On the Noether charge form of the first law of black hole mechanics

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(November 19, 1998)

Abstract

The first law of black hole mechanics was derived by Wald in a general covariant theory of gravity for stationary variations around a stationary black hole. It is formulated in terms of Noether charges, and has many advantages. In this paper several issues are discussed to strengthen the validity of the Noether charge form of the first law. In particular, a gauge condition used in the derivation is justified. After that, we justify the generalization to non-stationary variations done by Iyer-Wald.

PACS number(s): 04.70.Dy
I. INTRODUCTION

Analogy with thermodynamics is one of the most interesting results in the theory of black holes. It can be summarized as laws of black hole mechanics [1]. In particular, the first law makes it possible to assign entropy to horizon area and temperature to surface gravity up to a coefficient [2]. It was Hawking’s discovery [3] of thermal radiation from a black hole that determined the coefficient.

In Ref. [4], the first law of black hole mechanics was derived in a general covariant theory of gravity for stationary variations around a stationary black hole. It is formulated as a relation among variations of those quantities such as energy, angular momentum and entropy, each of which is defined in terms of a Noether charge. The first law was extended to non-stationary variations around a stationary black hole in Ref. [5].

These first laws in the Noether charge form have many advantages over the original first law. For example, it gives a general method to calculate stationary black hole entropy in general covariant theories of gravity [5]; it connects various Euclidean methods for computing black hole entropy [6]; it suggests a possibility of defining entropy of non-stationary black holes [5,7]; etc.

However, in their derivation there are several issues to be discussed in more detail.

(a) In Ref. [4], unperturbed and perturbed stationary black holes are identified so that horizon generator Killing fields with unit surface gravity coincide near the horizons and that stationary Killing fields and axial Killing fields coincide at infinity. This corresponds to taking a certain gauge condition in linear perturbation theory. For a complete understanding of the first law, we have to clarify whether such a gauge condition can be imposed or not. If it can, then we like to know whether such a gauge condition is necessary. Note that, on the contrary, the original derivation in general relativity by Bardeen, Carter and Hawking [1] is based on a gauge condition such that the stationary Killing fields and the axial Killing fields coincide everywhere on a spacelike hypersurface whose boundary is a union of a horizon cross section and
(b) In Ref. [5], the first law is extended to non-stationary perturbations around a stationary black hole. In the derivation, change of black hole entropy is calculated on a \((n-2)\)-surface, which is a bifurcation surface for an unperturbed black hole, but which is not a cross section of an event (nor apparent) horizon for a perturbed non-stationary black hole in general. Does this mean that black hole entropy would be assigned to a surface which is not a horizon cross section for a non-stationary black hole? It seems more natural to assign black hole entropy to a horizon cross section also for a non-stationary black hole.

In this paper these two issues are discussed and it is concluded that there are no difficulties in the derivation of the Noether charge form of the first law for both stationary and non-stationary perturbations about a stationary black hole. On the way, we give an alternative derivation of the first law based on a variation in which a horizon generator Killing field with unit surface gravity is fixed.

In Sec. II gauge conditions near horizon are investigated. In Sec. III the first law of black holes is derived for stationary variations around a stationary black hole. In Sec. IV the derivation is extended to non-stationary variations around a stationary black hole. Sec. V is devoted to a summary of this paper.

**II. GAUGE CONDITIONS**

Consider a stationary black hole in \(n\)-dimensions, which has a bifurcating Killing horizon. Let \(\xi^a\) denote a generator Killing field of the Killing horizon, which is normalized as \(\xi^a = t^a + \Omega^{(\mu)} H \varphi_{(\mu)}\), and \(\Sigma\) be the bifurcation surface. Here, \(t^a\) is the Killing field of stationarity with unit norm at infinity, \(\{\varphi_{(\mu)}\} (\mu = 1, 2, \cdots)\) is a family of axial Killing fields, and \(\{\Omega^{(\mu)} H\}\) is a family of constants (angular velocities).

Now let us show that it is not possible in general to impose a gauge condition such that
\( \delta \xi^a = 0 \) near the bifurcation surface. For this purpose we shall temporarily assume that \( \delta \xi^a = 0 \) and show a contradiction.

On \( \Sigma \), the covariant derivative of \( \xi^a \) is given by

\[
\nabla_b \xi^a = \kappa \epsilon^a_b,
\]

where \( \kappa \) is the surface gravity corresponding to \( \xi^a \) and \( \epsilon_{ab} \) is binormal to \( \Sigma \). However, the variation of the l.h.s. is zero:

\[
\delta(\nabla_b \xi^a) = \delta \Gamma^a_{bc} \xi^c = 0
\]

since \( \xi^a = 0 \), where \( \delta \Gamma^a_{bc} \) is given by

\[
\delta \Gamma^a_{bc} = \frac{1}{2} g^{ad}(\nabla_c \delta g_{db} + \nabla_b \delta g_{dc} - \nabla_d \delta g_{bc}).
\]

Hence,

\[
\delta \epsilon^a_b = -\frac{\delta \kappa}{\kappa} \epsilon^a_b.
\]

Substituting this into the identity \( \delta(\epsilon^a_b \epsilon^b_a) = 0 \), we obtain

\[
0 = \delta(\epsilon^a_b \epsilon^b_a) = -\frac{4 \delta \kappa}{\kappa}.
\]

Thus, the assumption \( \delta \xi^a = 0 \) leads \( \delta \kappa = 0 \), which implies, for example, that \( \delta M = 0 \) for the vacuum general relativity in staticity, where \( M \) is mass of Schwarzschild black holes.

This peculiar behavior can be understood as appearance of a coordinate singularity at the bifurcation surface of a coordinate fixed by the gauge condition \( \delta \xi^a = 0 \) since in the above argument finiteness of \( \delta \Gamma^a_{bc} \) has been assumed implicitly. Therefore, it is impossible to impose the condition \( \delta \xi^a = 0 \) near the bifurcation surface whenever \( \delta \kappa \neq 0 \).

As mentioned above, the original derivation of the first law in Ref. [1] adopt the gauge condition \( \delta t^a = \delta \varphi^a = 0 \). This leads \( \delta \xi^a = 0 \) when \( \delta \Omega = 0 \) (for example, when we consider static black holes). Of course, in Ref. [1], a general horizon cross section (not necessary a bifurcation surface) is considered as a surface on which black hole entropy is calculated.
Hence, the above argument arises no difficulties unless the cross section is taken to be the bifurcation surface. The derivation in Ref. [1] suffers from the above argument if and only if black hole entropy is estimated on the bifurcation surface.

On the other hand, arguments like the above do not lead to any contradiction if we adopt a gauge condition such that $\tilde{\xi}^a$ is fixed near the bifurcation surface under variations, where $\tilde{\xi}^a = \xi^a / \kappa$ is a horizon generator Killing field with unit surface gravity. Moreover, it is concluded that, if we intend to fix a horizon generator Killing field, then it must have the same value of surface gravity for unperturbed and perturbed black holes. Hence, the gauge condition $\delta \tilde{\xi}^a = 0$ near the bifurcation surface adopted in Ref. [4,5] is very natural one.

In fact, it is always possible to identify unperturbed and perturbed stationary black holes so that the Killing horizons and the generator Killing fields with unit surface gravity coincide. As stated in Ref. [4], such an identification can be done at least in a neighborhood of the horizon by using the general formula for Kruskal-type coordinates $(U, V)$ given in Ref. [10]. (The identified Killing horizon is given by $U = 0$ and $V = 0$. The identified Killing field with unit surface gravity is given by $\tilde{\xi}^a = U(\partial/\partial U)^a - V(\partial/\partial V)^a$.)

The purpose of the next section is to discuss about the remaining gauge condition $\delta t^a = \delta \varphi^a = 0$ at infinity. It is evident that this gauge condition at infinity can be imposed by identifying the perturbed and unperturbed specetimes suitably. So, our question now is whether this gauge condition is necessary or not. For this purpose we temporarily adopt a gauge condition such that $\tilde{\xi}^a$ is fixed everywhere on a hypersurface connecting the bifurcation surface and spatial infinity. In deriving the first law in this gauge condition, the gauge condition $\delta t^a = \delta \varphi^a = 0$ at infinity is found to be necessary for a proper interpretation of the first law. On the other hand, as shown in Sec. IV, it is not necessary to fix $\tilde{\xi}^a$ near the bifurcation surface, strictly speaking. Hence, it can be concluded that the minimal set of gauge conditions necessary for the derivation of the first law is that $t^a$ and $\varphi^a$ are fixed at spatial infinity.
III. THE FIRST LAW FOR STATIONARY BLACK HOLES

Before deriving the first law, we review basic ingredients of the formalism.

In this paper, we consider a classical theory of gravity in \( n \) dimensions with arbitrary matter fields, which is described by a diffeomorphism invariant Lagrangian \( n \)-form \( \mathbf{L}(\phi) \), where \( \phi \) denotes dynamical fields in the sense of Ref. [5].

The Noether current \((n - 1)\)-form \( j[V] \) for a vector field \( V^a \) is defined by

\[
 j[V] \equiv \Theta(\phi, \mathcal{L}_V \phi) - V \cdot \mathbf{L}(\phi),
\]

where the \((n - 1)\)-form \( \Theta(\phi, \delta \phi) \) is defined by

\[
 \delta \mathbf{L}(\phi) = \mathbf{E}(\phi) \delta \phi + d \Theta(\phi, \delta \phi).
\]

It is easily shown that the Noether current is closed as

\[
 dj[V] = -\mathbf{E}(\phi) \mathcal{L}_V \phi = 0,
\]

where we have used the equations of motion \( \mathbf{E}(\phi) = 0 \). Hence, by using the machinery developed in Ref. [8], we obtain the Noether charge \((n - 2)\)-form \( Q[V] \) such that

\[
 j[V] = dQ[V].
\]

Hereafter, we assume that in an asymptotically flat spacetime there exists an \((n - 1)\)-form \( \mathbf{B} \) such that

\[
 \int_{\infty} V \cdot \delta \mathbf{B}(\phi) = \int_{\infty} V \cdot \Theta(\phi, \delta \phi).
\]

By using \( \mathbf{B} \), we can write a Hamiltonian \( H[V] \) corresponding to evolution by \( V^a \) as follows [4].

\[
 H[V] \equiv \int_{\infty} (Q[V] - V \cdot \mathbf{B}).
\]

The symplectic current density \( \omega(\phi, \delta_1 \phi, \delta_2 \phi) \) is defined by

\[
 \omega(\phi, \delta_1 \phi, \delta_2 \phi) \equiv \delta_1 [\Theta(\phi, \delta_2 \phi)] - \delta_2 [\Theta(\phi, \delta_1 \phi)]
\]
and is linear both in $\delta_1 \phi$ and its derivatives, and $\delta_2 \phi$ and its derivatives [9].

Now we define a space of solutions in which we take a variation to derive the first law.

Let $\tilde{\xi}^a$ be a fixed vector field, which vanishes on a $(n-2)$-surface $\Sigma$. (Note that $\tilde{\xi}^a$ and $\Sigma$ can be defined without referring to any dynamical fields, e.g. the metric $g_{ab}$.) In the following arguments, we consider a space of stationary, asymptotically flat solutions of the equations of motion $E(\phi) = 0$, each of which satisfies the following three conditions. (a) There exists a bifurcating Killing horizon with the bifurcation surface $\Sigma$. (b) $\tilde{\xi}^a$ is a generator Killing field of the Killing horizon. (c) Surface gravity corresponding to $\tilde{\xi}^a$ is 1:

$$\tilde{\xi}^b \nabla_b \tilde{\xi}^a = \tilde{\xi}^a,$$

(13)
on the Killing horizon.

For each element in this space, there exist constants $\kappa$ and $\Omega_H^{(\mu)}$ ($\mu = 1, 2, \cdots$) such that

$$\kappa \tilde{\xi}^a = t^a + \Omega_H^{(\mu)} \varphi^a_{(\mu)},$$

(14)

where $t^a$ is the timelike Killing field of stationarity with unit norm at infinity, $\{\varphi^a_{(\mu)}\}$ ($\mu = 1, 2, \cdots$) is a family of axial Killing fields. Hence, $\kappa$ is surface gravity and $\Omega_H^{(\mu)}$ are angular velocities of the horizon.

Note that, by definition, the vector field $\tilde{\xi}^a$ is fixed under a variation of dynamical fields. We express this explicitly by denoting the variation by $\tilde{\delta}$:

$$\tilde{\delta} \tilde{\xi}^a = 0.$$

(15)

We now derive the first law of black hole mechanics.

First, by taking a variation of the definition (6) for $j[\tilde{\xi}]$ and using (15) and (7), we obtain

$$\tilde{\delta} j[\tilde{\xi}] = \tilde{\delta} \left( \Theta(\phi, L_{\tilde{\xi}} \phi) \right) - \tilde{\xi} \cdot \left( E(\phi) \tilde{\delta} \phi + d \Theta(\phi, \tilde{\delta} \phi) \right)$$

$$= \omega(\phi, \tilde{\delta} \phi, L_{\tilde{\xi}} \phi) + d \left( \tilde{\xi} \cdot \Theta(\phi, \tilde{\delta} \phi) \right).$$

(16)

Here we have used the equations of motion $E(\phi) = 0$ and the following identity for an arbitrary vector $V^a$ and an arbitrary differential form $\Lambda$ to obtain the last line.
\[ \mathcal{L}_V \Lambda = V \cdot d \Lambda + d(V \cdot \Lambda). \]  
(17)

Since \( \omega(\phi, \tilde{\delta} \phi, \mathcal{L}_\xi \phi) \) is linear in \( \mathcal{L}_\xi \phi \) and its derivatives, we obtain

\[ d(\tilde{\delta} \mathcal{Q}[[\tilde{\xi}]]) = d \left( \tilde{\xi} \cdot \Theta(\phi, \tilde{\delta} \phi) \right) \]  
(18)

by using \( \mathcal{L}_\xi \phi = 0 \) and Eq. (9).

Next we integrate Eq. (18) over an asymptotically flat spacelike hypersurface \( \mathcal{C} \), which is orthogonal to \( t^a \) at infinity and the interior boundary of which is \( \Sigma \). Since \( \tilde{\xi}^a = 0 \) on \( \Sigma \), we obtain

\[ \tilde{\delta} \int_{\Sigma} \mathcal{Q}[\tilde{\xi}] = \tilde{\delta} H[\tilde{\xi}] \]  
(19)

Finally we transform the r.h.s. and the l.h.s. of (19) to forms useful to be estimated at infinity and the horizon, respectively.

A relation among variations of \( \kappa, \Omega_{H}^{(\mu)} \), \( t^a \) and \( \varphi_{(\mu)}^a \) is obtained by substituting (14) to (15).

\[ t^a \tilde{\delta} \left( \frac{1}{\kappa} \right) + \varphi_{(\mu)}^a \tilde{\delta} \left( \frac{\Omega_{H}^{(\mu)}}{\kappa} \right) = - \frac{1}{\kappa} \kappa \tilde{\delta} t^a - \frac{\Omega_{H}^{(\mu)}}{\kappa} \kappa \tilde{\delta} \varphi_{(\mu)}^a. \]  
(20)

By using this relation and the fact that \( H[V] \) is linear in the vector field \( V \), we can rewrite the r.h.s. of (19) as follows.

\[ \tilde{\delta} H[\tilde{\xi}] = \frac{1}{\kappa} \delta_\infty H[t] - H[\tilde{\delta} t] + \frac{\Omega_{H}^{(\mu)}}{\kappa} \left( \tilde{\delta} H[\varphi_{(\mu)}] - H[\tilde{\delta} \varphi_{(\mu)}] \right) \]

\[ = \frac{1}{\kappa} \delta_\infty H[t] + \frac{\Omega_{H}^{(\mu)}}{\kappa} \delta_\infty H[\varphi_{(\mu)}], \]  
(21)

where the variation \( \delta_\infty \) is defined for linear functionals \( F[t] \) and \( G_{(\mu)}[\varphi_{(\mu)}] \) so that

\[ \delta_\infty F[t] = \tilde{\delta} F[t] - F[\tilde{\delta} t], \]

\[ \delta_\infty G_{(\mu)}[\varphi_{(\mu)}] = \tilde{\delta} G_{(\mu)}[\varphi_{(\mu)}] - G_{(\mu)}[\tilde{\delta} \varphi_{(\mu)}]. \]  
(22)

This newly introduced variation corresponds to a variation at infinity such that \( t^a \) and \( \varphi^a \) are fixed.
\[ \delta_\infty t^a = \delta_\infty \varphi^a_{(\mu)} = 0. \] (23)

In Ref. [5] a useful expression of the Noether charge was given as follows.

\[ Q[V] = W_e(\phi)V^e + X^{cd}(\phi)\nabla_c V_d + Y(\phi, \mathcal{L}_V \phi) + dZ(\phi, V), \] (24)

where \( W_e, X^{cd}, Y \) and \( Z \) are locally constructed covariant quantities. In particular, \( Y(\phi, \mathcal{L}_V \phi) \) is linear in \( \mathcal{L}_V \phi \) and its derivatives, and \( X^{cd} \) is given by

\[ \left( X^{cd}(\phi) \right)_{c_1 \cdots c_n} = -E^{abcd}_R \epsilon_{abc_1 \cdots c_n}. \] (25)

Here \( E^{abcd}_R \) is the would-be equations of motion form [5] for the Riemann tensor \( R^{abcd} \) and \( \epsilon_{abc_1 \cdots c_n} \) is the volume \( n \)-form.

By using the form of \( Q \) we can rewrite the integral in the l.h.s. of (19) as

\[ \int_\Sigma Q[\tilde{\xi}] = \int_\Sigma X^{cd}(\phi)\nabla_c \tilde{\xi}_d, \] (26)

where we have used the Killing equation \( \mathcal{L}_{\tilde{\xi}} \phi = 0 \) and the fact that \( \tilde{\xi}^a = 0 \) on \( \Sigma \).

Using the relation

\[ \nabla_c \tilde{\xi}_d = \epsilon_{cd} \] (27)

on \( \Sigma \), for any stationary solutions we can eliminate explicit dependence of Eq. (26) on \( \tilde{\xi} \), where \( \epsilon_{cd} \) is the binormal to \( \Sigma \). Hence, at least within the space of stationary solutions, we can take the variation \( \tilde{\delta} \) of the integral without any difficulties.

Therefore, we obtain the first law for stationary black holes by rewriting Eq. (19) as

\[ \frac{\kappa}{2\pi} \delta_\infty S = \delta_\infty \mathcal{E} - \Omega^{(\mu)}_H \delta_\infty \mathcal{J}_{(\mu)}, \] (28)

where entropy \( S \) is defined by

\[ S \equiv 2\pi \int_\Sigma X^{cd}(\phi)\epsilon_{cd}, \] (29)

and energy \( \mathcal{E} \) and angular momenta \( \mathcal{J}_{(\mu)} \) are defined by
\[ \mathcal{E} \equiv H[t] = \int_{\infty}^{\infty} (Q[t] - t \cdot B), \]
\[ \mathcal{J}(\mu) \equiv -H[\varphi(\mu)] = -\int_{\infty}^{\infty} Q[\varphi(\mu)]. \]

Note that, in the r.h.s. of Eq. (28), variations of \( \mathcal{E} \) and \( \mathcal{J}(\mu) \) are taken with \( t^a \) and \( \varphi(\mu) \) are fixed. This condition is explicitly implemented by the definition (22) of \( \delta_{\infty} \) and is necessary for a proper interpretation of the first law.

We conclude this section by giving another expression of the entropy.

Since \( \tilde{\xi}^a \) is a generator Killing field of the Killing horizon, we have \( \mathcal{L}_{\tilde{\xi}} \phi = 0 \) and the pull-back of \( \tilde{\xi} \cdot \mathcal{L}(\phi) \) to the horizon vanishes. Hence, the definition (6) says that the pull-back of \( j[\tilde{\xi}] \) to the horizon is zero [7]. Thus, the integral of \( Q[\tilde{\xi}] \) is independent of the choice of the horizon cross section.

Moreover, it can be shown that the integral in (29) is the same even if we replace the integration surface \( \Sigma \) by an arbitrary horizon cross section \( \Sigma' \) [7]. Therefore we obtain

\[ S = 2\pi \int_{\Sigma'} \mathbf{X}^{cd}(\phi) \epsilon'_{cd}, \]  

where \( \epsilon'_{cd} \) denotes the binormal to \( \Sigma' \).

**IV. NON-STATIONARY PERTURBATION**

In this section, we shall derive the first law for a non-stationary perturbation about a stationary black hole with a bifurcating Killing horizon. Unfortunately, for non-stationary perturbations, \( \delta\kappa \) and \( \delta\Omega_H^{(\mu)} \) do not have meaning of perturbations of surface gravity and angular velocity of the Killing horizon, even if they are defined. However, since the first law (28) does not refer to \( \delta\kappa \) and \( \delta\Omega_H^{(\mu)} \) but only to the unperturbed values of \( \kappa \) and \( \Omega_H^{(\mu)} \), we expect that the first law holds also for non-stationary perturbations. In the following, we shall show that it does hold.

First, we specify a space of solutions in which we take a variation.

Let \( \tilde{\xi}_0^a \) be a fixed vector field, which vanishes on an fixed \((n - 2)\)-surface \( \Sigma \). In this section, we consider a space of asymptotically flat solutions of the field equation \( \mathbf{E}(\phi) = 0, \)
for each of which \( \tilde{\xi}_0^a \) is an asymptotic Killing field.

For each solution in this space, there exist constants \( \kappa \) and \( \Omega_H^{(\mu)} (\mu = 1, 2, \cdots) \) such that at spatial infinity

\[
\kappa \tilde{\xi}_0^a = t^a + \Omega_H^{(\mu)} \varphi_{(\mu)}^a, \tag{32}
\]

where \( t^a \) is a timelike asymptotic Killing field with unit norm at infinity, \( \{ \varphi_{(\mu)}^a \} (\mu = 1, 2, \cdots) \) is a family of axial asymptotic Killing fields orthogonal to \( t^a \) at infinity and \( \{ \Omega_H^{(\mu)} \} \) is a family of constants. Note that the constants \( \kappa \) and \( \Omega_H^{(\mu)} \) do not have meaning of surface gravity and angular velocities unless we consider a stationary solution. Moreover, in general, \( \tilde{\xi}_0^a \) and \( \Sigma \) have no meaning but an asymptotic Killing field and a fixed \((n - 2)\)-surface, respectively.

Note that, by definition, the vector field \( \tilde{\xi}_0^a \) is fixed under the variation. We denote the variation by \( \tilde{\delta} \):

\[
\tilde{\delta} \tilde{\xi}_0^a = 0. \tag{33}
\]

On the contrary, \( t^a, \varphi_{(\mu)}^a, \kappa \) and \( \Omega_H^{(\mu)} \) are not fixed under the variation since definitions of them refer to dynamical fields, which are varied. Their variations are related by (20).

Suppose that an element \( \phi_0 \) of the space of solutions satisfies the following three conditions. (a') \( \phi_0 \) is a stationary solution with a bifurcating Killing horizon with the bifurcation surface \( \Sigma \). (b') \( \tilde{\xi}_0^a \) is a generator Killing field of the Killing horizon of \( \phi_0 \). (c') Surface gravity of \( \phi_0 \) corresponding to \( \tilde{\xi}_0^a \) is 1:

\[
\tilde{\xi}_0^b \nabla_b \tilde{\xi}_0^a = \tilde{\xi}_0^a, \tag{34}
\]

on the Killing horizon.

Now we derive the first law for the non-stationary perturbation \( \tilde{\delta} \phi \) around the stationary solution \( \phi_0 \).

First, we mention that the validity of Eq. (19) in the previous section depends on the following three facts. (i) The equations of motion \( E(\phi) = 0 \) hold for both unperturbed and perturbed fields. (Unless they hold also for perturbed fields, \( \tilde{\delta} j \) can not be rewritten
as $d(\tilde{\delta}Q)$. (ii) $\tilde{\xi}^a$ (corresponding to $\tilde{\xi}_0^a$) is a Killing field of the unperturbed solution. (iii) $\tilde{\xi}^a = 0$ (corresponding to $\tilde{\xi}_0^a = 0$) on $\Sigma$ for unperturbed solution.

These three are satisfied for the unperturbed solution $\phi_0$ and the non-stationary variation $\tilde{\delta}\phi$ around $\phi_0$, too. Thus, Eq. (19) is valid.

Since $\tilde{\delta}t^a$, $\tilde{\delta}\varphi_{(\mu)}$, $\tilde{\delta}\kappa$ and $\tilde{\delta}\Omega_{(\mu)}^H$ are related by Eq. (20), we can transform the r.h.s. of (19) to obtain

$$\kappa \tilde{\delta} \int_{\Sigma} Q[\tilde{\xi}_0] = \delta_\infty E - \Omega_{(\mu)}^H \delta_\infty J_{(\mu)},$$

(35)

where, as in the previous section, energy $E$ and angular momenta $J_{(\mu)}$ are defined by (30), and the variation $\delta_\infty$ is defined at infinity so that $t^a$ and $\varphi_{(\mu)}$ are fixed. Here note that $\kappa$ and $\Omega_{(\mu)}^H$ are surface gravity and angular velocities, respectively, for $\phi_0$.

Up to this point we have not yet used explicitly the fact that $\tilde{\xi}_0^a = 0$ on $\Sigma$ for the perturbed solution, although we have used it implicitly. By using it explicitly, we can rewrite the l.h.s. of (35) in a useful form. The result is

$$\tilde{\delta} \int_{\Sigma} Q[\tilde{\xi}_0] = \frac{1}{2\pi} \tilde{\delta} S,$$

(36)

where $S$ is defined by (29). (For explicit manipulations, see the proof of Theorem 6.1 of Iyer-Wald [5].)

Finally, we obtain the first law (28) for non-stationary perturbations $\tilde{\delta}\phi$ about a stationary black hole solution $\phi_0$.

Now we comment on entropy for the perturbed, non-stationary black hole.

As stated above, the $(n - 2)$-surface $\Sigma$ has no meaning for the perturbed solution. (It is a surface on which $\tilde{\xi}_0^a$ vanishes.) In general, it does not lie on the event (or apparent) horizon for the perturbed solution. Hence, entropy evaluated on $\Sigma$ may not coincide with that on a cross section of the perturbed horizon, provided that the entropy is defined as $2\pi$ times an integral of $Q[\tilde{\xi}_0]$ for both $(n - 2)$-surfaces. Note that it is in general impossible to make gauge transformation so that $\Sigma$ lie on a horizon cross section, if entropy (e.g. a quarter of area in general relativity) on $\Sigma$ is different from entropy on a horizon cross section. The
difference is given by $2\pi$ times an integral of the Noether current $j[\tilde{\xi}_0]$ over a hypersurface whose boundary is a union of $\Sigma$ and a cross section of the perturbed horizon. Since it is natural to assign black hole entropy to the horizon cross section [5], it might be expected that there appears an extra term corresponding to the integral of $j[\tilde{\xi}_0]$ in the first law.

However, as shown in the next paragraph, the integral of $j[\tilde{\xi}_0]$ vanishes to first order in $\tilde{\delta}\phi$ [12]. Thus, $\tilde{\delta}S$ evaluated on $\Sigma$ gives the correct variation of entropy defined on the horizon to first order in $\tilde{\delta}\phi$. This means that the extra term does not appear and that the first law of Ref. [5] derived in this section for non-stationary perturbation about a stationary black hole is the correct formula.

Let us show the above statement. Since $\tilde{\xi}_a^0 = 0$ on $\Sigma$ and $L_{\tilde{\xi}_0} \phi_0 = 0$, the Noether current $j[\tilde{\xi}_0]$ vanishes on $\Sigma$ for the unperturbed solution by the definition (6). Hence, for the perturbed solution, the Noether current is at least first order in $\tilde{\delta}\phi$ on $\Sigma$. On the other hand, deviation of a horizon cross section from $\Sigma$ is at least first order. Therefore, the integral of $j[\tilde{\xi}_0]$ over a hypersurface connecting $\Sigma$ and the perturbed horizon cross section is at least second order in $\tilde{\delta}\phi$.

Finally, let us apply the first law of this section to a stationary perturbation. The result is the same as that derived in the previous section. It is evident that the gauge condition used in this section is weaker than that used in the previous section. In fact, $\tilde{\xi}_a (\neq \tilde{\xi}_a^0$ for a perturbed solution) is not fixed in the former condition. Hence, it can be concluded that the minimal set of gauge conditions necessary for the derivation of the first law is that $t^a$ and $\varphi_\mu^{\nu}$ are fixed at spatial infinity.

V. SUMMARY AND DISCUSSION

In this paper we have re-analyzed Wald and Iyer-Wald derivation of the first law of black hole mechanics. In particular, two issues listed in Sec. I have been discussed in detail: (a) gauge conditions and (b) near-stationary black hole entropy. We can conclude that there are no difficulties in the derivation of the Noether charge form of the first law for both stationary
and non-stationary perturbations about a stationary black hole.

Unfortunately, the first law investigated in this paper cannot be applied to a purely dynamical situation. However, at least in general relativity, it seems possible to formulate a dynamical version of the first law as a law of dynamics of a trapping (or apparent) horizon [13–15].

**Acknowledgments**

The author thanks Professors H. Kodama and W. Israel for their continuing encouragement. He also thanks S. A. Hayward and M. C. Ashworth for helpful discussions. He was supported by JSPS program for Young Scientists, and this work was supported partially by the Grant-in-Aid for Scientific Research Fund (No. 9809228).
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[12] The author thanks Professor R. M. Wald for helpful comments on this point.

