STRICTLY ISOSPECTRAL SUSY POTENTIALS 
AND THE RICCATI SUPERPOSITION PRINCIPLE

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Summary. - The connection of unbroken SUSY quantum mechanics in its strictly isospectral form with the nonlinear Riccati superposition principle is pointed out.

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Generating quasi-isospectral potentials by the methods of supersymmetric quantum mechanics (SUSYQM) is a simple and well-known mathematical technique [1]. Its relationships with intertwining, factorization, Darboux covariance, and inverse scattering procedures have been recently reviewed by the author [2]. Also, the author, either in collaboration or alone [3] dealt extensively with a SUSYQM procedure based on the general Riccati solution [4], that may be called the double Darboux general Riccati method (henceforth DDGR). For details the reader is directed to my previous works on DDGR and the references therein. The purpose of this note is to display the connection of the DDGR method with the Riccati nonlinear superposition principle.

The DDGR procedure has been first used in physics by Mielnik for the quantum harmonic oscillator case [5]. For simplicity reasons, I shall deal with the case of unbroken SUSY, i.e., when one has at hand a true zero mode $u_0$ of the initial Schrödinger problem. DDGR leads to families of strictly isospectral potentials to the initial (bosonic) one. Asking for the most general superpotential (i.e., the general Riccati solution) such that the fermionic potential fulfills $V_+(x) = w_+^2 + \frac{dw_+}{dx}$, it is obvious that one particular solution to this equation is $w_+$, i.e., Witten’s superpotential. Thus, one is led to consider the following Riccati equation $w_+^2 + \frac{dw_+}{dx} = w_+^2 + \frac{d^2w_+}{dx^2}$, whose general solution can be written down as $w_+(x) = w_p(x) + \frac{1}{v(x)}$, where $v(x)$ is an unknown function. Using this ansatz, one obtains for the function $v(x)$ the following Bernoulli equation

$$\frac{dv(x)}{dx} - 2v(x)w_p(x) = 1,$$  \hspace{1cm} (1)

that has the solution

$$v(x) = \frac{I_0(x) + \lambda}{u_0^2(x)},$$  \hspace{1cm} (2)
where \( I_0(x) = \int_x u_0^2(y) \, dy \), and \( \lambda \) is an integration constant considered as the free parameter of the method. Thus, \( w_g(x) \) can be written as follows

\[
w_g(x; \lambda) = w_p(x) + \frac{d}{dx} \left[ \ln \left( I_0(x) + \lambda \right) \right] = w_p(x) + \sigma_0(\lambda) = -\frac{d}{dx} \left[ \ln \left( \frac{u_0(x)}{I_0(x) + \lambda} \right) \right],
\]

where in (2) and (3) the well-known relationship \( D = \frac{d}{dx} \)

\[
w_g(x; \lambda) = w_p(x) + \sigma_0(\lambda) = -\frac{d}{dx} \left[ \ln \left( \frac{u_0(x)}{I_0(x) + \lambda} \right) \right],
\]

(3)

has been used. The \( \sigma_0 \) notation for the logarithmic derivative is borrowed from the book of Matveev and Salle [6], and the subscript has been appended to show that the fixed solution on which the scheme is carried on is the Schroedinger ground state. Finally, one gets the family of “bosonic” potentials

\[
V_-(x; \lambda) = w_g^2(x; \lambda) - \frac{dw_g(x; \lambda)}{dx} = V_-(x) - 2D^2 \ln[I_0(x) + \lambda]
\]

\[
V_-(x) = V_-(x) - 2\sigma_{0,x}(\lambda)
\]

\[
V_-(x) = V_-(x) - \frac{4u_0(x)u_0'(x)}{I_0(x) + \lambda} + \frac{2u_0^4(x)}{I_0(x) + \lambda}^2.
\]

(5)

All \( V_-(x; \lambda) \) have the same supersymmetric partner potential \( V_+(x) \) obtained by deleting the ground state. By inspection of Eqs.(4) and (3) we can obtain the ground state wave functions for the potentials \( V_-(x; \lambda) \) as follows

\[
u_0(x; \lambda) = f(\lambda) \frac{u_0(x)}{I_0(x) + \lambda},
\]

(6)

where \( f(\lambda) \) is a normalization factor of the form \( f(\lambda) = \sqrt{\lambda(\lambda + 1)} \). A connection with other isospectral methods has been found, by noticing that the limiting values -1 and 0 for the parameter \( \lambda \) lead to the Abraham-Moses procedure [7], and Pursey’s one [8], respectively. Here, we notice furthermore that by writing the parametric family in terms of their unique “fermionic” partner

\[
V_-(x; \lambda) = V_+(x) - 2D^2 \ln \left( \frac{I_0(x) + \lambda}{u_0(x)} \right)
\]

(7)

the inverse character of the parametric Darboux transformation is manifest [6]. Therefore, a two step interpretation is possible. In the first step, one goes to the fermionic partner system, and in the second one returns to a \( \lambda \)-distorted bosonic system. This is why it can be called a double Darboux transformation, although this may be somewhat confusing because of Adler’s method [9], which is also a two step construction and is known by the same name. The strictly isospectral DDGR
method can be applied to any one-dimensional system whose dynamics is dictated by a Schroedinger-like equation. Moreover, one can employ combinations of any pairs of Abraham-Moses procedure, Pursey’s one, and the Darboux one. However, only the DDGR leads to reflection and transmission amplitudes identical to those of the original potential.

We now make use of the fact that Riccati equations are differential equations admitting a nonlinear superposition principle [10], i.e., it is possible to write down the general solution as a combination of particular solutions without using any quadrature. This is due to the well-known property [11] that given three particular solutions one can obtain the general Riccati solution from the following k invariant

$$w_3 - w_1 : w_3 - w_2 = k,$$

as the superposition

$$w = \frac{kw_1(w_3 - w_2) + w_2(w_1 - w_3)}{k(w_3 - w_2) + w_1 - w_3}.$$  \hspace{1cm} (9)

In particular, we have seen that within DDGR the general Riccati solution can be written in the form $w(\lambda) = -\sigma_0 + \sigma_0(\lambda)$. Considering three particular solutions $w(\lambda_i), i=1,2,3$, one can write the above invariant as

$$\frac{\sigma_0(\lambda) - \sigma_0(\lambda_1)}{\sigma_0(\lambda) - \sigma_0(\lambda_2)} : \frac{\sigma_0(\lambda_3) - \sigma_0(\lambda_1)}{\sigma_0(\lambda_3) - \sigma_0(\lambda_2)} = k.$$  \hspace{1cm} (10)

i.e., in terms of the $\lambda$-dependent logarithmic derivatives. The nonlinear Riccati invariant $k$ can be seen as a constraint on the Darboux contributions to the parametric bosonic potentials or as a constraint on a sequence of their free parameters $\lambda_i$ if written as follows

$$\frac{\int \sigma_0, x(\lambda) - \int \sigma_0, x(\lambda_1)}{\int \sigma_0, x(\lambda) - \int \sigma_0, x(\lambda_2)} : \frac{\int \sigma_0, x(\lambda_3) - \int \sigma_0, x(\lambda_1)}{\int \sigma_0, x(\lambda_3) - \int \sigma_0, x(\lambda_2)} = k.$$  \hspace{1cm} (11)

It is worthwhile to notice that an even more general form of the Riccati k invariant can be written down if one do not stop at the one-parameter type of bosonic Riccati solution. In another work [12], I have provided a simple multiple-parameter (iterative) generalization of Mielnik’s construction. At an arbitrary i-order of iteration, one can write the general Riccati solution as $w^{(i)}_g = w_0 + w_{\lambda_1} + \ldots w_{\lambda_{i-1}} + w_{\lambda_i}$, where up to $i-1$ the $\lambda$ parameters are fixed and only the last parameter is a free one, whereas $w_{\lambda_i} = D \ln(\lambda_i + \int F_{j-1}).$ The $F$’s are integration factors of the type $F_j = e^{-\int 2w_{j}^{(i)}},$ where $w_{j}^{(i)}$ is the particular Riccati solution at order $j$, which is of the same form as $w_{g}^{(i)}$ but with no free parameter. Since all these Riccati solutions correspond to the same fermionic Schroedinger potential, any triplet of multiple-parameter particular solutions that may be of arbitrary hierarchical order $j$ can enter in the Riccati k invariant and may be used to get the general solution.
at an arbitrary order. Thus, one can write

$$\frac{\int \sigma_{0,x}^{(i)}(..., \lambda)}{\int \sigma_{0,x}^{(j_1)}(..., \lambda_{j_1})} - \frac{\int \sigma_{0,x}^{(j_3)}(..., \lambda_{j_3}) - \int \sigma_{0,x}^{(j_2)}(..., \lambda_{j_2})}{\int \sigma_{0,x}^{(j_1)}(..., \lambda_{j_1})} = k.$$  \tag{12}

Finally, let us notice that instead of the zero mode solution one can employ any other solution of the initial Hamiltonian $H_b u(x) = \epsilon u(x)$, where the so-called factorization energy $\epsilon$ should be less than the ground state energy of $H_b$, and an analytic continuation is performed in the independent variable [13] to turn $u$ into a nodeless function. In this way no singularities with respect to the initial potential are introduced. For this more general case the zero subindex in all the above equations is to be dropped.

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References