Pseudotensors and quasilocal energy-momentum

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Early energy-momentum investigations for gravitating systems gave reference frame dependent pseudotensors; later the quasilocal idea was developed. Quasilocal energy-momentum can be determined by the Hamiltonian boundary term, which also identifies the variables to be held fixed on the boundary. We show that a pseudotensor corresponds to a Hamiltonian boundary term. Hence they are quasilocal and acceptable; each is the energy-momentum density for a definite physical situation with certain boundary conditions. These conditions are identified for well-known pseudotensors.

Energy-momentum can be regarded as the most fundamental conserved quantity, being associated with a symmetry of the space-time geometry. From Noether’s theorem and translation invariance one obtains a tensor, $T^{\mu\nu}$, which describes the density of energy-momentum and defines a conserved energy-momentum as a consequence of its satisfying the differential conservation law $\partial_\nu T^{\mu\nu} = 0$. But it is not unique, since one can add an arbitrary “curl” — which shifts the zero of energy-momentum. The gravitational response, however, via Einstein’s equation $G^{\mu\nu} = \kappa T^{\mu\nu}$, detects the total energy-momentum density for matter and interaction fields, and thereby removes the classical uncertainty in the energy-momentum expressions—the gravitational field is trivial only if the total energy momentum vanishes. Thus it seems ironic that the proper identification of the contribution to the total energy-momentum from the gravitational field itself has proved to be so elusive. It was natural to expect that, since these other sources exchange energy-momentum with the gravitational field locally, gravity should also have its own local energy-momentum density.

Attempts at identifying an energy-momentum density for gravity, however, led only to various energy momentum complexes which are pseudotensors, including those of Einstein [1], Papapetrou [2], Bergmann [3], Landau and Lifshitz [4], Möller [5], and Weinberg [6]. Pseudotensors are not covariant objects; they inherently depend on the reference frame, and thus by their very nature cannot provide a truly physical local gravitational energy-momentum density. Indeed any such quantity is precluded by the equivalence principle itself, since a gravitational field should not be detectable at a point. Consequently many have criticised the whole idea; e.g., an influential textbook states: Anyone who looks for a magic formula for “local gravitational energy-momentum” is looking for the right answer to the wrong question. Unhappily, enormous time and effort were devoted in the past to trying to “answer this question” before investigators realized the futility of the enterprise. [7] p. 467. Hence the pseudotensor approach has been largely abandoned (although interest continues, see e.g., [8]). A new idea, quasilocal (i.e., associated with a closed 2-surface), was proposed and has become widely accepted. In view of the role of gravity in identifying energy, this then becomes the most fundamental notion of classical energy-momentum. There have recently been many quasilocal proposals [9,10]. Various criteria have been advanced (see, e.g., [11]), in particular good limits to flat spacetime, weak field, spatial infinity (ADM), and null infinity (Bondi). However it has now been recognized that there are an infinite number of expressions satisfying these requirements [12]. Clearly, additional principles and criteria are needed. The Hamiltonian formalism includes such a principle, with the Hamiltonian boundary term determining both the quasilocal quantities and the boundary conditions [10,13]. Here we show that this Hamiltonian approach to quasilocal energy-momentum rehabilitates the pseudotensors.

We identify energy with the value of the Hamiltonian. The Hamiltonian for a finite region, $H(N) = \int_{\Sigma} N^\mu H_\mu + \oint_{S=\partial\Sigma} B(N)$, (1) generates the spacetime displacement of a finite spacelike hypersurface $\Sigma$ along a vector field $N^\mu$; it includes a surface and a boundary term. Noether’s theorems guarantee that $H_\mu$ is proportional to the field equations. Consequently the value depends only on the boundary term $B$, which gives the quasilocal energy-momentum. But the
boundary term can be modified. (This is a particular case of the usual Noether conserved current non-uniqueness.) Indeed it is necessary to adjust $B$ to give the correct asymptotic values [14]. Fortunately, $B$ is not arbitrary. A further principle of the formalism controls its form: choose the Hamiltonian boundary term $B$ so that the boundary term in $\delta H$ vanishes when the desired fields are held fixed on the boundary (technically necessary for the variational derivatives to be well defined). Hence, we find a nice division: the Hamiltonian density $H_\mu$ determines the evolution and the constraint equations, the boundary term $B$ determines the boundary conditions and the quasilocal energy-momentum. There still remain many possible boundary condition choices [15]. Consequently there are various kinds of energy, each corresponding to a different choice of boundary condition; this situation can be compared with thermodynamics with its various energies: internal, enthalpy, Gibbs, and Helmholtz.

For geometric gravity theories including general relativity, the Hamiltonian can be succinctly obtained in terms of differential forms [17], which readily displays the boundary term and its connection to pseudotensors. Since we wish to connect with traditional work we shall use a coordinate (holonomic) basis $dx^\mu$ and sometimes the (Hodge) dual basis $\eta^\mu$, where $\eta^\mu := (1/\sqrt{g})\sqrt{|g|}dx^\mu$, is the connection one-form. The simplest analysis uses the connection along with the metric as the dynamic variables. The Lagrangian density is the Einstein-Hilbert scalar curvature of the metric as the dynamic variables. The Hamiltonian can be succinctly obtained in brevity:

$$i_N \mathcal{L} = i_N \Omega \wedge \eta + \nabla \Omega \wedge \eta_N = \mathcal{L}_N \Omega \wedge \eta - \mathcal{H}(N);$$

(2)

here the time derivative is given by the Lie derivative (on form components $\mathcal{L}_N := di_N + \nabla d$); the Hamiltonian 3-form (without discarding total derivatives) is

$$\mathcal{H}(N) = -\nabla \wedge i_N \eta - i_N \omega D\eta + dB_M(N).$$

(3)

The Hamiltonian density term includes $N^\mu H_\mu = -\nabla^\beta \wedge N^\mu \eta_\mu = 2N^\mu G^\mu_\nu \eta_\nu$, a covariant expression which projects to the usual ADM Hamiltonian density (see e.g., Ch 21 in [7]), along with a frame gauge transformation generating term, $i_N \omega D\eta$, which vanishes because the connection is symmetric and metric compatible. When integrated over a finite spatial hypersurface $\Sigma$, the value of the Hamiltonian comes from the total differential term which (via the generalized Stokes theorem) yields a boundary term with the 2-form integrand $B_M(N) = i_N \omega_\mu \eta_\nu = N^\lambda M_\lambda^{\mu\nu}(1/2)dS_{\mu\nu}$, where $dS_{\mu\nu} := (1/2)\epsilon_{\mu\nu\beta\gamma} dx^\beta \wedge dx^\gamma$; the coefficients

$$M_\lambda^{\mu\nu} = 2\sqrt{-g}g^{\beta\gamma}[\Gamma^\nu_{\beta\lambda}] = 2\sqrt{-g}g^{\beta\gamma}g^{\alpha\delta}\partial_{[\beta}g_{\delta\lambda]},$$

(4)

turn out to be the superpotential whose divergence gives the Møller pseudotensor [5]. (Note the extreme directness of this derivation of Møller’s expression.) The variation of the Hamiltonian density (3) yields (with $N$ fixed)

$$\delta H_M = field\;equation\;terms + di_N(\delta \omega^\alpha_\beta \wedge \eta_\alpha \beta),$$

(5)

showing that the boundary condition implicit in (3) is of the Neumann type: the connection $\Gamma \sim \partial g$ is to be held fixed. This calculation also reveals a serious deficiency: the boundary term in the variation of the Møller Hamiltonian will not vanish with the standard asymptotics: $\delta g \sim O(1/r), \delta \Omega \sim O(1/r^2)$.

An almost obvious alternative (in view of the problem just mentioned) is simply to replace the Møller Hamiltonian boundary term by $B_E := \omega^\alpha_\beta \wedge i_N \eta_\alpha \beta = N^\lambda M_\lambda^{\mu\nu}(1/2)dS_{\mu\nu}$. Now the coefficients

$$U^{\mu\nu}_E = (-g)^{1/2}g^{\beta\gamma}\Gamma_{\alpha\beta\gamma}^{\mu/\nu\lambda} \equiv (-g)^{-\frac{1}{2}}g_{\lambda\gamma}\partial_{[\gamma}[-g(g^{\mu\tau}g^{\nu\gamma} - g^{\nu\tau}g^{\mu\gamma})],$$

(6)

are the Freud superpotential, whose divergence gives the Einstein pseudotensor [1,18]. The Hamiltonian variation now contains a boundary term of the form $d\mathcal{N}C$ where

$$C = -\omega^\alpha_\beta \wedge \delta \eta_\alpha \beta = -\Gamma^\alpha_{\beta\gamma}(\sqrt{-g}g^{\beta\gamma})\delta_{\alpha\beta}(d\mathcal{N})_\nu,$$

(7)

showing that this choice of Hamiltonian boundary term is associated with a Dirichlet type boundary condition: the contravariant metric density is to be held fixed.

The possible forms of the Hamiltonian boundary term have been considered in some detail elsewhere [13]. There are various choices involved including: (1) the representation or dynamic variables, such as the metric, orthonormal frame, connection, spinors; (2) the control mode: such as a Dirichlet or Neumann boundary condition for each dynamic variable; (3) the reference configuration: this is given by the field values that give vanishing energy-momentum (the standard choice is Minkowski space). (4) the displacement vector field $N$: which timelike displacement gives the energy? which spatial displacement gives the linear momentum?

We have shown how two famous pseudotensors naturally arise from Hamiltonian boundary terms and how they are associated with field boundary values. In like manner we shall now explain how the Hamiltonian boundary term approach to quasilocal energy-momentum rehabilitates all of the other pseudotensors so they can be recognized as legitimate. There is a direct relationship between pseudotensors and quasilocal expressions. Every pseudotensor corresponds to some acceptable choice of boundary expression. Conversely, every boundary expression defines a pseudotensor [19].

We consider the pseudotensor idea in some detail: a suitable superpotential $H_\mu^{\lambda/\alpha} \equiv H_\mu^{[\lambda/\alpha]}$ is selected and
used to split the Einstein tensor thereby defining the associated gravitational energy-momentum pseudotensor:

$$\kappa \sqrt{-g} N^\mu \psi^\nu := -N^\nu \sqrt{-g} G^\mu_\nu + \frac{1}{2} \partial_\lambda (N^\mu H^\nu_\mu)^\lambda, \quad (8)$$

where \( \kappa = 8\pi G c^{-4} \) and we have inserted a vector field to make the calculation more nearly covariant. The usual formulation is recovered by taking the components of the vector field to be constant in the present reference frame; then Einstein’s equation, \( G^\mu_\nu = \kappa T^\mu_\nu \), can be rearranged into a form where the source is the total effective energy-momentum pseudotensor

$$\partial_\lambda H^\nu_\mu = 2\kappa (-g)^{\frac{1}{2}} T^\nu_\mu := 2\kappa (-g)^{\frac{1}{2}} (t^\nu_\mu + T^\nu_\mu). \quad (9)$$

An immediate consequence of the antisymmetry of the superpotential is that \( T^\nu_\mu \) is a conserved current: \( \partial_\nu [(-g)^{\frac{1}{2}} T^\mu_\nu] = 0 \), which integrates to give a conserved energy-momentum, \( N^\nu T^\mu_\nu := \int N^\nu T^\mu_\nu (-g)^{\frac{1}{2}} (d^3x)_\nu \). This should be contrasted with the covariant formula

$$\nabla_\nu T^\mu_\nu := \partial_\nu T^\mu_\nu - \Gamma^\lambda_\nu T^\mu_\lambda + \Gamma^\nu_\mu T^\lambda_\nu = 0, \quad (10)$$

which does not lead to a conserved energy-momentum unless \( \Gamma = 0 \) (flat space).

A minor variation on the preceding analysis results from choosing a superpotential with a contravariant index: \( H^{\mu\nu\lambda} \equiv H^{[\mu\nu]\lambda]} \). A further variation:

$$H^{\mu\nu\alpha} := \partial_\beta H^{[\mu\nu\alpha]}, \quad (11)$$

along with the symmetries \( H^{\mu\alpha\beta} \equiv H^{[\mu\alpha]\beta} \equiv H^{[\mu\beta]\alpha} \) and \( H^{[\mu\alpha\beta]} \equiv 0 \), leads to a symmetric pseudotensor— which then allows for a simple definition of angular momentum, see [7]. We can cover these options simply by introducing a vector field, then, in the following computations, we can easily make modifications like \( N^\mu H^{\rho\nu\lambda} \rightarrow N^\mu H^{[\rho\nu\lambda]} \).

The energy-momentum within a finite region

$$-P(N) := - \int_{\Sigma} N^\mu T^\nu_\mu \sqrt{-g}(d^3x)_\nu \equiv \int_{\Sigma} [N^\mu \sqrt{-g}(\frac{1}{\kappa} G^\mu_\nu - T^\mu_\nu) - \frac{1}{2\kappa} \partial_\lambda (N^\mu H^\nu_\mu)^\lambda] (d^3x)_\nu \equiv \int_{\Sigma} N^\mu H^\nu_\mu + \int_{\partial \Sigma} B(N) \equiv H(N), \quad (12)$$

is seen to be just the value of the Hamiltonian. Note that \( H^\nu_\mu \) is the covariant form of the ADM Hamiltonian density, which has a vanishing numerical value, so that the value of the Hamiltonian is determined purely by the boundary term \( B(N) = -N^\mu (1/2\kappa) \partial_\lambda (N^\mu H^\nu_\mu)^\lambda \). Thus for any pseudotensor the associated superpotential is naturally a Hamiltonian boundary term. Moreover the energy-momentum defined by such a pseudotensor does not really depend on the local value of the reference frame, it is actually quasi-local—it depends (through the superpotential) on the values of the reference frame (and the fields) only on the boundary of a region.

Even more important, the Hamiltonian approach endows these values with a physical significance. To understand the physical meaning of the quasilocalization, calculate the boundary term in the Hamiltonian variation:

$$- \frac{1}{4\kappa} \left[ \delta T^{\rho\beta\lambda} N^\mu \sqrt{-g} g^{\beta\sigma} \delta g_{\rho\lambda}^\sigma + \delta(N^\mu H^{[\rho\beta\lambda]} \right] dS_T^\rho. \quad (13)$$

For the Einstein pseudotensor, we use the Freud superpotential (6) as the Hamiltonian boundary term in (12). Then the boundary term in the Hamiltonian variation [20] has the integrand \( \delta \sqrt{-g} g^{\beta\sigma} N^\mu \Gamma^\alpha_\beta \delta g_{\rho\lambda}^\sigma \), which shows not only that \( \sqrt{-g} g^{\beta\sigma} \) is to be held fixed on the boundary, but also that the appropriate displacement vector field is \( N^\mu = \text{constant} \), and the reference configuration here (as well as in the other cases below) is Minkowski space with a Cartesian reference frame.

This calculation is easily adapted to some other cases just by adjusting \( N^\mu \). The Bergmann pseudotensor [3], given by \( 2\sqrt{-g} g T^{\mu\nu} := \partial_\alpha (g^{\nu\sigma} U^{\mu\alpha}) \), leads to the Hamiltonian boundary variation term \( \delta g^{\beta\sigma} g^{\nu\mu} N^\mu \Gamma^\alpha_\beta \delta g_{\rho\lambda}^\sigma dS_T^\rho \), revealing that we are to fix \( \sqrt{-g} g^{\beta\sigma} g^{\nu\mu} \) on the boundary and use the displacement (co)vector \( N_\alpha = N_0^\alpha = \text{constant} \). This last statement takes on a more proper geometric form in terms of an auxiliary (background) metric \( g_{\mu\nu} \), having the Minkowski values in this coordinate system, then the desired displacement vector is \( N^\alpha = \bar{g}^{\alpha\sigma} N^\sigma \).

The Landau-Lifshitz pseudotensor [4] is slightly more complicated, being given by a weighted density \( 2\kappa \sqrt{-g} T_{LL}^{\mu\nu} := \partial_\lambda (\sqrt{-g} g^{\nu\alpha} U^{\mu\alpha}) \). The easiest way to handle this is to introduce, where necessary to obtain the proper geometric density weights, “extra” factors of \( (-g)^{1/2} \), the Jacobian factor for the flat metric (numerically constant in Cartesian coordinates). This leads to the conclusion that the displacement vector should be \( N^\mu = g^{\nu\alpha} (g/\bar{g})^{1/2} N_0^\alpha \) and the quantity to be held fixed on the boundary is \( (-g) g^{\beta\sigma} g^{\nu\mu} \).

The three pseudotensors just discussed are associated with similar but distinct Dirichlet type boundary conditions which are algebraic in terms of the metric. On the other hand, the Møller pseudotensor has a simple Neumann type condition. While the detailed physical significance of these conditions has not yet been probed, such an investigation seems straightforward. In contrast, the remaining pseudotensors in our survey are associated with more complicated boundary conditions.

In the context of Eq. (11), Goldberg [21] discussed the general form \( H^{[\mu\alpha\beta]} \equiv H^{[\mu\alpha]\beta} \equiv H^{[\mu\beta]\alpha} \) including various weighted densities. Because of the symmetries, the associated pseudotensor, \( \sqrt{-g} T^{[\mu\alpha\beta]} := \partial_\lambda \partial_\sigma H^{[\mu\alpha\beta]} \), is guaranteed to be symmetric and conserved for all symmetric \( H^{[\mu\alpha]} \). This pattern can nicely accommodate the Landau-Lifshitz version with \( H^{[\mu\alpha]} := \sqrt{-g} g^{\mu\alpha} \). More
generally, we note that $T(H_1 + H_2) - T(H_1) - T(H_2)$ is also identically conserved leading to the more general pattern $H_{\mu}^{\nu\alpha\beta} = H_1^{\mu\nu\alpha\beta} - H_1^{\mu\nu\alpha\beta}$. With $H_2^{\mu\nu\alpha\beta} = \delta^{\mu\nu\alpha\beta}$ we can now accommodate the pseudotensors of Papapetrou [2] $(H_1^{\mu\nu} = -\sqrt{g}g^{\mu\nu})$ and Weinberg [6] $(H_1^{\mu\nu} = -h^{\mu\nu} + \frac{1}{2}g^{\mu\nu}h_\alpha^\lambda, where h_{\mu\nu} := g_{\mu\nu} - \delta_{\mu\nu}$ and indices are raised with $\tilde{g}^{\mu\nu}$). The Hamiltonian variations then lead to boundary conditions involving rather complicated combinations of $\delta\Gamma^\alpha_{\mu\nu}$ and $\delta(\sqrt{g}g^{\mu\nu})$ or $\delta h_{\mu\nu}$, respectively.

In summary, because of the very nature of the gravitational interaction and its elusive contribution, the localization of total energy-momentum has remained an outstanding fundamental puzzle. The earlier pseudotensor approach was considered to be unsatisfactory. A newer idea is quasilocal. Quasilocal energy-momentum can be obtained from the Hamiltonian. For a finite region it includes a boundary term which plays the key role, determining both the boundary conditions and the quasilocal values. Consequently there are (as in thermodynamics) many different physical kinds of energy, each corresponding to a different boundary condition. We have shown that every energy-momentum pseudotensor is associated with a legitimate Hamiltonian boundary term. Hence the pseudotensors are quasilocal and acceptable. Each is the energy-momentum density for some definite physical situation. Via the Hamiltonian approach one can identify the necessary boundary conditions and thereby appreciate the physical significance of the associated energy-momentum quasilocalization.

Our analysis reclaims the pseudotensor work of the past to its rightful place: concerning a special class of quasilocal energy-momentum. Moreover it is additional evidence that the Hamiltonian boundary term approach to energy provides an effective ordering principle for the various quasilocal energy-momentum expressions.

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[15] Some further principle is needed. We have advocated covariance. For each dynamical field it was found [13], using symplectic techniques [16], that there are only two choices for $B$ corresponding to holding a covariant quantity fixed on the boundary (essentially Dirichlet or Neumann “control mode”). The pseudotensor boundary terms discussed here do not satisfy this covariance principle.


[20] This differs slightly from Eq. (7) because we have not included the $i\omega D\eta$ term of Eq. (3) in Eq. (12).