Symmetrizing Evolutions

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We introduce quantum procedures for making $G$-invariant the dynamics of an arbitrary quantum system $S$, where $G$ is a finite group acting on the space state of $S$. Several applications of this idea are discussed. In particular when $S$ is a $N$-qubit quantum computer interacting with its environment and $G$ the symmetric group of qubit permutations, the resulting effective dynamics admits noiseless subspaces. Moreover it is shown that the recently introduced iterated-pulses schemes for reducing decoherence in quantum computers fit in this general framework. The noise-inducing component of the Hamiltonian is filtered out by the symmetrization procedure just due to its transformation properties.

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The importance of the notion of symmetry in quantum theory cannot be overstated [1]. The associated state-space decomposition into dynamically invariant sectors is a highly desirable property in that it can strongly simplify the analysis of the system evolution. Suppose that on the state-space $\mathcal{H}$ of a quantum system $S$ acts a group $G$ via a representation $\rho$. In general the Hamiltonian $H$ of $S$ is not $G$-invariant i.e., $[H, \rho(g)] \neq 0$. The goal of this letter is to present a quantum procedure for generating an effective dynamics on $\mathcal{H}$ ruled by an operator $\tilde{H}$ that is the $G$-invariant component of $H$. It amounts to a sort of generalized Fourier transform which is one discards all the non-zero (i.e., non-translation invariant) components. We first discuss a procedure that involves frequently iterated measurements. The key idea is very simple: introducing an auxiliary space and resorting to the intrinsic parallelism of quantum dynamics one can simultaneously evolve all the group-rotated copies of an initial state. Then by repeated measurements one singles out the $G$-invariant component of the dynamics. After discussing several applications to state preparation, decoherence avoiding/suppression and constrained dynamics, we show that symmetrization can be achieved by purely unitary means and without additional space resources.

We first discuss a procedure that involves non-trivial invariant subspaces in $\mathcal{H}$. The space $\mathcal{H}$ splits according the $G$-irreps: $\mathcal{H} = \bigoplus_n \mathcal{H}_n$, where $n$ is the multiplicity of invariant subspace $\mathcal{H}_J$ associated to the $J$-th irrep of $G$. For instance the abelian (additive) group $\mathbb{Z}_2 = \{0, 1\}$ has two (1-d) irreps $\rho_J(\alpha) = e^{iJ\pi\alpha}$, the identical ($J = 0$) and the antisymmetric one ($J = 1$).
In this letter we shall mainly focus on the sector corresponding to the identity irrep. This is the subspace spanned by the vectors in $\mathcal{H}$ invariant under the action of $G$:

$$\mathcal{H}_{\text{inv}}^{G} := \{ |\psi\rangle \in \mathcal{H} : \rho_g |\psi\rangle = |\psi\rangle, \forall g \in G \}. \quad (2)$$

It is easy to check that the operator

$$\pi_{\rho} := |G|^{-1} \sum_{g \in G} \rho_g$$

is the projector onto $\mathcal{H}_{\text{inv}}^{G}$ [1]. In the very same way of all projections, $\pi_{\rho}$ has a clear geometrical meaning: from the elementary property $\| \pi_{\rho} |\psi\rangle - |\psi\rangle \| = \min_{\phi \in \mathcal{H}_{\text{inv}}^{G}} \| |\psi\rangle - |\phi\rangle \|$, it follows that $\pi_{\rho} |\psi\rangle$ represents the optimal $G$-invariant approximation of $|\psi\rangle$ and $\| \pi_{\rho} |\psi\rangle \|$ is a measure of the degree of $G$-invariance of the vector $|\psi\rangle$.

Since in the following it will play the role of ancilla, we consider the so-called Group Algebra $\mathcal{CG}$ of $G$. It is a $|G|$-dimensional vector space generated by an orthonormal basis $\{|g\rangle\}$ that is in a one-to-one correspondence with the elements of $G$. The following two elements also will have a major role in this paper

$$|0\rangle := |G|^{-1/2} \sum_{g \in G} |g\rangle, \quad W_{\rho} := \sum_{g \in G} \rho_g \otimes \Pi_g. \quad (4)$$

where $\Pi_g = |g\rangle\langle g|$. It is immediate to check that $W_{\rho} = e^{iK_{\rho}}$ is a unitary operator over $\mathcal{H} \otimes \mathcal{CG}$, with generator given by [5] $K_{\rho} = \sum_{g \in G} h_{g} \otimes \Pi_g$. The physical meaning of the entangling operator $W_{\rho}$ should be quite clear: it performs, conditionally on the group element encoded in the ancillary factor, the associated unitary rotations in computational space $\mathcal{H}$. Concerning $|0\rangle$ we observe that also the appearance of this vector is very natural in that the uniform superposition structure makes it the unique $G$-invariant element of the group algebra.

With these two ingredients one can design a simple quantum algorithm for extracting the $G$-invariant component of $|\psi\rangle$. Let $|\psi\rangle$ an arbitrary element of $\mathcal{H}$. Apply $W_{\rho}$ to the initial state $|\Psi_0\rangle := |\psi\rangle \otimes |0\rangle$:

$$W_{\rho} |\Psi_0\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \rho_g |\psi\rangle \otimes |g\rangle. \quad (5)$$

By projecting over $|0\rangle$ i.e., applying $I \otimes \Pi_0$, one finds

$$|\Psi_0\rangle \mapsto |G|^{-1} \sum_{g \in G} \rho_g |\psi\rangle \otimes |0\rangle = (\pi_{\rho} \otimes I) |\Psi_0\rangle. \quad (6)$$

Discarding the ancillary factor one gets the $G$-invariant component of $|\psi\rangle$ with probability of success given by $\| (I \otimes \Pi_0) W_{\rho} |\Psi_0\rangle \|^2 = \| \pi_{\rho} |\psi\rangle \|^2$. The procedure is illustrated by the following commutative diagram

$$\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{CG} & \xrightarrow{W_{\rho}} & \mathcal{H} \otimes \mathcal{CG} \\
\I \otimes |0\rangle & \mapsto & \I \otimes \Pi_0 \\
\mathcal{H} & \xrightarrow{\pi_{\rho}} & \mathcal{H} \\
\end{array}$$

Example 0 Let $\mathcal{H} = \mathcal{C}^{2 \otimes N}$ a $N$-partite quantum system, $G = \mathcal{S}_N$ the symmetric group and $\rho$ the natural action of permutations on a tensor product $[\rho(\sigma) \otimes \sigma_{j=1}^{N} |j\rangle = \otimes_{j=1}^{N} |\sigma(j)\rangle]$. Then $\pi_{\rho}(|\psi\rangle)$ is the totally symmetric component of $|\psi\rangle$. Here we have an exponentially large ancilla ($|\mathcal{S}_N| = N!$). Any permutation can be realized by a sequence of transpositions $t_{ij} (|\sigma\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\phi\rangle$. In the qubit case i.e., $\mathcal{H}_c = \mathcal{C}^{2}$ the $\{ t_{ij} \}_{i,j=1}^{N}$ can be implemented in $\mathcal{H}$ by switching on, for a suitable time, the two-qubit Hamiltonians $H_{ij} = s_i \cdot s_j [s_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)]$.

The described procedure can be immediately extended to the general $J$-th irrep of $G$. The corresponding projectors are given by $\pi_{\rho}^{(J)} = d_{j} |j\rangle \langle j| [\mathcal{C}^{d}]$, where $d_{j} := \text{tr} \rho_{j}^{\alpha} (d_{j})$ is the character (dimension) of the $J$-th irrep. Now one has to project over $|J\rangle := |G|^{-1} \sum_{g \in G} \chi^{j}\rho_{g}^{J} |g\rangle$, eventually obtaining $d_{J}^{-1} \pi_{\rho}^{(J)} |\psi\rangle \otimes |J\rangle$. This result is useful, for example, in providing a preparation procedure for the $sl(d)$-singlets introduced in ref. [6] for noiseless quantum encoding against collective decoherence in quantum computers.

Example 1 Let $\mathcal{H}, \mathcal{G}$ and $\rho$ as in Example 0, with $\mathcal{H}_c = \mathcal{C}^{d}$ and $N = m \cdot d (m \in \mathbb{N})$. Then there exists a (unique) $\mathcal{S}_N$-irrep $J$ associated with the rectangular Young tableaux with $d$ rows. $\pi_{\rho}^{(J)}$ is the projector over the singlet sector of $N$-fold tensor power of the defining irrep of $sl(d)$ [1].

Next example shows how a simple group-theoretic structure is associated to any linear subspace.

Example 2 Let $P$ be a projector in $\mathcal{H}, \mathcal{G} = \{ 0, 1 \} \cong \mathbb{Z}_2$, and $\rho: \alpha \mapsto e^{i \pi \alpha} P$ (that is a parity operator). In the simplest instance of this situation, when $\mathcal{H} = \mathcal{C}^{2}$ and $P^{+} = |0\rangle \langle 0|$, one has $K_{\rho} = |1\rangle \langle 1| \otimes |1\rangle \langle 1|$, in terms of the Pauli operator $\sigma^x$ (and neglecting a trivial shift) this reads $2K_{\rho} = \sigma^x \otimes I + I \otimes \sigma^x + 1/2 \sigma^z \otimes \sigma^z$. This expression shows that the interactions required for generating the unitary $W_{\rho}$ can be physically reasonable.

Unitary evolutions. Given the representation $\rho$ one can transform operators via the adjoint action: $\tilde{\rho}_{g} X \mapsto \rho_{g} X \rho_{g}^{-1}$. The subspace of $G$-invariant operators is then defined in the obvious way. Now we present a procedure for $G$-symmetrizing unitary evolutions. This is the natural operator extension of the projection/preparation procedures discussed above and similarly it involves the group algebra as ancillary space and repeated measurements.

Let $|\psi\rangle$ be an arbitrary element of $\mathcal{H}$.

I) Apply $W_{\rho}$ to the initial state $|\Psi_0\rangle := |\psi\rangle \otimes |0\rangle$:
\[
W_\rho |\Psi_0\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \rho_g |\psi\rangle \otimes |g\rangle.
\]

II) Evolve infinitesimally by \(H \otimes 1\), i.e., apply \(\delta U \otimes 1\) where \(\delta U \approx 1 - i \delta t \hat{H}\) (\(\delta t = t/M\)).

III) Apply \(W_\rho\),

\[
\frac{1}{\sqrt{|G|}} \sum_{g \in G} \rho_g \delta U \rho_g |\psi\rangle \otimes |g\rangle.
\]

IV) Project on \(|0\rangle\)

\[
\frac{1}{|G|} \sum_{g \in G} \rho_g \delta U \rho_g |\Psi_0\rangle \delta t \sim (1 - i \delta t \hat{H}) |\Psi_0\rangle.
\]

Here \(\hat{H} =: |G|^{-1} \sum_{g \in G} \rho_g S^g H \rho_g = \pi_\rho(H)\) is by construction \(G\)-invariant.

V) Iterate I–IV \(M\)-times with \(M \to \infty\).

Steps I–IV amount to the operation \(T(\rho_0) := S \rho_0 S^\dagger = (1 \otimes W_\rho) \delta U \otimes 1 W_\rho^{\dagger} \rho_0 |\Psi_0\rangle \langle \Psi_0| \). The overall success probability is given by

\[
\text{tr} T^M(\rho_0) = \text{tr} (S^M \rho_0 S^{M^\dagger}) = |S^M| |\langle \Psi_0|\Psi_0\rangle|\to \infty \approx |(1 - i \delta t \hat{H})^{M/2} |\Psi_0\rangle|^2 = 1 \quad (10)
\]

The global evolution is then \(\rho_0 \to T^M(\rho_0) / \text{tr} T^M(\rho_0) \approx S^M \rho_0 S^{M^\dagger}\), but

\[
S^M |\Psi_0\rangle = [(1 - \frac{i t}{M} \hat{H})^M \otimes 1] |\Psi_0\rangle \to \frac{e^{-i M^\dagger \hat{H} t}}{\text{tr}(e^{-i M^\dagger \hat{H} t})} (e^{-i t \hat{H}} \otimes 1) |\Psi_0\rangle = e^{-i t \hat{H}} |\psi\rangle \otimes |0\rangle \quad (11)
\]

Summarizing the above procedure [in the limit \(M \to \infty\)] induces, in the computational factor, an effective dynamics generated by the \(G\)-invariant Hamiltonian \(\hat{H} = \pi_\rho(H)\). As argued above, \(H\) represents the optimal \(G\)-invariant approximation of \(H\); from this point of view one can say that \(\hat{U}_t = e^{-i t \hat{H}}\) is the natural \textit{unitary} symmetrization of \(U_t\). One has to exploit a sort of quantum Zeno effect [7] [repeatedly measuring \(|0\rangle\)] in that in order to obtain an admissible quantum dynamics in the computational factor, evolution has to be symmetrized any infinitesimally small amount of time. For example the naïvely symmetrized evolution \(\hat{U}_t = |G|^{-1} \sum_g \rho_g U \rho_g^\dagger =: \pi_\rho(U)\) is not allowed, in that it is not unitary. The symmetrization has to be “exponentiated”. If in step IV) projection over \(|0\rangle\) were replaced by projection over \(|J\rangle\) followed by the application of the unitary extension of \(|0\rangle\langle J|\), eventually one would obtain the effective Hamiltonian \(H^J := \sum_g \chi^J_g |g\rangle \langle g| H \rho_g\), that transforms according the \(J\)-th irrep of \(G\).

\textbf{Example 3} With data like in Ex. 2 one finds \(\hat{H} = P^\dagger H P^\dagger + P H P\). This shows that constraining the dynamics, by measurements, to a subspace is a very special case of the general procedure introduced. Notice that the projection measurement are over a single qubit ancilla.

\textbf{Example 4} \(H = H_c \otimes N \otimes H_E\), \(G = S_N\), \(\rho\) is the natural action over the first factor (like in Ex. 0) times the identity in \(H_E\). The dynamics thus obtained is \textit{replica symmetric}. This case, in principle, relevant for quantum computation. Indeed let us suppose that the computational factor is a quantum register made of \(N\) cells with state-space \(H_c\), and \(H_E\) the state-space of environment. Then the resulting (permutation invariant) effective dynamics admits decoherence-free subspaces suitable for noiseless quantum encoding [6], [8]. The minimal implementation of this example would require a setup consisting of two qubits (interacting with an environment) and a third ancillary qubit (coding for the symmetric group \(S_2\)). By performing the above procedure the singlet \(2^{-1/2}(|01\rangle - |10\rangle)\) should be completely stabilized against decoherence.

\textbf{Example 5} \(H = H_c \otimes H_E\), and \(H = \sum_{i=1}^N H_i\), where \(H_i\) has non-trivial action only on \(H_c \otimes H_E\). To make this system \(S_N\)-invariant one only needs to consider the subgroup \(Z_N \subset S_N\) of cyclic permutations [acting on \(H_c \otimes H_E\)]. Indeed if \(H_i = \sum_{i=1}^N H_i^\dagger \). Then \(\hat{H} = \sum_i (H_i^\dagger \otimes B_i + \text{h.c.})\) where \(A_i^\dagger : = \sum_{i=1}^N A_i\), \((A = X, B)\). The latter example show that when \(H\) has some symmetry from the beginning one can achieve full \(G\)-invariance by resorting to an ancillary space smaller than \(CG\). Here one just needs an ancilla that is exponentially smaller than \(CS_N\). This result can be extended to the case in which \(H\) is \(G'\)-invariant where \(G' \subset G\) is a (normal) subgroup. To exemplify this situation let us consider a lattice Hamiltonian \(H\) over a regular polygon \(P\) with \(N\) vertices. Suppose \(\hat{H}\) to be invariant with respect to the group \(Z_N\) of cyclic permutations of the sites of \(P\). To make \(H\) invariant under the full group \(D_N\) of isometries of \(P\) just a two-dimensional ancilla (one qubit) is required. This stems from the fact that the coset space \(G' / G'' \approx D_N / Z_N \approx Z_2\) has order two.

The symmetrization procedure can be used for getting rid of unwanted terms in a system Hamiltonian. Let us suppose that \(H = H_0 + H_1\) where \(H_0\) is \(G\)-invariant and \(H_1\) transforms according the \(i\)-th row of the \(J\)-th irrep of \(G\), i.e., \(\rho_J^i H_1 \rho_g = \sum_j \rho_J^i (g) H J^j\). Then from the orthogonality relation \(\sum_g \rho_J^i (g) = 0\) [1] one obtains \(\pi_\rho(H_1) = 0\) and therefore \(\hat{H} = H_0\). This result can be in principle used for suppressing decoherence in a quantum computer. This issue is illustrated in the next two examples that deal respectively with \(N\) qubits and with an harmonic oscillator coupled with a dissipating environment [2], [3].

\textbf{Example 6} Let \(H = \mathbb{C}^{2N} \otimes H_E\), \(H_0 = H_0 + \sum_{i=1}^N (\sigma_i^x \otimes E_i + \sigma_i^y \otimes E_i^\dagger)\), and \(\rho Z_2 \rightarrow \{1, \sigma^z \otimes N \otimes 1\}\). Suppose that \(H_0\) is \(Z_2\)-invariant, from \(\sigma_i^x \sigma_i^y = -\sigma_i^y \sigma_i^x\), \(\alpha = \pm 1\) it follows that \(H_1 = \sum (\sigma_i^x \otimes E_i + \sigma_i^y \otimes E_i^\dagger)\) transforms according the antisymmetric irrep. The result can be easily generalized to different kind of interactions, for example if \(G\) is the Pauli group \(\{1, i\sigma^x, i\sigma^y, i\sigma^z\}\) and \(\rho\) the \(N\)-fold tensor representation \([i\sigma^a, i\sigma^b, i\sigma^c] \rightarrow -i\alpha \sigma^a\) one can elimi-
nate general couplings with the form \( \sum_{i=x,y,z} \sigma_i^a \otimes E_i^a \). In particular if only the \( \sigma_i^a \)'s are present one is dealing with a purely decohering environment. Since \( \sigma_i^a \) is \( N \) \( \otimes \exp(i \pi \alpha) = \exp(i \pi \sum_{\alpha=1}^{N} \alpha_i^a) \), here the \( \rho_0 \) corresponds to collective “\( \pi \)-pulses” along the \( \alpha = x, y, z \) directions. Notice that the invariance of free Hamiltonian holds for operators with the form \( H_0 = H_S \otimes 1 + \mathbb{1} \otimes H_B \), where \( H_S = \sum_{j} G_{ij} S_i \otimes \mathbb{1} \), that can be used for providing the conditional dynamics required for, along with single-qubit operations, universal quantum computation.

Example 7 Let \( \mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_E \) where \( \mathcal{H}_B = \text{span}\{\{|n\rangle\}_{n=0}^{\infty}\} \) is a single boson mode Fock space (with field operator \( a \)) and \( \mathcal{H}_E \) an environment state-space. We set \( H_0 = \omega a^\dagger a \otimes \mathbb{1} + \mathbb{1} \otimes H_E \), \( H_1 = a^\dagger \otimes E + a \otimes E^\dagger \). Now the relevant representation is \( \rho : Z_2 \mapsto \{1, \exp(i \pi a^\dagger a \otimes 1)\} \). Once again the system-environment interaction Hamiltonian \( H_1 \) is averaged away in that it has odd parity i.e., \( \exp(i a^\dagger a) \alpha \exp(i a^\dagger a) = -\alpha \). Notice that this example corresponds to \( \mathcal{E} \). \( \mathcal{E} \) being the subspace given by the even sector \( \mathcal{H}_E = \text{span}\{\{|2n\rangle\}_{n=0}^{\infty} = \mathcal{H}^{\text{inv}}_E \} \).

The strict relation with the frequent pulse control of decoherence proposed in refs. [2] and [3] should be clear. In fact this analogy allows to reformulate the whole symmetrization strategy by a procedure that does not resort to any ancilla and measurement.

Unitary symmetrization. Let \( \rho_i = \rho_{0i} = 1, \ldots, |G| \) the group representatives. Consider a time interval \( \delta t_N = t(N |G|)^{-1} \) and let \( \delta U_N = \exp(-i \delta t_N H) \) then apply the following sequence of transformations

\[
U_N(t) = \prod_{i=1}^{\frac{|G|}{2}} \rho_i^\dagger \delta U_N \rho_i = \prod_{i=1}^{\frac{|G|}{2}} e^{-i \delta t_N \rho_i^\dagger H \rho_i \frac{N}{N_{\alpha}}} \exp\left( -i \frac{t}{N} \hat{H} \right),
\]

implying \( U(t) = \lim_{N \to \infty} [U_N(t)]^N = \exp(-i t \hat{H}) \). Notice that here, for simplicity, we assumed that the unitaries \( \rho_0 \)'s can be realized in a vanishingly small amount of time in which the evolution induced by \( \hat{H} \) is negligible. A detailed analysis of the physical requirements needed in order to achieve the limit (12) can be found, for specific cases, in refs. [2], [3].

This unitary realization of \( G \)-symmetrization could be, from the point of view of feasibility, much better than the procedure based on iterated measurements. Indeed the latter implies extra space resources, the capability of carrying on unitary transformations (the \( W_r \)’s) that are possibly highly non-trivial and iterated measurements. In these respects the first procedure resembles the Error Correction techniques [9].

Our analysis sheds light on the structure underlying the decoherence-suppression strategies: the application of the symmetrization procedure can be viewed as an harmonic filter that selects out the decoherence-inducing part of the Hamiltonian in view of its representation-theoretic structure. This phenomenon is connected to the fact that, in the above examples, the symmetry content of a subspace is related to the number of “elementary excitations” contained in it. Since the interaction Hamiltonian \( H_1 \) describes the exchange of such elementary objects, it couples different symmetry sectors, therefore it cannot belong to the set of \( G \)-invariant operators.

The experimental realization of the scheme analysed in this letter is in general extremely demanding. One should able to perform unitary operations (and measurements), each one requiring a time \( \tau \), with a frequency \( \nu \) much greater than the one associated to the fastest time scale of the evolution generated by \( H \). For instance in case 1 one must have \( \tau^{-1} \gg \nu \gg \omega_c \), where \( \omega_c \) is the bath frequency cut-off (see refs. [2], [3]). In the scheme involving measurements one could turn on, for a time \( \tau \) and with frequency \( \nu \), the Hamiltonians \( H(t) = f(t) K^r \), where \( \int_{\tau} dt f(t) = 1 \). If these requirements are not exactly fulfilled one obtain a partial symmetrization for which, as far as the last examples are concerned, the noise is just reduced rather than eliminated. Moreover, for general \( G \) and \( \rho \) the “pulses” \( \rho_{0i} \)'s will be quite difficult to implement. Roughly speaking, this amounts to the capability of switching on the Hamiltonians \( b^a \) that in general will correspond to non-trivial collective interactions. On the other hand all the up-to-date proposals for maintaining coherence in a quantum computer are known to be quite challenging from the point of view of implementation. Conceptually it is intriguing to realize that all these techniques have at their root a group-theoretic structure.

I thank M. Rasetti for stimulating discussions and critical reading of the manuscript, Elsag, a Finmeccanica Company, for financial support.

[4] For reviews, see D.P. DiVincenzo, Science 270, 255 (1995); K. Pati takes a simple simple form when \( \rho_0 = \exp(i \phi_0 h) \), where \( h = h^l + \phi_{\phi h} = \phi_{\phi h} + \phi_h \in \mathbb{R} \). Now, by defining \( \phi := \sum_{\phi} \phi_{\phi h} \Pi_{\phi} \), one finds \( \Phi = h \otimes h \). In the Fourier case, for example, \( G \) is the (additive) group \( R \) with action \( \rho_a [x] = [x + a] \), one finds \( \Phi = \hat{p} \otimes \hat{a} \), where \( \hat{p} (\hat{a}) \) is the momentum (position) in the first (second) factor. A. Ekert and R. Josza, Revs. Mod. Phys. 68, 733, (1996).
