$$e^{-h\rho(0, \tau)} (\partial_\sigma)^h \phi(0, \tau).$$  \hspace{1cm} (2.19)
For $h \sim \frac{1}{G_N}$, the semiclassical saddle point must include the $\rho$ insertion on the boundary. This problem was considered in [26] (the bulk problem is considered in [14]) where it was shown semiclassically that the corresponding operator insertions produce kinks or deficit angles in the boundary of order $h G_N$. This follows from the $\rho$ equation of motion which has a delta function source at the operator insertion. Constraining the deficit angle to be less than $\pi$ leads to maximum value $h_{\text{max}}$ of order $\frac{1}{G_N}$ for allowed operators. Using the relation (2.9) between the weight and the mass gives a bound on the mass

$$m_{\text{max}} \sim \frac{1}{1G_N}.$$  \hspace{1cm} (2.20)

In order to get a precise value for $m_{\text{max}}$ one must specify the gravitational dynamics. The values will be numerically different for the pure Liouville theory and the dilaton-modified theories encountered in string compactification. For the moment we simply remark that a bound of the form (2.20) is in qualitative agreement with the stringy exclusion principle of [4], and might provide a generalization of the principle to non-BPS states.

2.4. Remarks

In this subsection we make several remarks on the preceding results.

Roughly speaking one might attempt to decompose the central charge as

$$0 = c_{\text{total}} = c_{\text{gravity}} + c_{\text{matter}} + c_{\text{ghosts}}.$$ \hspace{1cm} (2.21)

The asymptotic growth of physical states, which is related to black hole entropy, should be determined by $c_{\text{matter}}$. Naively $c_{\text{matter}}$ could be computed from the two-point function of the matter stress tensor at short distances in the bulk. One might expect that every field $\phi$ is an independent quantum field and contributes to $c_{\text{matter}}$ because the mass term does not affect the two-point function at sufficiently short distances. However this cannot be correct since there can be an infinite number of such fields e.g. from Kaluza-Klein compactification or massive string modes. A bound of the form (2.20) alleviates, but does not seem in general to eliminate the problem. Presumably there are further quantum relations between the fields (e.g. of the kind encountered in [4]) which become very significant at short distances.

An alternate and possibly more efficient method of determining $c_{\text{matter}}$ is from the two point function of the stress tensor on the boundary. The relation (2.17) implies that $\phi$ does not contribute to the two-point function of the stress tensor at the boundary if it
has weight $h > 1$. Hence we need only consider zero or negative \((\text{mass})^2\) fields in such a computation.

It is interesting to contrast the preceding $AdS_2$ analysis with the $AdS_3$ analysis of [3] and [4]. In that context the states are also described by a conformal field theory, but it lives on the boundary of $AdS_3$ [2], whereas in the present $AdS_2$ context the conformal field theory is in the bulk of spacetime. The analysis of the scalar wave equation in [4] paralleled that of the previous paragraph. However it employed $SL(2, R)_L \times SL(2, R)_R$ rather than just a single $SL(2, R)$. Furthermore the form of the differential operators used in the $AdS_3$ case was not the familiar one of conformal field theory: the operators acted on a three dimensional space. The action reduces to the familiar form only on the boundary of $AdS_3$. In the $AdS_2$ context the connection with conformal field theory is more direct.

3. $AdS_3 \rightarrow AdS_2$

IIB string theory on $K3$ (similar statements pertain to $T^4$) has black string solutions characterized by D-onebrane (D-fivebrane) charge $Q_1 (Q_5)$ and momentum density $\frac{n}{R^2}$. These objects have a dual description as a $c = 6Q_1Q_5$ conformal field theory whose target space is the symmetric product of $Q_1Q_5$ copies of $K3$ [20]. The near-horizon geometry of the spacetime solutions is $AdS_3 \times S^3 \times K3$. Compactifying the black string on a circle to yield a five-dimensional black hole corresponds to $S^1$ compactification of $AdS_3$ [21]. In this section we describe the appearance of $AdS_2$ in the very-near-horizon limit of this compactification.

3.1. $S^1$ Compactification

We begin with the near-horizon $AdS_3$ geometry for the extremal five-dimensional black hole with charges $(Q_1, Q_5, n)$

$$ds^2 = \ell^2 T^2 (dx^5 + \frac{dt}{R})^2 + U^2 \left( (Rdx^5)^2 - dt^2 \right) + \ell^2 \frac{(dU)^2}{U^2}. \quad (3.1)$$

t here is the asymptotic time coordinate in the full spacetime solution, while the normalization of $x^5$ is such that

$$x^5 \sim x^5 + 2\pi, \quad (3.2)$$
and all factors of the asymptotic $S^1$ radius $R$ are explicit in the metric. We further define

\[ T = \sqrt{\frac{n}{k}}, \]
\[ \ell^4 = g_6^2 k, \]
\[ k = Q_1 Q_5, \tag{3.3} \]

where $g_6$ is the six-dimensional string coupling and the dimensionless quantity $T$ is related to the temperature $T_L$ of the left-moving modes of the symmetric product conformal field theory by $\pi R T = T_L$. The classical supergravity limit is $g_6 \to 0$ with $g_6^2 n$, $g_6 Q_1$ and $g_6 Q_5$ fixed and large.

To get from AdS$_3$ to AdS$_2$ we decompose

\[ ds_3^2 = ds_2^2 + \ell^2 e^{2\psi}(dx^5)^2 + \ell^2 e^{2\psi} A_t^2 dt^2 + 2\ell^2 e^{2\psi} A_t dx^5, \tag{3.4} \]

with

\[ e^{2\psi} = T^2 + \frac{R^2 U^2}{\ell^4}, \]
\[ A_t = \frac{T^2}{R} e^{-2\psi}. \tag{3.5} \]

This yields

\[ ds_2^2 = -\frac{R^2 U^4}{\ell^6 T^2 + \ell^2 R^2 U^2} dt^2 + \frac{\ell^2}{U^2} dU^2. \tag{3.6} \]

### 3.2. The Very-Near-Horizon Region

The metric (3.6) reduces to the AdS$_2$ metric in the “very-near horizon” limit $\lambda \equiv \frac{U^2 R^2 k}{\ell^4 n} \to 0$, in which the $S^1$ radius $\psi$ is constant. Radii of order $U_0$ correspond to excitation energies $\mu$ above extremality (see section 3.4 below) of order $\mu \sim \frac{k^2 R^3 U_0^4}{n \ell^8} \sim \lambda^2 \frac{U_0}{R}$. Hence for $\lambda \to 0$ one is below the black hole mass gap [27]. The usual excited states are absent in this region. In terms of the dual symmetric-product conformal field theory this means the number of right-moving excitations obeys $n_R = 0$. The ground states (characterized by $n_L$) remain and should be described in this limit.

In addition to the ground state there is the possibility of a continuum of arbitrarily low-lying modes corresponding to the separation of the black hole into separate pieces. These modes are ignored in most discussions of black hole dynamics. In terms of the dual conformal field theory these are excitations of the Coulumb branch. This is an interesting subject and perhaps ultimately essential for a complete understanding of AdS$_2$. However
at present we do not understand the description of these modes in detail and simply ignore them for the rest of the paper.

One may also consider the region of $U^2$ much, but not infinitely smaller than $\ell_4^n$, in which case $\lambda$ is small but nonzero and the metric is nearly $\text{AdS}_2$. AdS$_2$/CFT$_1$ duality should yield approximate relations in this region in the spirit of [28]. This corresponds to excitation energies $\mu$ much less than $n_R$, or $n_R \ll n_L$. This is the regime of interest for the purpose of analyzing the near-extremal entropy.

In the very-near-horizon limit (3.6) reduces to

$$ds^2 = - \frac{\ell^2}{(t^+ - t^-)^2} dt^+ dt^-,$$

with $t^\pm = t \pm \frac{\ell^4 R}{RU^2}$. The length of the internal circle is

$$\ell e^{\psi} = \ell \sqrt{\frac{n}{k}}.$$  (3.8)

The Kaluza-Klein gauge field strength is

$$F_{+-} = \partial_+ A_- - \partial_- A_+ = - \frac{1}{T(t^+ - t^-)^2} = \frac{2\epsilon^+ - \ell^2}{\ell^2} \sqrt{\frac{k}{n}}.$$  (3.9)

Note that the asymptotic modulus $R$, which is a fixed scalar, drops out of the very-near-horizon geometry.

The very-near-horizon geometry can be described directly as a quotient of AdS$_3$ without a limiting procedure. It is useful to transform to Poincare coordinates on AdS$_3$ given by

$$w^+ = \frac{R}{2T} e^{2T(x^5 + \frac{\pi}{4})},$$

$$w^- = (Rx^5 - t) - \frac{\ell^4 T}{RU^2},$$

$$y = \frac{\ell^2}{U} e^{T(x^5 + \frac{\pi}{4})}$$

in which the metric becomes

$$ds_3^2 = \frac{\ell^2}{y^2} (dw^+ dw^- + dy^2).$$  (3.10)

The identification $x^5 \sim x^5 + 2\pi$ corresponds to

$$w^+ \sim e^{4\pi T} w^+, \quad w^- \sim w^- + 2\pi R,$$

$$y \sim e^{2\pi T} y.$$  (3.11)
The very-near-horizon limit can be reached by the diagonal $SL(2,R)_L \times SL(2,R)_R$ transformation
\begin{align}
w^\pm & \to w^\pm, \\
y & \to \frac{y}{\lambda'},
\end{align}
for $\lambda' \to 0$, since the horizon is at $y = \infty$. This transformation takes $R \to \lambda' R$, so the limit is equivalent to $R \to 0$. In this case (3.12) is a purely $SL(2,R)_L$ transformation, and contains no $SL(2,R)_R$ action. Hence we recover the construction of [29] of the near-horizon $AdS_2$ geometry as an $SL(2,R)_L$ quotient of $AdS_3$. As we shall see later, this leads to a somewhat subtle relation between the unbroken $SL(2,R)_R$ isometries of $AdS_3$ and the $SL(2,R)$ isometries of $AdS_2$.

The two-dimensional $AdS_2$ geometry resulting from the quotient (3.12) with $R$ set to zero is most simply expressed in terms of\footnote{Alternately, $u = x^5 + (t/R) + (\ln R/2T)$, $t^\pm = Rx^5 - t \pm (\ell^4 T/R^2)$.}
\begin{align}
w^- &= t^-, \\
w^+ &= \frac{1}{2T} e^{2Tu}, \\
y &= \sqrt{\frac{t^+ - t^-}{2T}} e^{Tu}.
\end{align}
The inverse transformation is
\begin{align}
t^- &= w^-, \\
t^+ &= \frac{y^2}{w^+} + w^-, \\
u &= \frac{1}{2T} \ln 2Tw^+.
\end{align}
One then has
\begin{equation}
ds_3^2 = \frac{\ell^2(dt^+ - dt^-)^2}{4(t^+ - t^-)^2} + \frac{\ell^2Tdudt^+ + dt^-}{(t^+ - t^-)} + \ell^2 T^2 du^2. \tag{3.16}
\end{equation}
Imposing the null identification
\begin{equation}
u \sim u + 2\pi, \tag{3.17}
\end{equation}
and using (3.4) gives
\begin{align}
e^{2\psi} &= T^2, \\
A_\pm &= \frac{1}{2T(t^+ - t^-)}, \\
ds_2^2 &= \frac{-\ell^2 dt^+ dt^-}{(t^+ - t^-)^2}. \tag{3.18}
\end{align}
The coordinates $t^\pm$ cover only the region outside the horizon, as illustrated. Coordinates which cover all of (the universal cover of) $AdS_2$ are\footnote{Alternately, for $\tau = u^+ + u^-$, $\sech \rho = \sin(u^+ - u^-)$, $4ds^2 = -\ell^2 \cosh^2 \rho d\tau^2 + \ell^2 d\rho^2$.} $t^\pm = \tan u^\pm$. The metric is
\[
d s^2 = -\frac{\ell^2 du^+ du^-}{\sin^2(u^+ - u^-)}. \tag{3.19}\]
$\sigma = (u^+ - u^-)$ runs from 0 to $\pi$ while $\tau = u^+ + u^-$ is unrestricted. The horizons are at $u^\pm = \pm \frac{\pi}{2}$.

Note that $AdS_2$ has two timelike boundaries $\sigma = 0, \pi$, one inside and one outside the horizon. The electric field strength (3.9) can be viewed as arising from charges $\frac{k\ell^2}{2} e^{3\psi} \epsilon^{++} F_{+-} = \pm n$ fixed at the boundaries.

3.3. The Dual Description

String theory on $AdS_3$ has a dual description as a $c = 6k$ (deformed) symmetric product conformal field theory. This theory lives on the cylindrical boundary of $AdS_3$ with $y = 0$. This boundary has null coordinates $w^\pm$. The action of the identification (3.12) in the very near horizon limit for which $R \to 0$ is simply a shift of $\ln w^+$, with no action on $w^-$. Hence the dual representation of the very-near horizon string theory is a DLCQ conformal field theory. This suggests a connection with the matrix model [30,31].

We note that the one dimension of the $D = 1$ conformal field theory dual to $AdS_2$ string theory is a null ($w^-$) coordinate. This provides a natural framework for the action of the Virasoro algebra, whose existence was argued for on general grounds in section 2. On the other hand the general arguments of [1,25] suggest a formulation of the dual theory as a conformal quantum mechanics on the two timelike boundaries of $AdS_2$. It would be interesting to understand this formulation in detail.

It is intriguing to observe that the general $AdS_D \leftrightarrow CFT_{D-1}$ duality boils down in this $D = 2$ context to a duality between two conformal field theories, one with $c = 0$ on the strip and another with $c = 6k$ on the DLCQ cylinder. Perhaps this $D = 2$ duality can be understood in more conventional field theoretic terms.
3.4. Two-Dimensional Effective Action

The gravity sector in $AdS_3$ is governed by the action

$$S_3 = \frac{k}{4\pi \ell} \int d^3x \sqrt{-g}(R + \frac{2}{\ell^2}).$$  \hfill (3.20)

Additional fields suppressed here include the $SU(2)_L \times SU(2)_R$ level $k$ Chern-Simons gauge fields. (3.20) reduces to a two-dimensional metric, scalar and $U(1)$ gauge field governed by

$$S_2 = \frac{k}{2} \int d^2x \sqrt{-g} \left( e^\psi (R + \frac{2}{\ell^2}) - \frac{\ell^2}{4} e^{3\psi} F^2 \right).$$  \hfill (3.21)

This action has a solution with $R = -\frac{8}{\ell^2}$ and constant $\psi$ and $F$ given by (3.8) - (3.9), corresponding to the extremal black hole. Additionally there are near extremal solutions given by

$$ds_2^2 = -\frac{U^2(U^2 - U_0^2)}{\ell^2(U^2 + U_0^2 \sinh^2 \sigma)} dt^2 + \frac{\ell^2}{U^2 - U_0^2} dU^2,$$

$$e^{2\psi} = \frac{U_0^2 + U^2 \sinh^2 \sigma}{\ell^4},$$

$$A_t = \frac{U_0^2 \sinh 2\sigma}{2(U^2 + U_0^2 \sinh^2 \sigma)},$$

with $U_0$ and $\sigma$ defined by

$$n = \frac{kU_0^2 \sinh 2\sigma}{2\ell^4},$$

$$\mu = \frac{kU_0^2 e^{-2\sigma}}{2\ell^4}.$$  \hfill (3.22)

$n$ is related to the momentum around the $S^1$ while $\mu$ is the energy above extremality.

Note that for large $U$ and nonzero $\mu$, the radius $\psi$ blows up. This means that asymptotically one effectively returns to $AdS_3$.\hfill (3.23)

Hence the near extremal solution can be understood in the $AdS_2$ context only for sufficiently small $U$ and $U_0$. This requires a small energy $\mu$. To leading non-trivial order for such small $\mu$ and fixed $n$ the entropy is

$$S_{BH} = 2\pi ke^{\psi(U_0)} = 2\pi \sqrt{k} n + \pi \sqrt{2k}\mu.$$  \hfill (3.24)

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10 In conformal gauge, $S_2 = \frac{k}{2} \int d^2x (4e^{\psi} \partial_- \partial_+ \rho + \frac{1}{\ell^2} e^{\psi+2\rho} + \ell^2 e^{3\psi-2\rho} F^2_{+-})$. For fixed charge the last term is $\frac{k}{2} \int d^2x e^{-3\psi+2\psi}(\frac{n}{\ell k})^2$.

11 $\psi$ is an $m^2\ell^2 = 8$ fixed scalar which according to (2.9) corresponds to an irrelevant $h = 2$ perturbation of the $AdS_2$ conformal field theory. Hence exciting this field corresponds to changing the $AdS_2$ boundary conditions.
4. Symmetries

In this section we consider the symmetries of $AdS_2$ string theory and its relation to those of $AdS_3$.

4.1. The Unbroken $SL(2,R)$

The quotient (3.12) in the very-near-horizon limit $R \to 0$ leaves intact the full $SL(2,R)_R$ isometry group of $AdS_3$. One expects this group is related to the $SL(2,R)$ isometry group of $AdS_2$. Using the coordinate transformations (3.15) and the $AdS_3$ isometries one finds that the $AdS_3$ and $AdS_2$ isometries are related by

\[
H_{-1} = i \frac{\partial}{\partial w} = i \left( \frac{\partial}{\partial t^+} + \frac{\partial}{\partial t^-} \right), \\
H_0 = i(w^- \frac{\partial}{\partial w^-} + \frac{1}{2} y \frac{\partial}{\partial y}) = i(t^+ \frac{\partial}{\partial t^+} + t^- \frac{\partial}{\partial t^-}), \\
H_1 = i((w^-)^2 \frac{\partial}{\partial w^-} + w^- y \frac{\partial}{\partial y} - y^2 \frac{\partial}{\partial w^+}) = i(t^+ \frac{\partial}{\partial t^+} + t^- \frac{\partial}{\partial t^-} - \frac{t^+ - t^-}{2T} \frac{\partial}{\partial u}).
\] (4.1)

We see that the $H_{-1}$ and $H_0$ generators are identified for $AdS_2$ and $AdS_3$, but there is a shift in $H_1$. Since $u$ is a coordinate on the compactified $S^1$, the extra term can be interpreted as a gauge transformation. Hence the $SL(2,R)_R$ $AdS_3$ isometries reduce to the $SL(2,R)$ $AdS_2$ isometry plus a gauge transformation.

Given such a relation between the $AdS_2$ and $AdS_3$ isometry groups one naturally expects a relation between the full conformal groups. To find this we note that the $AdS_2$ action in (4.1) can be written as

\[
H_j = i(\epsilon_j^+ \partial_+ - \frac{\partial_+ \epsilon_j^+}{4T} \partial_u + \epsilon_j^- \partial_- + \frac{\partial_- \epsilon_j^-}{4T} \partial_u),
\] (4.2)

with

\[
\epsilon_j^\pm = (t^\pm)^{j+1},
\] (4.3)

for $j = 0, \pm 1$. For general integral values of $-\infty < j < \infty$ one easily finds that the $H_k$ obey the classical Lie bracket relation

\[
\{H_j, H_k\}_{L.B.} = i(j - k)H_{j+k}.
\] (4.4)

One also finds for all values of $j$ using the coordinate transformation (3.15) that the action of $H_k$ near the boundary $y = 0$ of $AdS_3$ is

\[
H_k = i(\zeta_k \frac{\partial}{\partial w^-} + \partial_- \zeta_k \frac{y \partial}{2 \partial y} - \partial_+ \zeta_k \frac{y^2}{2 \partial w^+}) \left( 1 + O(y^2) \right),
\] (4.5)

14
where $\zeta_k^- = (w^-)^{k+1}$. This expression - including the subleading terms indicated - is the same as that given in [2] for diffeomorphisms preserving the AdS$_3$ boundary conditions. Hence the $H_k$ generate the right-moving AdS$_3$ conformal transformations.

The precise form for all values of $y$ of the generators (4.1) of the SL(2,R) subgroup is fixed by demanding that they are Killing vectors of the AdS$_3$ metric. The other generators are identified only by their action at infinity and in general have no preferred form in the interior. (4.2) for general $j$ is a convenient choice for our purposes.

Let us now assume that we can define a quantum theory on AdS$_2$ with operators $T$ and $U$ corresponding to the full $c = 0$ stress tensor and the appropriately normalized conformal current associated to the Kaluza-Klein $U(1)$ gauge symmetry. Then the diffeomorphisms (4.2) are generated by moments of the twisted stress tensor

$$\tilde{T}_{\pm\pm} = T_{\pm\pm} \mp \partial_{\pm} U_{\pm}. \quad (4.6)$$

Twisted Virasoro generators $\tilde{L}_n$ can be defined in terms of $\tilde{T}$. This yields the general relation between the right Virasoro generators $L_n^{(3)}$ of AdS$_3$ and those of AdS$_2$:

$$L_n^{(3)} = \tilde{L}_n = L_n + (n + 1)U_n. \quad (4.7)$$

Energy conservation $T_{--} = T_{++}$ as well as $\tilde{T}_{--} = \tilde{T}_{++}$ on the boundary requires that the current $U$ obeys Dirichlet boundary conditions:

$$U_+(\sigma) = -U_-(\sigma) \quad \text{for} \quad \sigma = 0, \pi, \quad (4.8)$$

with respect to which the modes of $U$ in (4.7) are defined. These boundary conditions correspond to fixed total charge or fixed electric flux (3.9). The level of the $U(1)$ current algebra can be deduced from (4.7) and the assumption that the AdS$_3$ central charge is $c = 6k$ while the AdS$_2$ central charge is $c = 0$. One finds

$$[U_m, U_n] = -\frac{km}{2}\delta_{m+n}. \quad (4.9)$$

It would be of interest to compute this level directly.

The assumption that a suitable quantum theory can be defined on AdS$_2$ is perhaps not as benign as it sounds. In order to do so one must specify boundary conditions on all the fields, and perhaps also understand the ground state entropy in (3.24). The problem

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12 The charge density $U_t$ does not include the fixed boundary charges $\pm n$. 

15
is analogous to that of finding the Hilbert space in DLCQ, and similar subtleties may be encountered. A further subtlety is the possibility of singleton degrees of freedom which are confined to the boundary of the strip but could interact with the bulk degrees of freedom. We note that the $AdS_2$ Hilbert space is not the same as the $AdS_3$ Hilbert space: they should agree only after the Virasoro constraints are imposed on the former. Indeed if the two-dimensional $L_n$s are translated back to $AdS_3$ they do not generate diffeomorphisms of the form (4.5) - the last term is missing - and hence create states which fail to obey the $AdS_3$ boundary conditions of [2] by a coordinate transformation. Hence the unconstrained $AdS_2$ Hilbert space is larger than $AdS_3$ by spurious states, which are hopefully eliminated by constraints.

In addition to the $U(1)$ gauge symmetry arising from the $S^1$ isometry there is an $SU(2)_L \times SU(2)_R$ gauge symmetry from the isometries of $S^3$. This appears to lead to the large $N = 4 \ A_\gamma$ algebra [32]. Since $c = 0$ it is the degenerate case $\gamma = 0$. This algebra is known [32] to contain the standard $N = 4$ algebra as a subalgebra. The relation between the Virasoro generators is exactly (4.7).

The twisting (4.7) of the stress tensor has also appeared as a central formula in the study of topological field theory [22]. In that context one begins with a conventional supersymmetric conformal field theory with a unitary Hilbert space. This corresponds to the $AdS_3$ conformal field theory in the present discussion. The stress tensor is then twisted to shift the central charge to zero exactly as in (4.7). Once the central charge is zero, the theory can be reinterpreted as a theory of gravity. Since the partition function for gravity is independent of a choice of background metric this is a topological field theory.

4.2. Charged Kaluza-Klein Modes

$AdS_3$ string theory has a tower of scalar fields in short BPS multiplets of various masses as described in [4,5,6]. For every scalar field of mass $m$ on $AdS_3$ there is a corresponding scalar field of mass $m$ on $AdS_2$ obtained from the constant mode on $S^1$. For each of these fields there is a highest weight state with weight given by (2.9) as $h = \frac{1}{2}(1 + \sqrt{1 + m^2 T^2})$, and a corresponding primary operator in the boundary conformal field theory.

The full Kaluza-Klein tower of fields $\phi_q$ on $AdS_2$ arises from $S^1$ modes $e^{i q u}$, where $q$ is an integer. These states have charge $q$ and mass

$$m_q^2 = m^2 + \frac{q^2}{\ell^2 T^2}.$$  \hspace{1cm} (4.10)
where \( m \) is the \( AdS_3 \) mass of the scalar field. We wish to find the weight of the highest weight state created by \( \phi_q \), following the \( q = 0 \) analysis of section 2. The \( \phi_q \) wave equation is

\[
D^2 \phi_q = m_q^2 \phi_q. \tag{4.11}
\]

Using \( D = \nabla - iqA \), the conformal gauge metric (2.3) on the strip, equation (4.10) and the gauge

\[
A_- = A_+ = \frac{\cos(u^+ - u^-)}{2T \sin(u^+ - u^-)}, \tag{4.12}
\]

(4.11) can be rewritten

\[
(-4 \sin^2(u^+ - u^-) \partial_+ \partial_- + \frac{iq}{T} \sin 2(u^+ - u^-)(\partial_+ + \partial_-) - \frac{q^2}{T^2} \sin^2(u^+ - u^-)) \phi_q = m^2 \phi_q. \tag{4.13}
\]

In terms of the twisted Virasoro generators on the strip

\[
\tilde{L}_n = \frac{i}{2}(e^{2niu^+} \partial_+ + e^{2niu^-} \partial_-) + \frac{in}{4T}(e^{2niu^+} - e^{2niu^-}). \tag{4.14}
\]

(4.13) has the simple form

\[
-4(-\tilde{L}_0(\tilde{L}_0 - 1) + \tilde{L}_{-1}\tilde{L}_1)\phi_q = m^2 \ell^2 \phi_q. \tag{4.15}
\]

It follows that the highest weight solutions obeying \( \tilde{L}_1 \phi_{h,q} = 0 \) have a weight \( h \) which is independent of \( q \) and given by (2.9). The wave function is

\[
\phi_{h,q} = (e^{-2iu^+} - e^{-2iu^-})^h e^{-\frac{qT}{2}(u^+ - u^-)}. \tag{4.16}
\]

We note that the boundary operators \( (\partial_\sigma)^k \phi_q(0, \tau) \) vanish for \( k < h \) independently of \( q \). To summarize, for every field on \( AdS_3 \) of mass \( m \) and associated weight \( h \), there is a tower of perturbative\(^{13}\) fields labeled by the Kaluza-Klein charge \( q \). These have \( q \) dependent masses (4.10) but \( q \)-independent weights (2.9). In particular all the charged Kaluza-Klein modes of the \( m^2 = 0 \) scalars on \( AdS_3 \) are \( h = 1 \) scalars on \( AdS_2 \). These will all have non-vanishing boundary stress tensor and contribute to boundary computation of the central charge.

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