ABSTRACT. This paper analyses the effect of low amplitude friction and noise in accelerating phase space transport in time-independent Hamiltonian systems that exhibit global stochasticity. Numerical experiments reveal that even very weak non-Hamiltonian perturbations can dramatically increase the rate at which an ensemble of orbits penetrates obstructions like cantori or Arnold webs, thus accelerating the approach towards an invariant measure, i.e., a microcanonical population of the accessible phase space region. An investigation of first passage times through cantori leads to three conclusions, namely: (i) that, at least for white noise, the detailed form of the perturbation is unimportant, (ii) that the presence or absence of friction is largely irrelevant, and (iii) that, overall, the amplitude of the response to weak noise scales logarithmically in the amplitude of the noise.

WHY CONSIDER FRICTION AND NOISE?

In general, very weak non-Hamiltonian perturbations will only have very weak effects on properties of flows in time-independent Hamiltonian systems which are integrable or near-integrable and admit no global stochasticity. This also seems to be true for systems completely dominated by chaos where regular orbits are virtually nonexistent. However, low amplitude non-Hamiltonian perturbations can have important qualitative effects on more complex Hamiltonian systems that admit significant measures of both regular and chaotic orbits. For example, weak noise will serve as a source of extrinsic diffusion that can dramatically accelerate phase space transport through cantori (for $D = 2$) or along Arnold webs (for $D \geq 3$).

Consider, e.g., flows in two-dimensional systems. Here one knows that, in the absence of friction and noise, cantori [1,2], fractured KAM tori associated with the breakdown of integrability that contain a cantor set of holes, can partition a single connected chaotic phase space region into separate parts which, albeit not completely disjoint, are distinct in the sense that a chaotic orbit starting in one part of the phase space will remain stuck in...
that part for a long time before wending its way through one or more holes in the cantori to access another region [3,4]. However, introducing even very weak friction and noise can dramatically increase the rate at which orbits pass through these holes, thus allowing orbits to probe the entire accessible phase space much more quickly. That noise can accelerate phase space diffusion has been long known to dynamicists studying various maps, dating back at least to the work of Lieberman and Lichtenberg [5] in the 1970’s. However, the details have not received all that much attention, especially for continuous systems.

But why should one care? Why is this phenomenon important in the real world? The crucial observation here is that there is no such thing as a truly isolated system. Every system in nature is coupled to at least some degree to its surrounding environment. The important point then is that, in many cases, one should expect that the coupling of a system to an external environment can be modeled as resulting in friction and noise, related by a Fluctuation-Dissipation Theorem [6,7] (although there are examples where such a picture not justified [8]). Indeed, the assumption of a system coupled by a Fluctuation-Dissipation Theorem to an environment, idealised as a heat bath characterised by some temperature $\Theta = k_B T$, is one powerful starting point for modern theories of nonequilibrium statistical mechanics (see, e.g., the textbook by Kubo, Toda, and Hashitsume [9]).

Most modelling of this sort involves the assumption of a composite entity of system plus environment which is characterised by a time-independent Hamiltonian $H$. One might therefore worry that this picture does not extend naturally to a cosmological setting where, in the average comoving frame, the Hamiltonian $H$ typically acquires an explicit time-dependence. Fortunately, however, the assumption of a time-independent $H$ is not essential [10]. Following Caldeira and Leggett [6,7], it is natural to write the composite Hamiltonian $H$ as a sum

$$H = H_{\text{sys}} + H_{\text{bath}} + H_{\text{int}},$$

where the system Hamiltonian $H_{\text{sys}}$ is completely arbitrary, $H_{\text{bath}}$ is idealised as the Hamiltonian for a collection of linearised excitations, i.e., “phonons,” with (in general) time-dependent frequencies, and the interaction $H_{\text{int}}$ is an arbitrary function of the system variables but linear in the bath variables. (These restrictions on $H_{\text{bath}}$ and $H_{\text{int}}$ assure that each mode of the environment is only weakly coupled to the system, so that the environment can be visualised as a thermal “bath.”) In this setting, one can always integrate out the explicit dependence on bath variables to derive an exact nonlocal (in time) Langevin equation for the system; and, if $H_{\text{bath}}$ is time-independent, this exact equation (and, presumably, any reasonable Markov approximation thereunto) will satisfy a Fluctuation-Dissipation Theorem, regardless of the possible time-dependence of $H_{\text{sys}}$ and $H_{\text{int}}$. This implies in particular that, for the case of a conformally static Friedmann cosmology, one has a cosmological Fluctuation-Dissipation Theorem whenever the environment can be approximated as a conformally coupled field, e.g., electromagnetic blackbody radiation.$^b$

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$^b$In this connection, it should be noted [11] that, in the dipole approximation, the Hamiltonian
The natural inference is that, in a variety of different settings, including many relevant to astronomical systems, weak couplings to an external environment will result in non-Hamiltonian perturbations which could have important physical implications. However, the proper form for the noise is not always obvious. The simplest model in terms of which to couple a system to its surroundings, the independent oscillator model [12,13] (which can be shown to be equivalent to many other phenomenological models that have been considered in the past [11]), leads immediately to additive white noise, i.e. state-independent noise that is delta-correlated in time. However, this simple form for the noise is a direct consequence of the assumptions (i) that $H_{\text{int}}$ is linear in the system variables and (ii) that the bath phonons are characterised by an ohmic distribution, i.e., a spectral distribution $\propto \omega^2 d\omega$ with appropriate cutoffs. Allowing $H_{\text{int}}$ to involve the system variables in a more complicated fashion leads to multiplicative (i.e., state-dependent) noise; allowing for a different spectral density leads to coloured noise (i.e., noise that is not delta-correlated in time).

When considering a galaxy embedded in a rich cluster, one is confronted with a system which, in many cases, is significantly impacted by its surrounding environment. Particularly close encounters between galaxies in a cluster, e.g., those resulting in physical collisions, probably cannot be viewed as “random” events. However, large numbers of relatively weak interactions probably can be viewed as a source of friction and noise, although there is no obvious reason why the noise associated with these interactions can be approximated as delta-correlated in time.

Another piece of physics which one might hope to model as friction and noise, again arising in galactic astronomy, is discreteness effects reflecting the fact that a galaxy is comprised of a collection of nearly point mass stars, rather than the smoothed-out continuum assumed in the context of a description based on the collisionless Boltzmann (i.e., gravitational Vlasov) equation. In the context of a smooth one-particle distribution function, these discreteness effects are typically described by a Fokker-Planck, or Landau, equation, which involves a velocity-dependent coefficient of dynamical friction and multiplicative noise (diffusion) related by a self-consistent Fluctuation-Dissipation Theorem [14]. Superficially this source of friction and noise might seem completely different from the aforementioned effects associated with an external environment. However this is not really so! In this setting, one can view the full many-particle dynamics as the composite entity of system plus environment, the reduced one-particle dynamics as the system, and couplings to higher order correlations ignored in a collisionless description as interactions that serve as a source of friction and noise [15].

In the past, a good deal of work has focused on the effects of relatively strong friction and describing the interaction of an electron with a radiation field is, when formulated in an inertial frame, equivalent to the independent oscillator model [12,13], which is perhaps the simplest “realistic” example of the Hamiltonian (1). Transforming to comoving coordinates makes $H_{\text{sys}}$ and $H_{\text{int}}$ time-dependent, but the conformal invariance of the electromagnetic field implies that $H_{\text{bath}}$ remains time-independent.

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noise in triggering barrier penetration and other phenomena which proceed on the natural relaxation time $t_R$ associated with the system’s approach towards thermal equilibrium (see, e.g., [14,16] and numerous references cited therein.). This is not the problem of interest here. Rather, the objective of the work described in this paper has been to focus on much weaker perturbations, where $t_R$ is much longer than any time scale of interest, and to determine the extent to which friction and noise have significant statistical effects on the evolution of ensembles of orbits already on time scales $\ll t_R$.

The numerical experiments described in the next two sections were performed with the aim of answering three basic questions:

1. How should one visualise the effects of accelerated phase space transport induced by friction and noise?

2. How does the size of the effect scale with the amplitude of the perturbation?

3. To what extent do the details of the perturbation matter? One knows, for example, that multiplicative noise can drive a system towards thermal equilibrium much more quickly than additive noise [17], and, as such, it would seem natural to ask whether multiplicative noise can also accelerate diffusion through cantori and Arnold webs more than additive noise.

**INVARIANT AND NEAR-IN Variant Distributions**

The computations described in this paper involved integrating Langevin equations of the form

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = -\nabla \Phi - \eta v + F,$$

these corresponding to motion in a time-independent Hamiltonian $H = \frac{v^2}{2} + \Phi(\mathbf{r})$ which is perturbed by friction and noise. The quantity $\eta$ represents a coefficient of dynamical friction which, in general, can be a nontrivial function of both $\mathbf{r}$ and $\mathbf{v}$. The quantity $F$ is a “random” force, idealised as Gaussian white noise, which is characterised completely by the statistical properties of its first two moments. Specifically,

$$\langle F_i(t) \rangle = 0 \quad \text{and} \quad \langle F_i(t_1)F_j(t_2) \rangle = 2\eta\Theta \delta_{ij}\delta_D(t_1 - t_2),$$

where $i$ and $j$ label vector components and angular brackets denote an ensemble average. The first of these conditions ensures that the average force vanishes identically. The second ensures that the autocorrelation function is delta-correlated in both direction and time. The normalisation in eq. (3) imposes a Fluctuation-Dissipation Theorem which ensures that, for $t \to \infty$, an arbitrary ensemble of orbits evolved with eqs. (2) will approach a canonical distribution with temperature $\Theta$. 

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To date, integrations have focused on three specific two-dimensional potentials, namely the sixth order truncation of the Toda lattice potential \[18\],
\[
\Phi(x,y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}x^3 + \frac{1}{2}x^4 + x^2 y^2 + \frac{1}{2}y^4 + x^4 y + \frac{2}{3}x^2 y^3 - \frac{1}{3}y^5 + \frac{1}{5}x^6 + x^4 y^2 + \frac{1}{3}x^2 y^4 + \frac{11}{45}y^6,
\]
the so-called dihedral potential [19] for one particular set of parameter values, i.e.,
\[
\Phi(x,y) = -(x^2 + y^2) + \frac{1}{4}(x^2 + y^2)^2 - \frac{1}{4}x^2 y^2,
\]
and the sum of isotropic and anisotropic Plummer potentials [20] for specified core radii and anisotropy parameters, i.e.,
\[
V(x,y) = - \frac{1}{(c^2 + x^2 + y^2)^{1/2}} - \frac{m}{(c^2 + x^2 + ay^2)^{1/2}},
\]
with \(c = 20^{2/3} \approx 0.136\), \(a = 0.1\) and \(m = 0.3\). In all three cases, the constants were so chosen that, in absolute units, a characteristic crossing time \(t_{cr} \sim 1\). This implies that, if one visualises these potentials as representing large galaxies like the Milky Way, the Hubble time \(t_H \sim 100 - 200\).

As discussed more carefully elsewhere [20], these three potentials manifest very different symmetries. Indeed, the only obvious feature which they share is that, for a variety of energies, they admit significant measures of both regular and chaotic orbits, so that the chaotic phase space regions are significantly impacted by cantori. The fact that, nevertheless, similar qualitative results were obtained for orbits evolved in all three potentials can thus be interpreted as evidence that the basic conclusions are probably robust.

Because all three potentials yielded similar results, the largest number of calculations were performed for the dihedral potential, the most inexpensive computationally, which corresponds physically to a slightly “squared” Mexican hat potential. In all cases, the orbits were computed using a fourth order Runge-Kutta algorithm, noise being implemented using an algorithm developed by Griner et al [21]. Most of the integrations were performed using a time step \(\delta t = 10^{-3}\). It was verified that a shorter time step \(\delta t = 10^{-4}\) does not yield significantly different results.

The first class of experiments to be performed involved tracking the evolution of ensembles of chaotic initial conditions of fixed energy \(E\), selected from some small phase space region in the center of the stochastic sea far from any important cantori. These initial conditions were first evolved into the future in the absence of any friction or noise by integrating the deterministic Hamilton equations. They were then reintegrated allowing for friction and noise of variable amplitude. All these experiments assumed additive white noise and friction characterised by a constant \(\eta\), the two quantities being related by a Fluctuation-Dissipation Theorem. The temperature was frozen at a value \(\Theta \sim E\) and the amplitude of the perturbing influences was varied by systematically changing the value of \(\eta\).
In the absence of friction and noise, such orbit ensembles exhibit a two-stage evolution [20,22]. The first stage involves a rapid coarse-grained evolution, proceeding exponentially in time, towards a phase space distribution which is near-invariant in the sense that, once achieved, it only exhibits significant changes on a much longer time scale. Basically, this near-invariant distribution corresponds to a distribution characterised by a nearly constant number density in those portions of the constant energy phase space hypersurface that are not blocked by cantori and a near-zero density everywhere else. The second stage involves a much slower evolution towards what appears to be a true invariant distribution, as orbits in the ensemble diffuse through cantori to access phase space regions that were avoided systematically over shorter time scales. This final invariant distribution corresponds to a microcanonical population of the accessible chaotic regions, i.e., a uniform (in canonical coordinates) population of those portions of the phase space that are accessible to orbits with the specified initial conditions. The time scale for the first stage of the evolution is set at least approximately by the largest short time Lyapunov exponents for the orbits in the ensemble, which determine how fast the ensemble will disperse. Typically this time $\sim t_{cr}$.

The time scale for the second stage is set by the time scale on which orbits diffuse through cantori, $t_{cr}$.

Suppose now that the orbits are perturbed by friction and noise with $\eta \sim 10^{-9} - 10^{-1}$, the limiting values here corresponding, respectively, to the typical amplitude associated with discreteness effects in very large and very small galaxies [23,24,25]. In this case, one finds that the time scale associated with the first stage of the evolution is essentially unchanged, i.e., ensembles still approach a near-invariant distribution on a time scale $\sim t_{cr}$, but that the time scale for the second stage decreases dramatically! In the absence of friction and noise, the time scale associated with diffusion through cantori typically satisfies $t_{\text{diff}}(\eta = 0) \sim 10^3 - 10^6 t_{cr}$, but even very weak friction and noise can decrease $t_{\text{diff}}(\eta)$ by orders of magnitude. For example, $\eta \sim 10^{-9} - 10^{-6}$ can result in a diffusion time as short as $\sim 100 t_{cr}$, an interval which, for large galaxies, corresponds to the Hubble time $t_H$. Indeed, for values of $\eta$ as large as $\eta \sim 10^{-4}$, the diffusion time scale $t_{\text{diff}}$ is often so short that one cannot clearly distinguish between two different stages of evolution. For values of $\eta$ that large, noisy ensembles exhibit a rapid approach towards a near-invariant distribution that differs significantly from the near-invariant distribution associated with a purely Hamiltonian evolution but is comparatively similar to the true invariant distribution associated with a Hamiltonian evolution. Grey-scale plots comparing representative deterministic and noisy near-invariant distributions in the dihedral and truncated Toda potentials are exhibited, respectively, in FIGURES 8 in [25] and FIGURES 2 in [24].

In the absence of friction and noise, one anticipates that a generic ensemble of initial conditions will ultimately evolve towards a microcanonical distribution, i.e., a uniform population of the accessible portions of the constant energy hypersurface, but this will only happen on the relatively long time scale $t_{\text{diff}}(\eta = 0)$. Alternatively, if one allows for friction and noise and integrates for a time $\sim t_R$, one anticipates an evolution towards a canonical
distribution with temperature $\Theta$. Noisy integrations performed for a time $\ll t_R$ but still much longer than the time required to breech cantori will result in a near-invariant distribution that can be reasonably visualised as a slightly “thickened” version of a constant energy microcanonical distribution. Because $E$ is not exactly conserved, this near-invariant distribution is not exactly microcanonical, i.e., not proportional to a delta function in energy. However, because $E$ is almost conserved and the orbits have succeeded in breeching cantori, this noisy near-invariant distribution is much closer to the purely Hamiltonian invariant distribution than to either a canonical distribution or the purely Hamiltonian near-invariant distribution [25]. In this sense, weak friction and noise can accelerate an approach towards a near-microcanonical equilibrium.

**FIRST PASSAGE TIME EXPERIMENTS**

To quantify the rate at which individual trajectories diffuse through cantori, a collection of first passage time experiments was also performed. These involved four components:

1. Select individual initial conditions corresponding in the absence of friction and noise to sticky or confined chaotic orbits, i.e., chaotic orbits which, because of cantori, are trapped near regular regions for relatively long times.
2. Specify the form and amplitude of the friction and noise.
3. For each choice of form and amplitude, perform a large number (~2000-5000) of different noisy realisations of the same initial condition; and, for each noisy realisation, determine the time at which the orbit escapes through one or more cantori to become unconfined.
4. Analyse the data to extract $N(t)$, the fraction of the orbits that have not yet escaped within a time $t$.

The results quoted below involve orbits in the dihedral potential (5) with $E = 10$ where, in the absence of any friction or noise, the diffusion time $t_{diff} \approx 1000$. Other choices of potential or energy can yield results that differ quantitatively, but the principal qualitative conclusions seem unchanged. Estimating when an orbit has escaped was done by identifying a “masked” region in the configuration space and recording the first time that the orbit left this region. That this mask criterion is reasonable was tested in two ways: (1) It was verified that changing slightly the shape and location of the mask had no appreciable effects. (2) For the case of purely Hamiltonian trajectories, escape from the masked region was shown to correspond to an abrupt increase in the value of the largest short time Lyapunov exponent. This is in accord with the fact that, albeit still chaotic, confined chaotic orbits are less unstable exponentially than are unconfined chaotic orbits [20]. One interesting variant of the preceding, also considered, involved tracking a localised ensemble of initial conditions corresponding to confined chaotic orbits evolved into the future both with and without friction and noise. These experiments yielded results very similar to those obtained from multiple integrations of individual initial conditions.

Six different forms of friction/noise were considered, namely: (1) additive white noise
and a constant coefficient of dynamical friction $\eta$, related by a Fluctuation-Dissipation Theorem at temperature $\Theta = E = 10$; (2) multiplicative white noise and dynamical friction with $\eta = \eta_0 e^2$, related by a Fluctuation-Dissipation Theorem with $\Theta = E = 10$; (3) multiplicative white noise and dynamical friction with $\eta = \eta_0 e^{-2}$, again related by a Fluctuation-Dissipation Theorem with $\Theta = E = 10$; and (4) - (6) the same noises as in (1) - (3) but vanishing friction. In all six cases, the individual noisy realisations were generated using the same pseudo-random seeds.

In analysing the effects of friction and noise, attention focused on three principal issues, namely:

1. What is the functional form of $N(t)$, the fraction of the orbits that have not yet escaped?
2. How does $N(t)$ depend on the amplitude of the perturbation?
3. To what extent does the form of the friction and the noise actually matter?

Overall, in these experiments escape is a two-stage process. Early on, there are no escapes. All that one sees is that, as one might expect [25,26], different noisy realisations of the same initial condition diverge exponentially at a rate set by the value of the largest short time Lyapunov exponent for the unperturbed deterministic trajectory. Eventually, however, once the noisy ensemble has dispersed to the extent that the root mean squared $\delta r_{rms} \sim 1.0$, individual noisy orbits begin to escape through holes in the cantori. This onset of escape is a comparatively abrupt phenomenon, the interval during which the first 5% of the orbits escape typically being only a small fraction of the time $T$ before the first escape occurs. It is also clear that, at least early on, escapes can be well approximated as a Poisson process, with the confined orbits becoming unconfined at a nearly constant rate, i.e.,

$$N(t) \approx \begin{cases} N(0), & \text{if } t \leq T; \\ N(0) \exp \left[ -\Lambda(t - T) \right], & \text{if } t > T. \end{cases}$$

This behaviour is illustrated in FIGURES 1 (a) and (b), which exhibit $\ln N(t)$ as a function of time $t$ for two different initial conditions integrated in the presence of a constant $\eta$ and additive white noise. Each panel summarises multiple noisy realisations of a single initial condition evolved for a total time $t = 1024$ with $\Theta = 10$. The six different curves in each panel, each summarising 4000 noisy realisations, represent six different values of $\eta$, namely $\ln \eta = -9, -8, -7, -6, -5, \text{ and } -4$.

It is clear from FIGURE 1 that, although $\ln N(t)$ originally decreases linearly in time, it eventually develops nontrivial curvature indicating that the escape rate is slowly decreasing. Exactly why this is so is not completely clear. However, two important points should be noted. (1) In every case where it is observed, this curvature arises at a time sufficiently late that changes in energy $\delta E$ have become appreciable, $\sim 10\%$ or more. This suggests strongly that, at least in part, this change in escape rate reflects changes in the “effective” Hamiltonian phase space in which the noisy orbits evolve. (2) In at least some cases, the curvature reflects the fact that, because of the perturbations, some originally chaotic orbits have become trapped by KAM tori. (If, for the nonescapers, the friction and noise are
turned off at some time $\tau$ and the trajectories integrated for a significantly longer time, it becomes apparent that many of the orbits have become regular.

So how, overall, does the form of $N(t)$ scale with $\eta$, the amplitude of the perturbation? When probing the time $T$ required before escapes begin or the rate $\Lambda$ at which escapes initially proceed after they begin, one finds a roughly logarithmic dependence on $\eta$. In other words, when plotting $T(\eta)$ or $\Lambda(\eta)$ the natural independent variable, i.e., the abscissa, is $\ln \eta$, not $\eta$. An example thereof is provided in FIGURES 2 a and b, which summarise data generated from a single initial condition with friction and additive white noise related by a Fluctuation-Dissipation Theorem. The top panel exhibits $T(0.01)$, the time required for 1% of the orbits in a 4000 orbit ensemble to become unconfined. The lower panel exhibits the best fit value of the escape rate $\Lambda$ of eq. (7), as fit to the interval $T(0.01) < t < 256$. The curvature observed in both panels is statistically significant, so one cannot assert that $T$ or $\Lambda$ are linear functions of $\ln \eta$. However, it is clear that, overall, $T$ and $\Lambda$ should be visualised as functions of $\ln \eta$ rather than $\eta$.

Perhaps the most important conclusion derived from these experiments is that the computed $N(t)$ is nearly independent of the presence or absence of friction, and that $N(t)$ is also largely independent of whether the noise is additive or multiplicative!

First perform 4000 noisy realisations of the same initial condition, all with the same $\Theta$ and the same $\eta(v)$, and analyse the resulting data to extract $N(t)$. Then repeat these experiments with exactly the same noise (generated from the same pseudo-random seeds!) but without friction, and once again compute $N(t)$. A comparison of the two $N(t)$'s then shows virtually no appreciable differences. At early times, there are absolutely no statistically significant differences. Later on, one can see some tiny differences. However, these can be attributed entirely to the fact that the energies of the orbits with and without friction will be slightly different, and that slightly different energies can give rise to slightly different escape statistics.

Comparing additive and multiplicative noise is a bit more subtle since one must worry about normalisations. Suppose, however, that, when introducing multiplicative noise, one selects $\eta_0$ so that the “average” $\eta \equiv \eta_0 \langle v^2 \rangle$ or $\eta \equiv \eta_0 \langle v^{-2} \rangle$ coincides with the white noise constant $\eta$. In this case, one finds that the form of the noise matters very little. Plots of $N(t)$ for additive white noise, multiplicative noise $\propto v^2$, and multiplicative noise $\propto v^{-2}$ yield no statistically significant differences.

Two examples of this behaviour are exhibited in FIGURES 3 (a) and (b) which compare the effects of additive and multiplicative noise for two different initial conditions at two different perturbation levels. Each panel contains four curves, representing (1) additive white noise and friction with a constant $\eta$, (2) the same additive white noise with vanishing friction, (3) multiplicative noise and friction with $\eta \propto v^2$, and (4) multiplicative noise and friction with $\eta \propto v^{-2}$. It is evident that, at late times, the curves do not completely overlap. However, it is also clear that none of the curves is extremely different from the others.

These numerical experiments suggest two obvious inferences which, however, remain to
be checked more carefully for larger orbit ensembles, different frictions and noise, and other potentials: (1) Smooth non-Hamiltonian perturbations like friction play only a minimal role in accelerating phase space transport through cantori. (2) At least assuming that the noise is white, its details seem comparatively unimportant. In particular, additive noise and multiplicative noise depending on the orbital velocity \( \mathbf{v} \) exhibit only minimal differences. Overall, when perturbing the Hamiltonian trajectories what seems important is that the orbits be subjected to highly “irregular” perturbations that violate Liouville’s Theorem at some given amplitude. In this context, it should perhaps be noted explicitly that a rapidly varying time-dependent Hamiltonian need not be as efficient in triggering accelerated phase space transport as random noise. Specifically, when considering a Hamiltonian of the form \( H = H_0 + \epsilon H_1(t) \), with \( H_1(t) \) periodic in time and \( \epsilon \) an adjustable parameter, one finds in at least some cases [27] that, for periods \( \ll t_{cr} \), one must allow for relatively large values of \( \epsilon \) to dramatically accelerate diffusion through cantori.

**WORK IN PROGRESS AND POTENTIAL IMPLICATIONS**

The experiments described above still need to be generalised in two important ways. One obvious tack involves extending the computations to three-dimensional systems. Arnold webs can serve as partial phase space obstructions in the same sense as can cantori; and one might anticipate that friction and noise could accelerate phase space transport through such barriers in three-dimensional systems in the same ways as they do through cantori in two dimensions. This remains, however, to be checked. Indeed, for orbit ensembles in three-dimensional Hamiltonian systems, even the purely deterministic evolution towards an an invariant or near-invariant distribution is not completely understood. Preliminary investigations performed by Merritt and Valluri [28] suggest that the approach towards equilibrium can closely resemble what is observed in two-dimensional systems [20,22]. However, one would anticipate that, for a generic three-dimensional system with two positive Lyapunov exponents, the situation could be more complicated – and interesting – than in two dimensions since unequal Lyapunov exponents could induce “mixing” that proceeds in different directions at different rates [29]!

Another equally important objective is to allow for the effects of coloured noise, where the autocorrelation function \( \langle F_i(t_1)F_j(t_2) \rangle \) is not delta-correlated in time. The assumption of singular, delta-correlated noise is an idealisation never exactly realised in nature, even when modeling high frequency phenomena like discreteness effects in systems interacting via short range forces; and the assumption seems especially unreasonable when trying to model objects like galaxies embedded in a dense cluster environment where individual “random” interactions would seem characterised dimensionally by a time scale \( \sim t_{cr} \) or even larger! Allowing for coloured noise could also be important by providing some insights into the question of exactly how and why non-Hamiltonian perturbations result in accelerated phase space transport. Diffusion through cantori, either in the absence or presence of noise, must
be related to some "natural" microscopic time scale(s), but the precise nature of these time scales has not yet been established. Systematically increasing the autocorrelation time from zero (white noise) to values substantially larger will allow one to determine the point at which a finite correlation time actually begins to matter.

In summary, the numerical experiments described in this paper lead to at least three tentative conclusions:
1. At least for systems that admit a coexistence of regular and chaotic behaviour, even weak couplings to an external environment, modeled as friction and noise, can dramatically accelerate evolution towards a (near-)microcanonical equilibrium. This suggests in particular that idealising a complex Hamiltonian system as a completely isolated entity may be a very bad idea.
2. There is reason to think that the detailed form of the friction and noise is comparatively unimportant. When assessing the effects of friction and noise in accelerating phase space transport, all that really matters may be the amplitude of the perturbation. If true, this would suggest that it may not be all that hard to satisfactorily model the coupling of one’s system to its surrounding environment. The details which are hard to determine may not be very important!
3. "Collisionality," i.e., discreteness effects, may be significantly more important in galactic dynamics than generally recognised. For example, such graininess could serve to destabilise quasi-equilibria which use confined chaotic orbits to support interesting structures such as bars [30] or triaxial cusps [31].

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FIGURE CAPTIONS

FIGS. 1 (a) $N(t)$, the fraction of confined chaotic orbits that have not yet escaped to become unconfined, for ensembles of 4000 noisy realisations of the same initial condition evolved in the dihedral potential with $E = 10.0$, $x = 0.0$, $y = 1.3$, $v_y = 1.75$, and $v_x = v_x(x, y, v_y, E) > 0$. Each orbit was subjected to additive white noise and friction with constant $\eta$, related by a Fluctuation-Dissipation Theorem with temperature $\Theta = 10$. Passing from top to bottom at small $t$, the six curves represent ensembles with $\eta = 10^{-9}$, $10^{-8}$, $10^{-7}$, $10^{-6}$, $10^{-5}$, and $10^{-4}$. (b) The same quantities generated for a different initial condition, namely $E = 10.0$, $x = 0.0$, $y = 2.7$, and $v_y = 2.25$.

FIGS. 2 (a) $T(0.01)$, the time required for 1% of the members of an ensemble of 4000 noisy realisations of the same unconfined chaotic orbit with $E = 10.0$, $x = 0.0$, $y = 1.3$, $v_y = 1.75$, and $v_x = v_x(x, y, v_y, E) > 0$ to become unconfined. Each orbit was subjected to additive white noise and friction with constant $\eta$, related by a Fluctuation-Dissipation Theorem with $\Theta = 10$. (b) $A$, the rate at which orbits in the ensemble escape, fit to the interval $T(0.01) < t < 256$.

FIGS. 3 (a) $N(t)$, the fraction of confined chaotic orbits that have not yet escaped to become unconfined, for ensembles of 4000 noisy realisations of the same initial condition evolved in the dihedral potential with $E = 10.0$, $x = 0.0$, $y = 1.1$, $v_y = 3.35$, and $v_x = v_x(x, y, v_y, E) > 0$. Each orbit was evolved with $\Theta = 10$ and $\eta_0 = 10^{-5}$. The four curves represent additive white noise and a constant $\eta$ (solid line), additive noise but no friction (dashed), multiplicative noise and friction with $\eta \propto v^2$ (dot-dashed), and multiplicative noise and friction with $\eta \propto v^2$ (triple-dot-dashed). (b) The same quantities generated for a different initial condition, namely $E = 10.0$, $x = 0.0$, $y = 1.3$, $v_y = 1.75$, and $v_x = v_x(x, y, v_y, E) > 0$, now allowing for $\Theta = 10$ and $\eta_0 = 10^{-7}$.
Figure 1.
Figure 2.
Figure 3.