Determination of the strange-quark mass from QCD pseudoscalar sum rules∗

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A new determination of the strange-quark mass is discussed, based on the two-point function involving the axial-vector current divergences. This Green function is known in perturbative QCD up to order $O(\alpha_s^3)$, and up to dimension-six in the non-perturbative domain. The hadronic spectral function is parametrized in terms of the kaon pole, followed by its two radial excitations, and normalized at threshold according to conventional chiral-symmetry. The result of a Laplace transform QCD sum rule analysis of this two-point function is: $ar{m}_s(1\text{GeV}^2) = 155\pm 25\text{MeV}$.

Past optimistic expectations to determine the values of the light quark masses with increasing precision, in the framework of QCD sum rules, are presently being brought into question. This is a result of the uncovering of various systematic uncertainties, previously unknown or underestimated. First, there is the problem of reconstructing the hadronic spectral function using experimental data on the masses and widths of the ground state hadrons and their (radially) excited states. This information is far from being sufficient to achieve the task. While narrow resonances may be reasonably parametrized by Breit-Wigner forms, this is not the case for broad states, e.g. the $a_1(1260)$ axial-vector meson. Still more important is the potential presence of non-resonant background which could interfere destructively or constructively with the resonance parametrization. While the by now standard procedure of normalizing hadronic spectral functions at threshold using chiral symmetry [1] does provide some form of non-resonant background, this may not be enough. To illustrate the point, let us consider the determination of $(m_u + m_d)$ using Finite Energy QCD sum Rules (FESR) in the pseudoscalar channel, and compare the result of

$$\langle m_u + m_d \rangle (1\text{ GeV}) = 15.5 \pm 2.0 \text{ MeV}, \quad (1)$$

with that of [3]

$$\langle m_u + m_d \rangle (1\text{ GeV}) = 12.0 \pm 2.5 \text{ MeV}. \quad (2)$$

While compatible within errors, these two results lead to rather different values for the strange quark mass. In fact, using the current algebra ratio [4]

$$\frac{m_s}{m_u + m_d} = 12.6 \pm 0.5, \quad (3)$$

one finds from Eqs. (1) and (2), respectively

$$m_s (1\text{ GeV}) = 195 \pm 28 \text{ MeV}$$

$$= 151 \pm 32 \text{ MeV}. \quad (4)$$

The problem here is that the same raw data for resonance masses and widths, plus the same threshold normalization from chiral-perturbation theory has been used in both analyses [2] and [3]. The difference in the results arises mainly from the choice of the functional form for the hadronic spectral function. Since there is no direct experimental information on this function over a wide range of energies, this issue remains unresolved. Results (1) and (2) should be considered together, with the spread in values providing an estimate

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of systematic uncertainties from the hadronic sector. A more dramatic illustration of the size of these uncertainties comes from the QCD Laplace sum rule determination of $m_s$ in the scalar channel. Here, the available experimental data on $K - \pi$ phase shifts \cite{5} should allow, in principle, for a clean reconstruction of the hadronic spectral function from threshold up to $s \simeq 7\text{GeV}^2$. Assuming that the only non-resonant background is the one provided by the chiral-symmetry normalization of the hadronic spectral function at threshold, two independent determinations give

$$m_s (1 \text{ GeV}) = 171 \pm 15 \text{ MeV} \quad (6)$$
$$178 \pm 18 \text{ MeV} \quad (7).$$

A recent reanalysis \cite{8}, in exactly the same framework, has uncovered a sizable systematic uncertainty in the hadronic sector. In fact, it is claimed in \cite{8} that after using the Omnès representation to relate the spectral function to the $K - \pi$ phase shifts, one should include a background interfering destructively with the resonances. As a result of this, the area under the hadronic spectral function is much smaller than that in \cite{6}-\cite{7}, leading to a smaller value of $m_s$, viz. \cite{8}

$$m_s (1 \text{ GeV}) = 140 \pm 20\text{MeV}. \quad (8)$$

Systematic uncertainties are also present in the theoretical, i.e., the QCD, sector. For instance, the perturbative QCD expression of the two-point function involving the vector or axial-vector divergences is generically of the form

$$\psi (Q^2) \propto m_s^2 (Q^2) \left(1 + a_1 \left( \frac{\alpha_s (Q^2)}{\pi} \right) + a_2 \left( \frac{\alpha_s (Q^2)}{\pi} \right)^2 + a_3 \left( \frac{\alpha_s (Q^2)}{\pi} \right)^3 + b_1 m_s^2 (Q^2) \left(1 + c_1 \frac{\alpha_s (Q^2)}{\pi} + \cdots \right) + b_2 m_s^4 (Q^2) \left(1 + c_2 \frac{\alpha_s (Q^2)}{\pi} + \cdots \right) \right). \quad (9)$$

Knowing both $m_s (Q^2)$ and $\alpha_s (Q^2)$ to a given order in perturbation theory, the question is: to expand or not in powers of the inverse logarithms of $Q^2$ appearing in Eq.(7)? It has been argued in \cite{9} that one should make full use of the perturbative expansions of the quark mass and coupling (known presently to 4-loop order) and thus not expand. A strong argument in favour of this alternative is that, numerically, the non-expanded expression is far more stable than the truncated one when going from one order in perturbation theory to the next. In addition, it was shown in \cite{9} that logarithmic truncation can lead to large overestimates of radiative corrections. In any case, repeating the analysis of \cite{6}-\cite{7}, but without expanding leads to a higher value of $m_s$, i.e. \cite{9}

$$m_s (1 \text{ GeV}) = 203 \pm 20\text{MeV}. \quad (10)$$

Combining all the above determinations gives the overall result

$$m_s (1\text{GeV}) = 170 \pm 50\text{MeV}, \quad (11)$$

which provides a realistic estimate of the underlying uncertainties.

Turning to the present determination, we consider the correlator

$$\psi_5 (q^2) = i \int d^4 x \, e^{iqx} < 0 | T (\partial^\mu A_\mu (x) \partial^\nu A_\nu (0)) | 0 \rangle, \quad (12)$$

where $A_\mu (x) = : \bar{s} (x) \gamma_\mu \gamma_5 u (x) :$, and $\partial^\mu A_\mu (x) = m_s : \bar{s} (x) i \gamma_5 u (x) :$. The QCD expression of this two-point function is known \cite{6,7,9} at the four-loop level in perturbative QCD, and up to dimension six in the non-perturbative sector. Also, the old problem of mass singularities has been satisfactorily solved in \cite{6,7}. As a result of this, quark mass corrections are also known up to quartic order. The QCD expression of the Laplace transform of Eq.(10), i.e.

$$\psi_5'' (M^2) = \hat{L} \left[ \psi_5'' (Q^2) \right] = \int_0^\infty e^{-s/M^2} \frac{1}{\pi} \text{Im} \psi_5 (s) \, ds, \quad (13)$$

given by \cite{10}

$$\psi_5'' (M^2) |_{QCD} = [\tilde{m}_s (M^2)]^2 M^4 \left[ \psi_5'' (0) (M^2) + \frac{\psi_5'' (2) (M^2)}{M^2} + \frac{\psi_5'' (4) (M^2)}{M^4} + \frac{\psi_5'' (6) (M^2)}{M^6} + \cdots \right]. \quad (14)$$
ψ''_{5(0)}(M^2) \equiv \hat{L} \left[ \psi''_{5(0)}(Q^2) \right] = \frac{3}{8\pi^2} \left\{ 1 + \bar{\alpha}_s(M^2) \right\}

\left( \frac{11}{3} + 2\gamma_E \right) + \left( \frac{\bar{\alpha}_s(M^2)}{\pi} \right)^2 \left( \frac{5071}{144} - \frac{35}{2} \zeta(3) \right)

+ \frac{17}{4} \gamma_E^2 + \frac{139}{6} \gamma_E - \frac{17}{24} \gamma_E^2 \cdot \frac{\bar{\alpha}_s(M^2)}{\pi} \left( \frac{16}{3} + 4\gamma_E \right)

+ \frac{823}{6} \zeta(3) + \frac{221}{24} \gamma_E^2 + \frac{695}{8} \gamma_E^2 - \frac{221}{48} \gamma_E \pi^2

+ \frac{2720}{9} \gamma_E - \frac{695}{48} \pi^2 \right\} , \quad (13)

\psi''_{5(2)}(M^2) \equiv \hat{L} \left[ \psi''_{5(2)}(Q^2) \right] = -\frac{3}{4\pi^2}

\left[ \bar{m}_s(M^2) \right]^2 \left[ 1 + \frac{\bar{\alpha}_s(M^2)}{\pi} \left( \frac{16}{3} + 4\gamma_E \right) \right] , \quad (14)

\psi''_{5(4)}(M^2) \equiv \hat{L} \left[ \psi''_{5(4)}(Q^2) \right] = \frac{1}{8} < \frac{\alpha_s}{\pi} G^2 >

\left( \frac{1}{2} < m_s \bar{s}s \right) \left[ 1 + \frac{\bar{\alpha}_s}{\pi} \left( \frac{11}{3} + 2\gamma_E \right) \right]

- \left( m_s \bar{u}u \right) \left[ 1 + \frac{\bar{\alpha}_s}{\pi} \left( \frac{14}{3} + 2\gamma_E \right) \right]

+ \frac{3}{28\pi^2} m_s^4 \left[ -\frac{233}{36} - \frac{15}{2} \gamma_E \right]

+ \frac{2\bar{\alpha}_s}{\pi} \left( \frac{37}{9} + 2\gamma_E \right) \left( \frac{\pi}{\alpha_s} - \frac{53}{24} \right) , \quad (15)

and where \( \gamma_E \) is Euler’s constant, \( \zeta(n) \) is Riemann’s zeta function, \( a_1 = 2795.0778 \), all numerical coefficients refer to three flavours and three colours, and we have neglected the up-quark mass everywhere. Given the uncertainties of the method, plus the size of systematic errors, it is not justified to keep \( m_u \) different from zero. In line with the discussion at the beginning, and following [9], we shall not expand the QCD expressions in inverse powers of logarithms, but rather substitute the numerical values of \( \alpha_s \) and \( \bar{m}_s \) for a given value of \( \Lambda_{QCD} \). The dimension-six non-perturbative term has been omitted as it is of no numerical importance. The hadronic spectral function associated with the correlator (10) is very different from that of the vector di-

\[ \frac{1}{\pi} \text{Im} \psi_{5}(s)|_{K^*} = \frac{M_K^2 - 3}{2f_K^2} \frac{3}{2\pi^4} \sum_{s = 0}^{\infty} \frac{I(s)}{s} \theta(s - M_K^2) , \quad (16) \]

where

\[ I(s) = \int_{M_K^2}^{s} \frac{du}{u} \left( u - M_K^2 \right) \left( s - u \right) \left( M_K^2 - s \right) \]

\[ \left[ u - \frac{(s + M_K^2)}{2} \right] - \frac{1}{8u} \left( u^2 - M_K^2 \right) \left( s - u \right) \]

\[ + \frac{3}{4} (u - M_K^2)^2 |F_{K^*}(u)|^2 \] , \quad (17)

and

\[ |F_{K^*}(u)|^2 = \frac{[M_{K^*} - M_K^2]^2 + M_K^2 - \Gamma_{K^*}^2}{(M_{K^*} - u)^2 + M_K^2 - \Gamma_{K^*}^2} \] . \quad (18)

The pion mass has been neglected above, in line with the approximation \( m_u = 0 \) made in the QCD
sector, and in our normalization \( f_K \simeq 93 \text{MeV} \).
The complete hadronic spectral function is then

\[
\frac{1}{\pi} \text{Im} \psi_5(s)|_{\text{HAD}} = 2 f_K^2 M_K^4 \delta(s - M_K^2) \\
+ \frac{1}{\pi} \text{Im} \psi_5(s)|_{K\pi} \frac{[BW_1(s) + \lambda BW_2(s)]}{(1 + \lambda)} \\
+ \frac{1}{\pi} \text{Im} \psi_5(s)|_{\text{QCD}} \theta(s - s_0),
\]

where \( f_K \simeq 1.2 f_\pi \), \( \text{Im} \psi_5(s)|_{\text{QCD}} \) is the perturbative QCD spectral function modelling the continuum which starts at some threshold \( s_0 \), \( BW_{1,2}(s) \) are Breit-Wigner forms for the two kaon radial excitations, normalized to unity at threshold, and \( \lambda \) controls the relative importance of the second radial excitation. The choice \( \lambda \simeq 1 \) results in a reasonable (smaller) weight of the K(1830) relative to the K(1460).

We have solved the Laplace transform QCD sum rules using the values:

\[
\langle \alpha_s G^2 \rangle \simeq 0.024 \text{GeV}^4 , \quad \langle s \bar{s} \rangle \simeq \langle \bar{u} u \rangle = -0.01 \text{GeV}^3 ,
\]

and allowing \( \Lambda_{\text{QCD}} \) and \( s_0 \) to vary in the range: \( \Lambda_{\text{QCD}} = 280 - 380 \text{MeV} \), and \( s_0 = 4 - 8 \text{GeV}^2 \). The results for \( m_s(1 \text{GeV}^2) \) are very stable against variations in the Laplace variable \( M^2 \) over the wide range: \( M = 1 - 4 \text{GeV}^2 \), as well as against variations in the value of \( s_0 \) in the above range. The combined result of this determination is

\[
m_s(1 \text{GeV}^2) = 155 \pm 25 \text{MeV}.
\]

The error given above originates exclusively from changes in the relevant parameters, and does not reflect possible systematic uncertainties from the hadronic sector. These could be large, as discussed in the introduction. Our result is consistent with the other determinations in the scalar channel, Eqs.(5),(6),(8), as well as with the result from combining the determination of \( (m_u + m_d) \) with the current algebra ratio of strange to non-strange quark masses, Eq.(4). It is also in very good agreement with recent lattice QCD results reported at this conference [12] : \( \bar{m}_s(1 \text{GeV}^2) = 155 \pm 15 \text{MeV} \).

REFERENCES

12. V. Lubicz, these Proceedings. See also, V. Gimenez, these Proceedings.