Spreading Width for Decay out of a Superdeformed Band

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The attenuation factor $F$ responsible for the decay out of a superdeformed (SD) band is calculated with the help of a statistical model. This factor is given by $F = (1 + \Gamma^0 / \Gamma_S)^{-1}$. Here, $\Gamma_S$ is the width for the collective $E2$ transition within the superdeformed band, and $\Gamma^0$ is the spreading width which describes the mixing between a state in the SD band and the normally deformed (ND) states of equal spin. The attenuation factor $F$ is independent of the statistical $E1$ decay widths $\Gamma_N$ of the ND states provided that $\Gamma_N \gg \Gamma^0, \Gamma_S$. This condition is generically met. Previously measured values of $\Gamma_N$ for the ND states have identical values denoted by $\Gamma_N$. The coupling matrix elements $V_{ij}$ connect the SD and the ND states and are responsible for decay out of the SD band. Following Refs. [2–5], we assume that the ND states $\{j\}$ can be modeled as eigenstates of the Gaussian Orthogonal Ensemble (GOE) of random matrices. The spreading width $\Gamma^0$ is defined as $\Gamma^0 = 2\pi \sigma^2 / D$, where $\sigma^2$ is the mean square of the matrix elements $V_{ij}$. Typically, we have $\Gamma_N, \Gamma_S \ll D$, see Table I. We note that in all cases, $\Gamma_S \ll \Gamma_N$. Because of the dominance of $\Gamma_S$ over all other transitions, this inequality is expected to hold generically. It is fair to expect that decay out of the SD band sets in whenever decay time $\tau = \Gamma_S / \Gamma_N$ and mixing time $\tau^0 = \Gamma^0 / \Gamma_S$ of the state $\{0\}$ are comparable, i.e., when $\Gamma^0$ is of the same order of magnitude as $\Gamma_S$. Thus, we expect that we also have $\Gamma^0 \ll \tau$. This is, indeed, borne out by the analysis described below.

The Hamiltonian $H$ of the system is modeled as a matrix of dimension $K + 1$ and has the form

$$H = \begin{pmatrix} E_0 & V_{0j} \\ V_{0l} & \delta \xi E_j \end{pmatrix}. $$ (1)

To $H$ must be added the diagonal width matrix $\Sigma$ with

$$\Sigma = -i/2 \begin{pmatrix} \Gamma_S & 0 \\ 0 & \delta \xi \Gamma_N \end{pmatrix}. $$ (2)

The effective Hamiltonian $\mathcal{H}$ is given by $\mathcal{H} = H + \Sigma$.

We first investigate an experimental situation which would require an energy resolution that is not available at present: We study the distribution in energy of the $E2$ transition intensity feeding the state $\{0\}$. Since $\Gamma^0 \neq 0$, the

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this state is mixed with the ND states, and the E2 transition intensity feeding the state is spread out over a number of eigenstates of $\mathcal{H}$. We calculate the ensemble average of this intensity distribution. The distribution cannot be measured at present because the required resolution is of the order of the average spacing $D$ of the ND states. However, the calculation defines concepts and yields results which are important later on.

We first consider the case $\Gamma_N = 0$. For $\Gamma^\downarrow < D$, the mixing of the SD state with the ND states will cause the E2 feeding strength to have a central peak located at (or near) $E_0$, and a number of much smaller and well-separated peaks located at (or near) the energies $E_j$. The central peak will obviously be reduced in height compared to the case $\Gamma^\downarrow = 0$. If observable, this phenomenon could be referred to as "decay out of the SD band". The peak E2 transition intensity is reduced. The reduction occurs even when the ND states are not capable of decaying by statistical E1 emission. If we allow for the statistical E1 decay of the ND states, the picture remains essentially the same except that all peaks become wider. We expect that the reduction of the E2 transition peak intensity is not dependent on $\Gamma_N$ and is governed by the competition between mixing time and decay time, i.e., by the ratio $\Gamma^\downarrow / \Gamma_S$. We show presently that this is indeed the case.

To calculate the distribution of transition strength, we again consider first the case $\Gamma^\downarrow = 0$. The state [0] is fed from the next higher SD state. The relative transition intensity $I(E)$ versus energy $E$ is given by

$$ I(E) = -2 \gamma \left[ \text{Im} \left( E - E_0 + (i/2)\Gamma_S \right) \right]^{-1} \gamma, $$

where $\text{Im}$ (or, later, $\text{Re}$) stands for the imaginary (the real) part. The symbol $\gamma$ denotes the E2 decay amplitude feeding the state [0], and $I(E)$ has the expected Lorentzian form. Integrating over all $E$, we obtain

$$ \int dE I(E) = 2 \pi \gamma^2. $$

We extend Eq. (3) to the case $\Gamma^\downarrow \neq 0$ by writing it as

$$ I(E) = -2 \gamma \left[ \text{Im} \left( E - \mathcal{H} \right) \right]^{-1}_{00} \gamma, $$

where $\mathbf{E} = E \times 1K_{+1}$ and where $1K$ is the $K$-dimensional unit matrix. To check the general validity of Eq. (4), we transform $\mathcal{H}$ to diagonal form with left (right) eigenfunctions $\psi_m^L (\psi_m^R)$ and complex eigenvalues $\mathcal{E}_m$ with $m = 0, \ldots, K$. Then, Eq. (4) takes the form

$$ I(E) = -2 \gamma \sum_{m=0}^{K} \text{Im} \left[ \langle 0 | \psi_m^L | (E - \mathcal{E}_m)^{-1} \psi_m^R | 0 \rangle \right] \gamma, $$

which is obviously a sum of Lorentzians. The $m$th Lorentzian is peaked at the energy $E = \text{Re} \mathcal{E}_m$. The area under the peak is given by $2 \pi$ times the probability $\langle 0 | \psi_m^L | \psi_m^R | 0 \rangle$ to find the state [0] admixed into the $m$th eigenstate. A change of the E1 width $\Gamma_N$ will result in a change of the peak widths of the $m$ resonances and the weight factors $\langle 0 | \psi_m^L | \psi_m^R | 0 \rangle$. The total intensity (the sum of the contributions of all $m$ resonances) is given by $2 \pi \gamma^2$, as in the case $\Gamma^\downarrow = 0$. It is independent of $\Gamma_N$. These results show that we are dealing with a spreading width phenomenon, which is not related to the E1 decay of the ND states.

It remains to average $I(E)$ over the GOE. We recall that the energies $E_j$ are the eigenvalues of a GOE matrix, and that, correspondingly, the matrix elements $V_{ij}$ are uncorrelated Gaussian-distributed random variables with common mean value zero and common variance $\nu^2$. We denote the ensemble average by a bar. Clearly, all resonance structure will be washed out in $\overline{I(E)}$, and physical intuition and the arguments given above lead us to expect that $\overline{I(E)}$ has Lorentzian shape and width $\Gamma_S + \Gamma^\downarrow$. This is, indeed, the result of the calculation. To see this, we put first $E = E_0$ and write

$$ \left[ \left( E_0 - \mathcal{H} \right)^{-1} \right]_{00} = -\left( i / \Gamma_S \right) \left[ 1 + S_00(E_0) \right]. $$

Here, $S_00(E)$ is given by

$$ S_00(E) = 1 - 2 \pi i \sum_{\beta \neq \gamma} w_{\beta \gamma} |D^{-1}(E)|_{\beta \gamma} w_{\gamma \beta}, $$

where $w_{\beta \gamma} = w_{\gamma \beta} = \sqrt{2/(\pi \Gamma_S)} V_{\beta \gamma}$ and

$$ D_{\beta \gamma} = (E - E_j + (i/2)\Gamma_N) \delta_{\beta \gamma} + i \pi w_{\beta \gamma} w_{\gamma \beta}. $$

Equations (7,8) show that $S_00$ is an element of a bona fide unitary scattering matrix. Aside from the channel denoted by zero, there are inelastic channels corresponding to the emission of E1 radiation by the ND states and leading to the appearance of the term $\Gamma_N$ in Eq. (8). We have introduced the form Eq. (6) because calculating the ensemble average over the diagonal element of a scattering matrix defined in terms of the GOE is a standard problem in stochastic scattering theory. The ensemble average is taken [11] over the distribution of the $V_{ij}$’s and $E_j$’s and yields

$$ \overline{S_00(E)} = \frac{E - E_0 + (i/2)(\Gamma_S - \Gamma^\downarrow)}{E - E_0 + (i/2)(\Gamma_S + \Gamma^\downarrow)}. $$

For the reader not familiar with the technicalities of ref. [11], we mention that Eq. (9) can be obtained in a simpler way: We replace the ensemble average by the substitution $E_j \to E_j + i I$ with $I \gg D$. This is patterned after Brown’s energy averaging procedure [12]. However, we keep $E_0$ fixed because the GOE average does not affect the SD state. The independence of $\overline{S_00}$ of $\Gamma_N$ is caused by the GOE average which replaces the discrete spectrum of the ND states by a quasi-continuum. Using this result for averaging Eqs. (6) and (4), we find

$$ \overline{I(E)} = -2 \gamma \left[ \text{Im} \left( E - E_0 + (i/2)(\Gamma_S + \Gamma^\downarrow) \right) \right]^{-1} \gamma. $$
Equation (10) shows that the average intensity has Lorentzian shape with width $\Gamma_S + \Gamma^4$ and is independent of $\Gamma_N$. The ratio of $\overline{I}(E_0)$ and the intensity for $\Gamma^4 = 0$ at $E = E_0$ given by Eq. (3) yields the average peak intensity attenuation factor $F$,

$$F = (1 + (\Gamma^4/\Gamma_S))^{-1} .$$

The peak attenuation factor $F$ does possess physical significance but cannot be measured in practice. However, $F$ also determines the reduction of the total intensity of E2 radiation down from the state $|0\rangle$ and the states mixed with it into the next state of the SD band. This information is presently available and used below to determine $F$ from the data. We have just shown that it is not necessary to ask in which way the next lower SD state is mixed with the ND states when we ask for the total intensity feeding it. It suffices to calculate the transition intensity feeding that next lower SD state.

In the simplest case where $\Gamma^4 = 0$, only the intermediate population of the state $|0\rangle$ is of interest. With $\Gamma_S = \gamma_S^2$, the transition amplitude has the form

$$T(E) = \gamma (E - E_0 + (i/2)\Gamma_S)^{-1} \gamma_S .$$

We have $\int dE|T(E)|^2 = 2\pi\gamma^2$. Feeding intensity and emitted intensity are, of course, equal. For $\Gamma^4 \neq 0$, we have

$$T(E) = \gamma [(E - \mathcal{H})^{-1}]_{00} \gamma_S .$$

For $\Gamma_N = 0$, the total strength $\int dE|T(E)|^2 = 2\pi\gamma^2$ is expected to be preserved. This is, indeed, the case and can be checked by using the identity

$$[(E - \mathcal{H})^{-1}]_{00} \Gamma_S [(E - \mathcal{H}^*)^{-1}]_{00} = -2 Im [(E - \mathcal{H})^{-1}]_{00} .$$

The identity Eq. (14), in turn, follows immediately when we use the form $[(E - \mathcal{H})^{-1}]_{00} = (E - E_0 + (i/2)\Gamma_S - \sum_j \nu_j(E - E_j)^{-1} v_j)^{-1}$. Averaging over the ensemble does not affect the conservation of total strength. A change does occur, however, for $\Gamma_N \neq 0$. Then, the identity Eq. (14) does not apply and E1 decay of the ND states weakens the E2 decay intensity. The question is: By how much?

To calculate $\int dE|T(E)|^2$ for $\Gamma_N \neq 0$, we use the form

$$[(E - \mathcal{H})^{-1}]_{00} = (1/2) (E - E_0 + (i/2)\Gamma_S - \sum_j \nu_j(E - E_j)^{-1} v_j)^{-1} .$$

see Eqs. (6,7), and write $\delta_{\mathcal{O}} = \delta_{\mathcal{O}} + \mathcal{S}_0$ where the upper index $\mathcal{I}$ denotes the fluctuating part. Then,

$$|T(E)|^2 = \gamma^2 \Gamma_S \times$$

$$(1/4)[(E - E_0)^2 + (1/4)\Gamma_S^{-1}][1 + \delta_{\mathcal{O}}^2 + |\mathcal{S}_{\mathcal{O}}(E)|^2] .$$

The entities on the rhs of Eq. (15) are all known: $\delta_{\mathcal{O}}$ is given in Eq. (9), and $|\mathcal{S}_{\mathcal{O}}(E)|^2$ is given explicitly in Ref. [11]. Integration over $|T(E)|^2$ then yields the E2 intensity down to the next SD state. The actual calculation would be rather involved. It would involve a fourfold numerical integration. Fortunately, the explicit calculation is not needed since for $\Gamma_N \gg \Gamma_S, \Gamma^4$, the term $|\mathcal{S}_{\mathcal{O}}(E)|^2$ can be neglected. This is seen as follows. As in any stochastic reaction problem, the fluctuating part of the scattering matrix describes those processes where the long-lived intermediate resonances (the ND states) undergo statistical decay. For $\Gamma_N \ll \Gamma_S, \Gamma^4$, such decay will overwhelmingly lead to E1 emission. The term $|\mathcal{S}_{\mathcal{O}}(E)|^2$ describes the statistical decay of the ND states back into the SD state and can, therefore, be neglected. We quantify this statement by comparing the sum $\Sigma_{\mathcal{O}}$ of the transmission coefficients for E1 decay with the transmission coefficient $t_0 = 1 - |S_{\mathcal{O}}|^2$ for decay back into the state $|0\rangle$. For $\Gamma_N \ll D$, we have $\Sigma_{\mathcal{O}} = \Gamma_S/D$, while $t_0$ depends upon energy and is given by $t_0 = \Gamma_S/[(E - E_0)^2 + (1/4)(\Gamma_S + \Gamma^4)^2]$. For ND resonances a distance $D$ from $E_0$ we have $t_0 \sim \Gamma_S/D^2 \ll t_0$. The inequality $t_0 \ll t_0$ remains valid down to distances $\sim 10\Gamma_S \ll D$.

Neglecting $|\mathcal{S}_{\mathcal{O}}(E)|^2$, using in Eq. (15) for $\delta_{\mathcal{O}}$ the value of Eq. (9), and integrating over all $E$, we obtain

$$\int dE|T(E)|^2 = 2\pi\gamma^2 F ,$$

with $F$ given by Eq. (11). This is our central result: The intensity attenuation factor due to decay out of the SD band is given by $F$.

Equation (16) has a simple interpretation. Decay out of the SD band is a sequential process with two intrinsic time scales, the spreading time $\hbar/\Gamma^4$ for populating the ND states from the SD state $|0\rangle$, and the time $\hbar/\Gamma_N$ for E1 emission. The larger of these two times defines the relevant overall time scale. This is $\hbar/\Gamma^4$. Thus, the spreading width $\Gamma^4$ signifies the effective partial decay width for the total E1 decay out of the SD band. The branching ratio for E2 decay is given by $F = \Gamma_S/(\Gamma_S + \Gamma^4)$. The branching ratio for the total E1 decay is correspondingly given by $(1 - F)$. This can be verified by extending Eq. (14) to the case $\Gamma_N \neq 0$. These facts make it possible to determine $\Gamma^4$ directly from the intensity attenuation within the SD band. We expect $\Gamma^4$ to increase strongly as we move down the SD band. We point out that in contrast to the problem studied in the first half of this paper (where $\delta_{\mathcal{O}}$ contains the spreading width $\Gamma^4$ of the state fed by E2 radiation), $\delta_{\mathcal{O}}$ now contains the spreading width $\Gamma^4$ of the state from which the E2 radiation is emitted.

For the population of the ND states $|j\rangle$ from the SD band, the state $|0\rangle$ acts like a doorway state. Because of the barrier separating the first and second minima, this doorway state has similarity to an isobaric analog resonance, where isospin conservation has the same function as the barrier. The two cases differ in that the width $\Gamma_S$
of the doorway state $|0\rangle$ is small compared to the widths $\Gamma_N$ of the ND states. For isobaric analogue resonances, the converse situation holds. A situation similar to the one studied above is met in vinylene, a molecule with a shape isomer which is analogous to the SD state [13]. Here, however, the inequality $\Gamma^4 \gg \Gamma_N$ applies.

To sum up: Our result, Eq. (11), is obtained by calculating $S_{0\alpha}$ and by showing that for $\Gamma_N \gg \Gamma_S, \Gamma^4, S_{0\alpha}$ is negligible. We have applied this result to data given in Refs. [7-10]. The measured quantities are $F_{\text{exp}}$, the intensity reduction for E2 decay from the state $|0\rangle$ with spin $I_0$, and the lifetime $\tau$ of that state. We equate $F_{\text{exp}}$ with the ensemble-averaged intensity reduction factor $F$ calculated above, $F_{\text{exp}} = F$. With $1 - F = \Gamma^4 / (\Gamma_S + \Gamma^4)$ and $(\Gamma_S + \Gamma^4) = \hbar / \tau$, we have

$$\Gamma^4 = \hbar (1 - F_{\text{exp}}) / \tau. \quad (17)$$

The decay width $\Gamma_S$ is given by $\Gamma_S = \hbar F_{\text{exp}} / \tau$. These formulae entail an error due to statistical fluctuations of $F_{\text{exp}}$. The fluctuations are caused by the fact that in any given nucleus, the level spectrum is discrete. A very conservative estimate of this error (based on the influence of the closest-lying ND states) leads to an uncertainty of a factor 2 or 3 for $\Gamma^4$. The results are shown in Table I. We note that measurable decay out of the SD rotational band sets in whenever $\Gamma^4 \geq 0.1 \Gamma_S$ or so. The values of $\Gamma_S$ are consistent with approximately constant quadrupole moments in the SD bands. Our results support the connection postulated in Refs. [1,7] between $\Gamma^4$ and the barrier separating the first and the second minima.

In Refs. [2-5] the decay by E1 (E2) emission is calculated by multiplying the squares of the projections of the eigenfunctions of the Hamiltonian Eq. (1) onto the ND (the SD) states with $\Gamma_N$ ($\Gamma_S$, respectively). This scheme is expected to apply in the regime $\Gamma^4 \gg \Gamma_N, \Gamma_S$. Motivated by the later results of Refs. [7-10], we have investigated the regime $\Gamma_N \gg \Gamma^4, \Gamma_S$. Naturally, decay out of the SD band follows different rules in the two regimes. Our analysis shows that the regime $\Gamma_N \gg \Gamma^4, \Gamma_S$ actually applies to the data summarized in Refs. [7-10].

In summary, we have shown that data on the attenuation of the transition strength in a SD rotational band yield direct information on the spreading width $\Gamma^4$ and, thus, on properties of the barrier separating the first and second minima. This information is practically model-independent and parameter-free.

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Table I. The spreading widths $\Gamma^4$ deduced from the data reviewed in Ref. [9] for a number of nuclei. The spin values of the decaying states are given in brackets. The units are eV for $\Gamma_D$ and $10^{-7}$ eV for $\Gamma_N, \Gamma_S$ and $\Gamma^4$. The results indicated with $\ddagger$ were calculated with estimated lifetimes in Ref. [9]. The total width $\Gamma = \Gamma_S + \Gamma^4$. 