**PARTICLE-PHOTON INTERACTION**

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**ABSTRACT**

We give a brief account of the theoretical and formal methods necessary to deal with the basic principles of electromagnetic, i.e., particle-photon, interactions. A short presentation of synchrotron and undulator radiation, Raman and Compton scattering, Bremsstrahlung and FEL interaction is included.

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1. **INTRODUCTION**

   Physical phenomena are believed to be completely described by four fundamental interactions mediated by fields acting between particles:

   a) gravitational interactions;
   b) electromagnetic interactions;
   c) strong interactions;
   d) weak interactions.

   Interactions c) and d) are short range, that is they are revealed only in microscopic phenomena; a) is important, to date, only at a macroscopic level. Only e.m. interactions are manifest in both macroscopic and microscopic effects. Their description can therefore require quantum theory and/or special relativity, according to the following picture.

   **QUANTUM MECHANICS**

   ![Quantum Mechanics Table]

<table>
<thead>
<tr>
<th>SPECIAL RELATIVITY</th>
<th>NO</th>
<th>YES</th>
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<tr>
<td>Rutherford scattering</td>
<td>Radiation from bound electrons ex &quot;conventional&quot; losses</td>
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<td>ex FEL</td>
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The mathematical and formal content of these theories is obviously very different. In classical mechanics one has to solve a system of ordinary differential equations arising from Newton's law and Lorentz forces. In radiation phenomena in relativistic mechanics the problem is reduced to the solution of partial differential equations (PDE) derived from Maxwell equations for e.m. potentials driven from known sources. The key equation of quantum mechanics is a parabolic PDE of dispersion type (Schrödinger equation) in the presence of a classical source (the potential). Finally, in quantum field theory, interaction is represented by the exchange of virtual (off-mass-shell) particles, that is, particle-photon interactions. Mathematically, the problem is the solution of a system of coupled PDE. Of course the description of a particular phenomenon can require more than one scheme. For example synchrotron radiation from one electron can be calculated classically, but quantum fluctuations are important in describing the distribution function of a beam in a circular accelerator; the FEL theory can be formulated in many equivalent ways. It is therefore natural to find features common to the four theories already mentioned, specifically,

a) the canonical (Lagrangian and Hamiltonian) formulation of physical laws;
b) the link between symmetry and conservation laws.

2. **CLASSICAL MECHANICS: A SHORT OVERVIEW**

In classical mechanics [1], we associate a Lagrangian function with every dynamic system:

\[ \mathcal{L} = \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \]

where the \( q_i \) are generalized coordinates. The equations of motion are derived from Hamilton's principle

\[ \delta \int_{t_1}^{t_2} \mathcal{L} dt = 0 \]  

where the time integral of Lagrangian function is known as action. Equation (1) states that the actual motion makes action stationary with respect to any variation of path and with the constraints of fixed end points, that is, \( \delta q(t_1) = \delta q(t_2) = 0 \).

If the Lagrangian is a function \( \mathcal{L}(q, \dot{q}, t) \) of generalized coordinates, velocity and time and nothing else, the Hamiltonian principle is satisfied if, and only if, the Euler-Lagrange equations are satisfied:
\[
\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = 0.
\]

To formulate Hamilton's equations, we first define the generalized or conjugate momenta

\[
p_j = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_j}
\]

and then introduce the Legendre transformation, which is just a change from \((q, \dot{q})\) to \((q, p)\) variables.

We now define a new function, the Hamiltonian:

\[
H(q, p, t) = \dot{q}_p - L(q, \dot{q}, t)
\]

Under Legendre transformations, the Lagrange equations reduce to Hamilton's equations:

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}
\]

We now have \(2n\) first order ODE instead of \(n\) second order ODE.

Canonical formalism has several advantages:

a) it allows an easy transition to formal formulations of quantum mechanics [1] and quantum field theory [2];
b) it shows the intimate correlation between symmetry and conservation laws.

Canonical transformations are defined as transformation from the variables \((q, p, H)\) to the variables \((Q, P, K)\) which leave Hamilton's equations unchanged:

\[
\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}
\]

From Hamilton's principle we infer that this happens if the following equality is true:

\[
p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}.
\]

Here \(F\) is any function of the \((q, p, Q, P)\) coordinates with continuous second derivatives. It is called the generating function of the transformation because it gives the link be-
tween old and new coordinates. A trivial example is the function $F = q_i p_i$ which generates the identity transformation

$$P_i = p_i, \quad Q_i = q_i.$$  \hspace{1cm} (8)

Another example is given by the generating function

$$F = q_i p_i + \epsilon G(q, p, t)$$  \hspace{1cm} (9)

where $\epsilon$ is a small parameter, which generates an infinitesimal canonical transformation (ICT):

$$\delta p_j = p_j - p_j = - \epsilon \frac{\partial G}{\partial q_j}$$ \hspace{1cm} (10)

$$\delta q_j = q_j - q_j = \epsilon \frac{\partial G}{\partial p_j}.$$  

Canonical transformation can also be characterized by the invariance of some expressions, one of which is the Poisson brackets

$$[u, v]_{q, p} = \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial q}.$$ \hspace{1cm} (11)

The simplest Poisson brackets are

$$[q_i, p_j] = \delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0.$$ \hspace{1cm} (12)

The evolution equation of the dynamical variable $u$ can be written, if we take into account (5),

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}.$$ \hspace{1cm} (13)

Dynamical variables have the interesting property of forming a Lie algebra with respect to Poisson bracket composition.

The transition to quantum mechanics can be formally accomplished, as well known, by considering dynamical variables as Hermitian operators and replacing the Poisson
brackets with commutators. In this way (13) becomes the evolution equation of operators in the Heisenberg scheme.

Consider now the infinitesimal canonical transformation (9) which has $G$ as generator, and denote by $\mathcal{G}$ the change in the value of a function under infinitesimal canonical transformations. For the Hamiltonian we have

$$\mathcal{G}H = -\varepsilon \int G \, dt.$$  \hspace{1cm} (14)

Now if $G$ is a constant of motion, it generates an ICT which leaves the Hamiltonian invariant. In this way we have a powerful method to deduce conservation laws from the symmetry of the system. In fact, we have merely to inspect the Hamiltonian to find its symmetries, that is, the transformations which leave it invariant. The corresponding constants of motion are now the generators of the canonical transformations.

**Gauge invariance**

Another kind of invariance is gauge invariance [4], which becomes increasingly important when we go from classical mechanics to quantum mechanics and to quantum field theory.

It is well known that in classical electrodynamics, physical laws must depend only on the variables $\mathbf{E}, \mathbf{B}$. Therefore physical laws must be invariant with respect to the gauge transformations

$$\mathbf{A}' = \mathbf{A} - \nabla \psi$$

$$\psi' = \psi + \frac{\partial \psi}{\partial t}$$  \hspace{1cm} (15)

which leave e.m. fields and Maxwell equations invariant. It is remarkable that the dynamical behaviour of a classical system of fields and particles can be derived from Hamilton's variational principle given a simple Lagrangian function which is gauge invariant [5].

To this purpose we consider the Lagrangian of a free particle:

$$L = m c^2 \sqrt{1 - \beta^2} \quad \beta = v/c.$$  \hspace{1cm} (16)

The simplest gauge invariant and quadratic Lagrangian we can write for the e.m. field is
The e.m. field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \frac{\partial \phi_{\mu}}{\partial x_{\nu}} - \frac{\partial \phi_{\nu}}{\partial x_{\mu}}$$

where the four-vector potential is $\phi^{\mu}(\Phi, cA)$.

The Lagrangian for the complete system is

$$L = L_e + L_{PH} + L_{\text{int}}$$

that is, $L$ is the sum of the Lagrangian of free subsystems of field and particles plus a suitable interaction term. The following prescription is known to work very well:

$$L_e + L_{\text{int}} = L_e(\Phi, \underline{\rho}, eA)$$

In quantum mechanics [6], the evolution equation for the wave function of an electron in an external e.m. field is the Pauli equation

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left[ \sigma \cdot \left( \underline{p} - eA \right) \right] \psi + e\Psi \psi$$

where the $\sigma$ are Pauli matrices. We can now consider space-dependent phase transformations of the wave function

$$\psi' = e^{-i\chi(x)} \psi$$

Gauge invariance means invariance of the Pauli equation with respect to these phase transformations, which is true only if we have, together with the phase transformations, a corresponding change of the four-vector e.m. potential

$$\underline{A}' = \underline{A} - \frac{\hbar}{e} \underline{\gamma}$$

$$\phi' = \phi + \frac{\hbar}{e} \frac{\partial x}{\partial t}$$
2.1 Radiation from unbound electrons

In this case, classical theory is well suited [7]. We therefore consider the Maxwell equations

\[ \square A = - \mu_0 J, \]
\[ \square \phi = -\varrho/\varepsilon_0. \]

(24)

If we define the four-current \( J^\mu = (\varrho_0, \varrho^u_0/c), \) they can be written

\[ \partial_\mu \partial^\mu A^\nu = - J^\nu/\varepsilon_0. \]

(25)

We can easily write the solution in the electron rest frame:

\[ \phi^\mu (s) = \left( \frac{e}{4 \pi \varepsilon_0 c^2}, 0 \right). \]

(26)

To find the solution in an arbitrary frame, we have merely to write this solution in a form which is Lorentz invariant and which reduces to \( \phi^\mu (s) \) in the rest frame. In our case we have the result

\[ \phi^\mu = \frac{e}{4 \pi \varepsilon_0 c^2} \left( \frac{1}{s}, \frac{u^\mu}{cs} \right), \quad s = r - r_c. \]

(27)

These are the Lienard-Wiechert potentials from which we can derive the expressions for e.m. fields by the well-known rules

\[ \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \]

(28)

We can now calculate the radiation from an accelerated charge. All we need to do is to link the differential operators in (28) which refer to the actual space-time point to the retarded space-time in which radiation was emitted. The result is

\[ \mathbf{B} = \frac{\mathbf{r} \times \mathbf{E}}{rc}, \]

(29)

\[ \mathbf{E} = \frac{e}{4 \pi \varepsilon_0} \left( \frac{1}{s^3} (r - r_c)(1 - \frac{u^2}{c^2}) + \frac{1}{c^2 s} \left[ \mathbf{r} \times (r - r_c x_0) \right] \right). \]
The fields are the sum of a nonradiative term, which varies as $1/r$ and does not contribute to a net energy flow, and a radiative term, which varies as $1/r$. We drop the nonradiative term and write the radiative fields [8] as

$$\vec{E} = \frac{e}{4\pi c_0} \left( \frac{n x (n - \vec{E})}{(1 - \vec{E} \cdot n)^2} \right) \hat{r} \quad \vec{B} = \frac{[n \times \vec{E}]}{c} \hat{r} \quad (30)$$

where $\hat{r} = \frac{\hat{n}}{c}, \hat{n} = \frac{\vec{r}}{r}$.

If we choose a reference frame where the particle motion is nonrelativistic, we can drop the term $\hat{r}$ in (30). The energy flux is given by the Poynting vector, $\vec{S} = \frac{1}{\mu_0} \frac{\vec{E} \times \vec{B}}{c}$, so we obtain the power radiated per solid angle:

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2 \theta}{16 \pi^2 \epsilon_0 c^3} \quad (31)$$

where $\theta$ is the angle between $\hat{n}$ and the acceleration vector $\vec{a}$. Integrating over the solid angle, we obtain Larmor's formula for the irradiated power:

$$P = \frac{e^2 a^2}{6 \pi \epsilon_0 c^3} \quad (32)$$

Following the method already quoted, we can generalize to an arbitrary frame to obtain

$$P = \frac{e^2 a^2}{6 \pi \epsilon_0 c} \left( \vec{a} \cdot \vec{B} \right)^2 - \frac{e^2 a^2}{6 \pi \epsilon_0 c} \left( \frac{\vec{E} \times \vec{B}}{c} \right)^2) \quad (33)$$

The spectral distribution of energy irradiated can be evaluated from the Fourier transform of the power irradiated per unit solid angle, so as to obtain

$$\frac{d^2 \tilde{I}}{d\omega d\Omega} = \frac{e^2 \omega^2}{6 \pi^2 \epsilon_0 c} \left| \int \vec{n} \cdot \vec{x} \hat{r} e^{j\omega (t - \vec{n} \cdot \vec{r}/c)} dt \right|^2 \quad (34)$$

2.2 Synchrotron radiation

Previous formulas can be applied to calculate synchrotron radiation (so called because it was detected for the first time in circular accelerators), that is, radiation emitted by a particle in circular motion [8]. It can be simply calculated from formulas (30) and (34) for the particular case $\vec{E} = \vec{B}$. For the radiated power, a straightforward evaluation gives
\[ P = \frac{e^2 c}{6\pi\epsilon_0 \rho} \beta^3 \gamma^3 \]  

(35)

where \( \rho \) is the radius of the particle orbit in a magnetic field, \( \phi = \rho/eB \). The energy loss per orbit is

\[ \Delta E = \frac{e^2}{3\pi \epsilon_0 \rho} \beta^3 \gamma^3. \]  

(36)

The power radiated per solid angle is

\[ \frac{dP}{d\Omega} = \frac{e^2}{16\pi^2 \epsilon_0 c^2} \frac{1}{(1 - \beta \cos \phi)^2} \left[ \frac{\sin^2 \beta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \phi)^2} \right]. \]

(37)

The spectral distribution, that is, the energy radiated in the frequency \( (\omega, d\omega) \) and in the solid angle \( d\Omega \) is

\[ \frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{12\pi^2 \epsilon_0 c} \omega \left( \frac{1}{\gamma^2 + \beta^2} \right)^2 \left[ K_{\frac{1}{2}}^2 \left( \frac{\omega}{c} \right) + \frac{\omega^2}{1 + \beta^2} K_{\frac{3}{2}}^2 \left( \frac{\omega}{c} \right) \right]. \]

(37)

where \( K_{\frac{1}{2}}, K_{\frac{3}{2}} \) are modified Bessel functions, and the variable \( \zeta \) is defined as follows:

\[ \zeta = \frac{\omega}{3c} \left( \frac{1}{\gamma^2 + \beta^2} \right)^{3/2}. \]

(38)

From the asymptotic exponential behaviour of the modified Bessel functions, we deduce that radiation is negligible for \( \zeta >> 1 \), so that we define a critical frequency \( \omega_c \), beyond which radiation is negligible. For \( B = 0, \zeta = 1 \), we have

\[ \omega_c = \frac{3c\gamma^3}{\rho}. \]

(39)
Therefore synchrotron radiation has a continuous spectrum with a broad maximum.

We can define a spectral bandwidth for synchrotron radiation which reads

$$\Delta \omega \sim \frac{3\pi e^2 v^2}{4} \frac{1}{\lambda}.$$  \hspace{1cm} (40)

2.3 Undulator radiation

The undulator is a device designed to improve the brightness of radiation emitted by an e-beam [9]. It consists, for example, of a periodic array of permanent magnets which produce a spatially alternate magnetic field. Undulators can be linear or helical: in the first case, the on-axis magnetic field is \( \vec{B} = [0, B \sin(2\pi z/\lambda_u), 0] \), where \( \lambda_u \) is the undulator period; in the second case the on-axis magnetic field is \( \vec{B} = [B \cos(2\pi z/\lambda_u), \sin(2\pi z/\lambda_u), 0] \). It can be shown that the bandwidth for an undulator is

$$\Delta \omega_u \sim \frac{2\pi c}{L} \frac{1}{1+K^2} \gamma^2 \hspace{1cm} (41)$$

so that \( \Delta \omega / \Delta \omega \sim 1/\gamma \).  \hspace{1cm} (42)

Undulator radiation can be evaluated from (34) once the electron trajectory is known.

In the case of a helical undulator, for the trajectory we have

$$r(t) = \left[ -\frac{K \lambda}{2\gamma} \sin(\beta z) \frac{2\pi c}{\lambda_u} t, \frac{K \lambda}{2\gamma} \cos(\beta z) \frac{2\pi c}{\lambda_u} t, \beta \varepsilon t \right]. \hspace{1cm} (43)$$

A long but straightforward calculation gives for the brightness

$$\frac{d^2 I}{d\omega d\Omega} = \frac{4(\varepsilon^2 N^2)}{c} \sum_{n=1}^{\infty} \left\{ \frac{n^5}{K} \frac{\sin \nu}{\nu} \right\} \left( J^2_{n+1}(n\chi) + J^2_{n-1}(n\chi) - \frac{2(1+K^2)}{K^2} \frac{J^2_n(n\chi)}{n} \right) \hspace{1cm} (44)$$

where \( J_n \) are the ordinary Bessel functions and

$$\chi = \frac{2K\varepsilon \theta}{1+K^2+\gamma^2 \theta^2} \hspace{1cm} \xi = \frac{1}{2} \frac{K^2}{1+K^2+\gamma^2 \theta^2}$$

$$\nu = \left( n - \frac{\omega}{2\gamma^2 \omega_u} (1 + K^2 + \gamma^2 \theta^2) \right) \pi$$

$$\omega_u = \frac{2\pi c}{\lambda_u} \frac{eB\lambda}{2\pi mc}.$$  \hspace{1cm} (45)
From the formulas we see that the spectrum is peaked at $\nu_n = 0$, and the situation is different according to the value of the undulator parameter $K$. For $K < 1$, only the first harmonic contributes to the emission, while for $K > 1$, the emission spectrum consists of many harmonics and increasingly resembles the synchrotron spectrum. Finally, we remark that on axis ($\theta = 0$), only the first harmonic is present.

Similar remarks can be made for the linear undulator, and reference should be made to [9] for further details.

3. A SHORT INTRODUCTION TO QUANTUM MECHANICS

While a classical description is well suited for radiation emitted by unbound electrons, quantum mechanics [10] is needed for the treatment of radiation emitted by bound electrons. Thus the FEL theory is classical, while the "conventional" laser theory is quantum. However, two comments are necessary: first, a quantum theory is required for the investigation of the coherence properties of FEL radiation; second, for lasers, a semiclassical description, in which the active medium is treated quantum mechanically and radiation classically, is often sufficient. We sketch only a few remarks on quantum mechanics, and further details can be found elsewhere (e.g.,[3],[10]).

a) An operator is associated to any dynamical variable, acting on the state vectors (kets) of a Hilbert space;

b) The state vectors completely describe the state of the system which is assumed to be a linear superposition of the auto-kets of a complete set of commuting observables:

$$|\psi> = \sum_n c_n |a_n>.$$  (46)

A measurement process of an observable $A$ filters the state vector which becomes an auto-ket of $A$ with eigenvalue $a_n$. The probability for this is $|c_n|^2$.

c) There exists an operator $U(t,t_0)$, the evolution operator, which obeys the Schrödinger equation

$$i \hbar \frac{\partial U}{\partial t} = HU \quad U(t_0,t_0)=1$$  (47)

where $H$ is the Hamiltonian of the system.

The main problem is to determine the temporal evolution of the state vectors and observables.

Different pictures can be used to solve the problem. In the Schrödinger picture, the ket $|\psi(t)>$ is determined by the ket $|\psi(0)>$ using the relation $|\psi(t)> = U(t,t_0)|\psi(0)>$. 
From the Schrödinger equation for the evolution operator we deduce that $|\psi_s(t)\rangle$ satisfies a Schrödinger-type equation:

$$i \hbar \frac{\partial |\psi_s \rangle}{\partial t} = H_s |\psi_s \rangle.$$  \hspace{1cm} (48)

The Heisenberg picture is obtained from the Schrödinger picture by performing a unitary transformation $|\psi_H(t)\rangle = U_s^+ |\psi_s(t)\rangle = |\psi_s(t_0)\rangle$. Therefore, in the Heisenberg picture, the ket are stationary in time. For operators, things are different. In the Schrödinger picture, operators which do not have an explicit time dependence are fixed. For example, if the system has a classical analogue $H_{\text{cl}} = T + V$, where $T$ is kinetic energy and $V$ is potential energy, we can deduce the quantum Hamiltonian $\hat{H}_q$ from the correspondence principle

$$\hat{x} \rightarrow x_{\text{op}}, \quad \hat{p} \rightarrow -i \hbar \frac{\partial}{\partial x}.$$  \hspace{1cm} (49)

The Heisenberg operators are defined as follows:

$$A_H^s(t) = U_s^+ A U_s.$$  \hspace{1cm} (50)

It can be shown that the observables evolve according to the Heisenberg equation

$$i \hbar \frac{dA_H^s}{dt} = [H_s, A_H^s].$$

Another picture, the interaction picture, is useful. If we have $H = H_0 + H_{\text{int}}$, we again perform a unitary transformation

$$|\psi_t \rangle = U_t^+ |\psi(t)\rangle; \quad A_{t} = U_t^+ A U_t.$$  \hspace{1cm} (51)

where $U_{t}(t)$ is the evolution operator corresponding to $H$, and obtain a Heisenberg equation for the operators

$$i \hbar \frac{dA_t^s}{dt} = [A_t^s, H_0^s]$$

and a Schrödinger equation for state vectors

$$i \hbar \frac{d|\psi_t \rangle}{dt} = H_{\text{int}}^t |\psi_t \rangle; \quad H_{\text{int}}^t = U_t^+ H_{\text{int}} U_t.$$  \hspace{1cm} (52)
Thus, to disentangle free Hamiltonian $H_0$ and interaction Hamiltonian $H_{\text{int}}$, Schrödinger or Heisenberg equations are seldom analytically solved, so we are interested in perturbative techniques. The problem is [10], given the Hamiltonian $H = H_0 + \lambda U$, to solve the equation $i \hbar \partial \Psi / \partial t = (H_0 + \lambda U)\Psi$ once the complete set of eigenfunctions of the unperturbed Hamiltonian is given, that is,

$$H_0 \phi_n = E_n \phi_n.$$  \hspace{1cm} (53)

The standard method consists in expanding $\Psi$ in eigenfunctions of $H_0$

$$\Psi(t) = \sum_n c_n(t) e^{i E_n t / \hbar} \phi_n.$$  \hspace{1cm} (54)

In the case where $\Psi(0) = \phi_k$, to the first order in $\lambda$ we have

$$c_n(t) = \frac{1}{i \hbar} \int_0^t dt' e^{i (E_n - E_k) t' / \hbar} \langle \phi_n | U(t') | \phi_k \rangle.$$  \hspace{1cm} (55)

We now consider an electron bound to the nucleus by a potential $V(t)$ in the presence of an e.m. wave $A(r,t)$ so that

$$H = \left( \frac{\dot{p} - eA}{2m} \right) + V(r)$$  \hspace{1cm} (56)

the perturbation term in the Coulomb gauge, $\dot{V} \cdot \dot{A} = 0$, is

$$\lambda U(t) = (e / m) \dot{A} \cdot \dot{r}$$  \hspace{1cm} (57)

where the vector potential is

$$A(r,t) = \frac{\sqrt{2 \pi e^2 m \hbar}}{\omega V} \{ \dot{a} e^{i (k \cdot r - \omega t)} + a e^{i (k \cdot r + \omega t)} \}.$$  \hspace{1cm} (58)

For the amplitude of photon emission we have

$$c_n(t) = \frac{1}{mc^2 \hbar} \int_0^t dt' e^{-i k \cdot r} e^{i (E_n - E_k + \omega) t' / \hbar} \langle \phi_n | \phi_k \rangle$$  \hspace{1cm} (59)

and for the transition probability
\[ P_{k,n}(t) = \frac{2\pi e^2}{m^2 \hbar^2} \left| \langle \phi_n | e^{-i \mathbf{k} \cdot \mathbf{r} + \mathbf{e}^* \cdot \mathbf{p} - \mathbf{e} \cdot \mathbf{p} + \mathbf{e} \cdot \mathbf{r} \rangle \langle \phi_k | \right|^2 \left| \int_0^t e^{i (E_k - E_n + \hbar \omega) t' / \hbar} dt' \right|^2. \]  

(60)

With the substitution \( \Delta = (E_k - E_n + \hbar \omega) / \hbar \), the time integral becomes

\[ \left| \int_0^t e^{i \Delta t'} dt' \right|^2 = \left( \frac{\sin(\Delta t/2)}{\Delta/2} \right)^2. \]  

(61)

For large \( t \) we have

\[ \left( \frac{\sin(\Delta t/2)}{\Delta/2} \right)^2 \to 2\pi t \delta(\Delta) = 2\pi t \delta(E_k - E_n + \hbar \omega) \]

In this way, for the transition probability per unit time we get

\[ \frac{\delta W}{\delta t} = \frac{2\pi}{\hbar} \left| \langle \phi_n | e^{-i \mathbf{k} \cdot \mathbf{r} + \mathbf{e}^* \cdot \mathbf{p} + i \mathbf{e} \cdot \mathbf{k} - \mathbf{e} \cdot \mathbf{r} \rangle \langle \phi_k | \right|^2 \delta(E_k - E_n - \hbar \omega). \]  

(62)

The meaning of the \( \delta \) function is the conservation of energy. If the final state lies in a continuum of states, there is an extra phase-space factor which represents the density of states. Formula (62) with this extra factor is known as the Fermi golden rule.

3.1 **Raman effect**

If a light of a given frequency travels a dielectric medium, weak sidebands of radiation can be observed close to the frequency of the incident light [11]. When light is travelling in a plasma, the Raman effect is characterized by the scattering of incident light with the excitation of plasma waves.

The Raman effect is a two-photon effect, so it must be treated with the aid of the 2nd-order quantum-mechanical perturbation theory.

The relevant Feynman diagrams are

![Feynman Diagrams](Fig.2)
where \( \omega \) is the frequency of the pump wave, \( \omega_r \) is the frequency of the scattered wave, \( W_1 \) and \( W_f \) are the energy of dielectric, and \( W_v \) is the energy in the intermediate virtual state.

The interaction Hamiltonian is given by \( H_{int} = - (e/m)^2 \frac{\mathbf{p} \cdot \mathbf{A}}{\hbar}. \) The initial and final states are

\[
|\phi_i> = |\psi_i> |n_r><n_r|
\]

\[
|\phi_f> = |\psi_f> |n_r>-|n_r>+.
\]

The transition probability is given by the 2nd-order perturbation theory:

\[
\frac{\delta W}{\delta t} = \frac{2\pi}{\hbar} \sum_{m} |M|^2 \delta(E_f - E_i)
\]

where

\[
M = \sum_{\nu} \left\{ \frac{<\phi_f| \mathbf{p} \cdot \mathbf{A}|\psi_i> |<\psi_i| \mathbf{p} \cdot \mathbf{A}|\phi_f>}{E_i - E_{\nu}} \right\}.
\]

Factor (65) can be evaluated using expression (58) for the vector potential and expression (63). In this way we obtain

\[
M = \frac{3}{2k} \sqrt{\frac{\hbar n(n+1)}{\omega_{\nu} \omega_i}} \sum_{\nu} \sum_{r} \sum_{p} \left\{ \frac{<\psi_i| \mathbf{p} \cdot \mathbf{r}|\psi_{\nu}> |<\psi_{\nu}| \mathbf{p} \cdot \mathbf{r}|\psi_i>}{E_i - E_{\nu}} + \frac{<\psi_i| \mathbf{p} \cdot \mathbf{r}|\psi_{\nu}> |<\psi_{\nu}| \mathbf{p} \cdot \mathbf{r}|\psi_i>}{E_i - E_{\nu}} \right\} e^{i(k - k_r) \cdot \mathbf{r}}.
\]

The first term in the sum corresponds to the first Feynman diagram, and the second to the second Feynman diagram. If we sum over all the intermediate states, for the transition probability for the emission of a Raman photon, we get

\[
\frac{W}{\epsilon} = \frac{\pi n}{2m^2 \hbar^2} \sum_{\nu} \sum_{r} \sum_{p} \left\{ \frac{<\psi_i| \mathbf{p} \cdot \mathbf{v}_{\nu}|\psi_i> |<\psi_{\nu}| \mathbf{p} \cdot \mathbf{v}_{\nu}|\psi_i>}{\omega_i - \omega_{\nu}} + \frac{<\psi_i| \mathbf{p} \cdot \mathbf{v}_{\nu}|\psi_i> |<\psi_{\nu}| \mathbf{p} \cdot \mathbf{v}_{\nu}|\psi_i>}{\omega_i - \omega_{\nu}} \right\}.
\]

where \( \rho \) is the density of the final states, \( M_1 \) and \( M_2 \) are the number of initial state and final state molecules, and

\[
\omega_{\nu} = \frac{W - W_{\nu}}{\hbar}.
\]
4. RELATIVISTIC QUANTUM THEORY

The generalization of quantum mechanics to include special relativity began as a single particle theory. The goal [6,10] was to generalize the Schrödinger equation for the probability amplitude \( \phi \) of a single particle.

Taking into account the relativistic relation

\[
E^2 = p^2 + m^2 \quad \text{\( \hbar = c = 1 \)}
\]  \hspace{1cm} (67)

and the principle of correspondence

\[
\vec{p} \leftrightarrow -i \vec{\nabla} \quad E \leftrightarrow i \frac{\partial}{\partial t}
\]  \hspace{1cm} (68)

one obtains the Klein-Gordon equation

\[
(\square + m^2) \phi = 0 \quad \square = \frac{\partial^2}{\partial t^2} - \nabla^2 .
\]  \hspace{1cm} (69)

This equation was first rejected because

a) The continuity equation

\[
\frac{\partial}{\partial t} j^\mu = 0
\]  \hspace{1cm} (70)

cannot be satisfied by a positive definite probability density. In fact, the four-current which satisfies (70) has a 0-th component

\[
j^0 = \phi \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}
\]  \hspace{1cm} (71)

b) The K-G equation "leads to a result that seemed so unreasonable at the time it was first brought to light: negative energy states"[6]. In fact, if we look for a solution of (69) in terms of plane waves, we obtain the dispersion relation

\[
E = \pm \sqrt{p^2 + m^2} .
\]  \hspace{1cm} (72)

The reason for both the failures is the appearance of the second time derivative
in the K-G equation as a consequence of the correspondence principle and relativistic energy-momentum (mass-shell) relation. Dirac tried to circumvent such difficulties writing a linear mass-shell relation:

$$E = \alpha \cdot p + \beta m.$$  \hspace{1cm} (73)

Clearly $\alpha, \beta$ cannot be c-numbers; they are matrices whose properties are fixed by the requirement that $E, p$ also satisfy (67), that is,

$$\alpha_1 \cdot \alpha_1 = \beta^2 = 1$$

$$\beta \alpha_1 + \alpha_1 \beta = 0$$ \hspace{1cm} (74)

$$\alpha_i \cdot \alpha_j + \alpha_j \cdot \alpha_i = 0 \quad i \neq j.$$

Therefore, the Dirac equation is

$$i \frac{\partial \psi}{\partial t} = - i \alpha \cdot \nabla \psi + \beta m \psi.$$ \hspace{1cm} (75)

A more symmetric form of (75) can be obtained by multiplying it by $\beta$ and defining the $\gamma$ matrices

$$\gamma^0 = \beta \quad \gamma^i = \beta \alpha_i \quad \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$$ \hspace{1cm} (76)

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \delta^{\mu \nu}$$

so as to obtain

$$(i \gamma^\mu \partial_\mu - m) \psi = 0.$$ \hspace{1cm} (77)

The $\gamma$ matrices constitute a Clifford algebra whose base has 16 elements: $I, \{\gamma^\mu\}, \{\gamma^\mu \gamma^\nu\}, \{\gamma^\mu \gamma^5\}, \gamma^5$. The center of this algebra is $I$. The number \[12\] of nonequivalent fundamental irreducible representations is therefore 1, i.e., the order of the center, and the order $J$ of the $\gamma$ matrices must be 4 because $J^2 = 16$. For a particle in an e.m. field, the gauge invariance fixes the form of the equation as
\[ \gamma_{\mu}(i\partial_{\mu} - eA_{\mu})\psi = m\psi. \quad (78) \]

Defining the slash vector as \( \not{a} = \gamma_{\mu}a_{\mu} \), we have

\[ (i\not{\gamma} - e\not{A})\psi = m\psi. \quad (79) \]

Dirac's equation allows us to correctly define a positive definite probability density which obeys the continuity equation but predicts again negative-energy states.

**A sketch of the quantum field theory**

The impossibility of obtaining a "correct" relativistic equation leads to a reinterpretation of the formalism and to the introduction of the quantum field theory [13].

The wave function now becomes an operator which acts on a Fock space of state vectors.

The operatorial aspect of \( \psi \) is better understood if we Fourier transform \( \psi \) as

\[ \psi = \frac{1}{\sqrt{2\pi}} \int e^{i\mathbf{k} \cdot \mathbf{x}} \left( a(k)e^{i\mathbf{k} \cdot \mathbf{x}} + a^{\dagger}(k)e^{-i\mathbf{k} \cdot \mathbf{x}} \right) \, dk. \quad (80) \]

The operators \( a, a^\dagger \) are formally defined by the commutation relations

\[ [a(k), a^{\dagger}(k')] = \delta(k - k') \]

\[ [a(k), a(k')] = [a^{\dagger}(k), a^{\dagger}(k')] = 0. \]

An elementary particle is now defined as an irreducible representation (IRREP) of the quantum-mechanical Poincaré group which has the following Casimir invariants: the mass

\[ M^2 = P_{\mu}P^{\mu}. \quad (81) \]

and the norm of the Pauli-Lubarski vector

\[ \rho_{\eta} = -\frac{1}{2} \epsilon_{\lambda \mu \nu} \gamma^{\lambda \mu}p^{\nu}. \quad (82) \]
(\(\epsilon_{\lambda\mu\nu\rho}\) is the Ricci tensor). For the eigenvalues we have

\[ M^2 \phi = m^2 \phi \quad m = \text{rest mass} \]

\[ W^2 \phi_{\text{rest}} = m^2 s(s+1) \phi_{\text{rest}} \quad s = \text{spin} \]

An elementary particle with a peculiar mass and spin corresponds to each IRREP.

We have now the K-G equation for the scalar field:

\[ \Box + m^2 \phi = 0 \quad \text{spin 0}, \quad (83) \]

the Dirac equation for the spinor field

\[ (i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \text{spin 1/2}, \quad (84) \]

the vectorial K-G equation for the vector field

\[ \Box + m^2 \Phi_\nu = 0 \quad \text{spin 1}, \quad (85) \]

with the subsidiary condition \( \partial^\nu \Phi_\nu = 0 \).

Field equations are formally derived from the Lagrange equation applied to an action functional which has the Lagrangian densities for the free fields.

a) Real scalar field

\[ \mathcal{L} = -\frac{1}{2} m^2 \phi^2 + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \quad (86) \]

b) Complex scalar field

\[ \mathcal{L} = -m^2 \phi^* \phi + \partial_\mu \phi^* \partial^\mu \phi \quad (87) \]

c) Spinorial field

\[ \mathcal{L} = -\bar{\psi}(i\partial_\mu + m)\psi \quad (88) \]

d) Electromagnetic field (vector field with \( m = 0 \))
\[ \mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

\[ F_{\mu\nu} = \partial_{\mu} \phi - \partial_{\nu} \phi. \]

Field equations for the interacting fields A and B, for example, can be derived from the Euler-Lagrange equations for an action derived by the Lagrangian:

\[ \mathcal{L} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_{AB}. \]

\( \mathcal{L}_A \) and \( \mathcal{L}_B \) are the free Lagrangians for fields A and B, while \( \mathcal{L}_{AB} \) is the interaction Lagrangian which is completely arbitrary for a system without a classical analogue. The Lorentz invariance and other symmetries restrict the choice for \( \mathcal{L}_{AB} \), but the only criterion to fix the interaction term is the principle of gauge invariance with respect to local phase transformations

\[ \phi \rightarrow e^{-i\chi(x)} \phi. \]

Invariance of the Lagrangian is restored if partial derivatives are changed to "covariant derivatives"

\[ \partial_{\mu} \rightarrow \nabla_{\mu} = \partial_{\mu} - ieA_{\mu}. \]

\( A_{\mu} \) transforms according to

\[ A_{\mu} + A_{\mu} = \frac{1}{e} \partial_{\mu} \chi. \]

**Feynman diagrams**

In quantum electrodynamics we can perform calculations in a consistent way without resorting to the full machinery of quantum field theory. The trick we use is the Feynman interpretation of negative energy states [6].

In fact, reverting the energy sign is the equivalent to changing the sign of the conjugate variable time, so negative energy states can be considered as electrons (positive energy states) moving backward in time.

The classical equation of motion
\[
\frac{d^2 \hat{z}}{ds^2} = e \frac{\hat{z}}{ds} \frac{\hat{v}}{\mu v}
\]  

shows that a negative charge moving backward in time is equivalent to a positive charge moving forward in time. In this way we have a consistent interpretation of negative energy states.

Following Feynman we define the propagation kernels for an interacting and a free particle from point 1 to point 2 as solutions of equations:

\[
[i \frac{\partial}{\partial t_2} - e\phi_2 - \hat{a} \cdot (-i \nabla_2 - eA_2) - m\beta] K^A(2,1) = i\delta(2,1)
\]  

\[
[-i \frac{\partial}{\partial t_2} + i\hat{a} \cdot \nabla_2 - m\beta] K_+(2,1) = i\delta(2,1).
\]

The kernel \( K^A \) can be expanded in terms of the free particle kernel \( K_+ \) as

\[
K^A(2,1) = K_+(2,1) - i\int K_+(2,3)e\hat{a}(3)K_+(3,1)dt_3 - \]

\[
- \int K_+(2,3)e\hat{a}(3)K_+(3,4)e\hat{a}(4)K_+(4,1)dt_4dt_5 + ...
\]

In a time-stationary field, if the wave functions of the stationary state are known, for the propagation kernel in the interacting case we can write

\[
k^A(2,1) = \sum_{\text{pos. energies}} \exp[-iE_n(t_2-t_1)]\phi_n(x_2)\phi_n(x_1) -
\]

\[
- \sum_{\text{neg. energies}} \exp[-iE_n(t_2-t_1)]\phi_n(x_2)\phi_n(x_1).
\]

The transition amplitude between states 1 and 2 is given as in the nonrelativistic case:

\[
M = \int \hat{g}(2)\hat{g}(2,1)\beta f(1)d^3x_1d^3x_2
\]

and by using definitions

\[
f(3) = \int K_+(3,1)\beta f d^3x_1
\]

\[
\bar{g}(3) = \int \bar{g}(2)\hat{g}(2,3)d^3x_2
\]
we can write the transition amplitude to the first order as

$$M^{(1)} = -ig(3)eA(3)f(3)\tau_1$$

and to the second order

$$M^{(2)} = -\int g(4)eA(4)K^+(4,3)eA(3)f(3)\tau_1\tau_2. \quad (101)$$

The transition probability per unit time is defined as

$$\frac{\delta W}{\delta t} = \frac{2\pi}{2E_1,2E_2} |M|^2 \rho(E) \quad (102)$$

where the factors in the denominator arise from the relativistic normalization of wave functions and $\rho(E)$ is the phase-space density of the final states which in the two-body case is

$$\rho(E) = \frac{p^2dpd\Omega}{(2\pi)^3dE}. \quad (103)$$

We can give a straightforward interpretation of the transition amplitude to all orders. From $M^{(2)}$ for example, we see that a free particle interacts with potential in point 3, propagates as a free particle from 3 to 4, interacts with potential in point 4, and emerges as a free particle.

Momentum representation allows much easier calculations. In fact the propagation kernel for a free electron is simply

$$K(p) = i\frac{(p\cdot n)}{p^2-m^2} \quad (104)$$

while for the potential we have

$$\delta(q) = \int A(x)e^{iqx}dx. \quad (105)$$

The first order matrix element is

$$M^{(1)} = -i\langle u_2e\delta(q)u_1 \rangle \quad (106)$$
and the second order matrix element is

\[
M^{(2)} = i \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\delta^4(p_1+q_1-m)}{(p_1+q_1)^2-m^2} \epsilon^{*}(q_1) u_1 \epsilon(q_2) v_2
\]

(107)

**Fig. 3**

In this way every matrix element contributing to a given scattering reaction can be visualized by Feynman diagrams which can be calculated according to the rules:

1) An electron in a virtual (intermediate) state of momentum \( \not{p} \) contributes with a term \( i(p \cdot m)/(p^2-m^2) \).
2) A potential containing the momentum \( \not{q} \) contributes a term \( -i\epsilon^{*}(q) \).
3) All indeterminate momenta \( q_i \) in the intermediate state are summed over \( d^4q_i/(2\pi)^4 \).

4.1 Compton scattering

**Fig. 4**

From the conservation laws we can write
\[ \frac{m c^2}{\mathcal{M}_0} - \frac{m c^2}{\mathcal{M}_1} = 1 - \cos \theta. \quad (108) \]

The Feynman diagrams, to the lowest order, which contribute to the Compton scattering are

![Feynman diagrams](image)

Fig. 5

The incoming photon is represented by the potential

\[ A_{1\mu} = e_{1\mu} \exp(-i q_1 x) \quad (109) \]

and the outgoing photon by

\[ A_{2\mu} = e_{2\mu} \exp(-i q_2 x). \quad (110) \]

The initial and final states of the electron are given by

\[ \psi_1 = u_1 \exp(-i p_1 x) \quad \psi_2 = u_2 \exp(-i p_2 x). \quad (110) \]

The incoming photon beam can be resolved into two types of polarization (A) and (B):

(A) \[ \hat{e}_1 = \gamma_z \quad \text{(B)} \quad \hat{e}_2 = \gamma_y \quad (112) \]

and for the outgoing photon beam

(A') \[ \hat{e}_1 = \gamma_z \quad \text{(B') } \hat{e}_2 = \gamma_y \cos \theta - \gamma_x \sin \theta. \quad (113) \]

The transition probability is given by
\[ \frac{\delta W}{\delta t} = \frac{2\pi}{2E_1 \cdot 2E_2 \cdot 2\omega_1 \cdot 2\omega_2} \rho(E) |M|^2. \]  

(114)

Taking into account \( \delta W/\delta t = 0 \) and

\[ \rho(E) = \frac{E \cdot \omega_2 \cdot d\Omega}{(2\pi)^2 \omega_2 \omega_1}. \]

(115)

for the cross section \( \sigma \) we have

\[ \sigma = \frac{\omega_2 \cdot d\Omega}{(2\pi)^2 \cdot 16m^2 \omega_1^2} |M|^2. \]

(116)

The matrix element for the first diagram is

\[ R = -i4\pi e^2 \{ u_2 \cdot \gamma_2 \cdot \frac{p_1 + q_1 + m}{(p_1 + q_1)^2 - m^2} \cdot \gamma_1 \cdot u_1 \}. \]

(117)

and for the second

\[ S = -i4\pi e^2 \{ u_2 \cdot \gamma_2 \cdot \frac{p_1 - q_1 + m}{(p_1 - q_1)^2 - m^2} \cdot \gamma_1 \cdot u_1 \}. \]

(118)

Computing matrix elements one obtains the Klein-Nishina formula for polarized light

\[ \sigma = \frac{e^2}{4m^2 \omega_1^2} \int \frac{d\omega_2}{\omega_2^2} \left[ \frac{\omega_2}{\omega_1} + \frac{1}{\omega_2} - 2 + 4(e_1^* e_2)^2 \right]. \]

(119)

For unpolarized beams, one must average over the polarizations of the incoming beam and sum over the outgoing beam polarizations, so that

\[ \sigma = \frac{1}{2} \left( AA' + AB' + BA' + BB' \right) = \frac{e^4}{2m^2 \omega_1^2} \int \frac{d\omega_2}{\omega_2^2} \left( \frac{\omega_2}{\omega_1} + \frac{1}{\omega_2} - \sin^2 \theta \right). \]

(120)

4.2 Bremsstrahlung

In the Born approximation the Feynman diagrams are
With standard techniques, the differential cross section is

\[ d\sigma = \frac{\alpha Z^2}{(2\pi)^2} \frac{1}{r_s^2} \cdot \frac{m}{|q|^2} \left| \frac{p}{p'} \right| \frac{d\omega}{\omega} d\Omega \cdot d\Omega_k \cdot \]

\[ \cdot \left[ (2\epsilon \frac{B'\sin\theta'\cos(\Phi+\Psi)}{1-\beta'\cos\theta'}) - 2\epsilon' \frac{B\sin\phi\cos\psi}{1-\beta\cos\theta} - q^2 \left( \frac{B'\sin\theta'\cos(\Phi+\Psi)}{1-\beta'\cos\theta'} - \frac{B\sin\phi\cos\psi}{1-\beta\cos\theta} \right)^2 \right. 

\[ + \left. \omega \left( \frac{\gamma'\gamma}{\gamma} \beta^2 \sin^2\theta' + (\gamma'\gamma) \beta^2 \sin^2\theta' - 2\beta'\beta'\sin\theta'\sin\theta' \cos\theta' \right) \right] \cdot (1-\beta'\cos\theta')(1-\beta\cos\theta) \] (121)

Integration over the direction of the outgoing particles gives the Bethe-Heitler formula

\[ d\sigma(\omega, \epsilon) = aZ^2 r_s^2 \frac{p'}{p} \frac{d\omega}{\omega} \left( \frac{4}{3} - \frac{2}{(3\beta')^2} \left( \frac{\gamma'}{\gamma} + \frac{\gamma}{\gamma'} - \frac{2}{\gamma \gamma'} \right) \right) \]

\[ + \left( \frac{\gamma'}{\gamma} + \frac{\gamma}{\gamma'} - \frac{2}{\gamma \gamma'} \right) + \left( \frac{8}{3} \gamma \gamma' + \frac{\omega^2}{m^2} \left( \frac{1}{(3\beta')^2} + 1 \right) \right) \]

\[ + \frac{1}{2} \frac{\omega}{m} \left( 2 \frac{\gamma}{\gamma'} - \frac{2}{\gamma \gamma'} \right) \] (122)
where

$$|p| = \beta \varepsilon = \beta \gamma \alpha, \ |p'| = \beta' \varepsilon' = \beta' \gamma' m, \ \gamma = \gamma' + \frac{\omega}{m}$$

(123)

and

$$\xi = \frac{1}{\beta' \gamma'} \ln \frac{1 + \beta}{1 - \beta}, \ \xi' = \frac{1}{\beta \gamma} \ln \frac{1 + \beta'}{1 - \beta'}$$

$$L = \frac{2}{\beta' \gamma'} \ln \frac{\gamma' (1 + \beta')}{(1 - \beta')} - 1 \ \omega/m$$

In the soft photon limit, that is, $\omega \ll m_e$, the cross section can be written

$$\lim_{\omega \rightarrow 0} \frac{d^2 \Gamma}{d\omega d\Omega} = \frac{e^2 \varepsilon}{4\pi^2 c} \left| e \cdot \left( \frac{\beta}{1 - n^2 \beta} - \frac{\beta'}{1 - n' \beta'} \right) \right|^2.$$  

(124)

In this case a classical calculation gives the same result.

5. THE FREE ELECTRON LASER (FEL)

The working principle of the "free electron laser" (FEL) [9] is completely different from that of "conventional" lasers. The active medium, indeed, does not consist of atoms or molecules, but is a beam of relativistic electrons.

In a FEL device an ultrarelativistic electron beam interacts with an "undulator magnet" which gives a spatially periodic magnetic field. Electrons undergo transverse oscillations and emit radiation similar to synchrotron radiation but with a much smaller bandwidth so that the e-beam can amplify a copropagating laser beam or, once the radiation is stored in an optical resonator and reinteracts with the e-beam, the system works as a self-sustained oscillator. Therefore, to understand the FEL, we are led to studying stimulated emission of radiation in the undulator field.

It is convenient to choose a reference frame [15] in which the electron motion is nonrelativistic. Transforming to this frame, we can operate the Weizsacker-Williams (W-W) approximation and consider the undulator magnetic field as a radiation field which in the laboratory frame has wavelength

$$\lambda = (1 + \beta) \lambda_u = 2 \lambda_u$$

(125)
where $\lambda_u$ is the undulator wavelength. We choose a reference frame (resonant frame), with speed relative to the laboratory frame

$$v = c \sqrt{1 - \frac{2\lambda_u}{\lambda_L}}$$

(126)

where $\lambda_L$ is the laser wavelength in which the undulator and radiation field have the same wavelength.

In this frame we can treat the FEL interaction as a stimulated Compton scattering of pump (undulator) photons into radiation photons. The matching of momenta is provided by the electron recoil which is backward at laser wavelengths greater than the spontaneous emission peak given by

$$\lambda_L = \frac{\lambda_u}{2\gamma^2} (1 + k^2)$$

(127a)

$$k = \frac{eB\lambda_u}{2\pi m_e c^2} \quad \text{undulator parameter,}$$

(127b)

and forward at lower laser wavelengths.

In the first case, electrons lose energy in the laboratory frame and we have radiation gain (FEL), while in the second case we have radiation loss (IFEL).

![Diagram of FEL and IFEL interactions](image)

Fig. 8

In the resonant frame the Hamiltonian of the system can be written as
\[ H = \frac{1}{2m} \left( \frac{\hbar^2}{c} \right)^2 + \frac{1}{2} \left( P_L^2 + \omega^2 Q_L^2 \right) + \frac{1}{2} \left( P_W^2 + \omega^2 Q_W^2 \right). \] (128)

The vector potential can be written in terms of canonical variables as

\[ \mathbf{A} = \frac{k}{V} \left( \omega (Q_L + Q_W) \cos k z - (P_L - P_W) \sin k z \right). \] (129)

It is convenient to operate a contact transformation to the dynamical variable action \( \Gamma \) and phase

\[ P = \sqrt{2\omega I} \cos(\omega t + \phi) \]

\[ Q = \sqrt{2I/\omega} \sin(\omega t + \phi). \] (130)

The Hamiltonian is now

\[ H = \frac{\hbar^2}{2m} + \frac{2\pi r_c^2}{\omega V} \left[ I_L + I_W + 2I_L I_W \cos(\Phi_L - \Phi_W - 2kz) \right] \] (131)

where \( V \) is the interaction volume and \( r_c \) the classical electron radius. From this Hamiltonian we can derive the following conservation laws

\[ p + k(I_L - I_W) = \mu \]

\[ I_L + I_W = I_0. \] (132)

From the Hamiltonian equations and energy conservation, for the electron momentum we can derive

\[ \dot{p}^2 = \left( \frac{4\pi r_c}{\omega V} \right)^2 \left[ (kI_0)^2 - (p - \mu)^2 \right] - 4k^2 \left[ E - \left( \frac{\hbar^2}{2m} + \frac{2\pi r_c^2}{\omega V} I_L I_W \right) \right]^2. \] (133)

In the limit of a small signal gain theory, that is, signal unchanged by the interaction \((I_L = I_L(0))\), the Hamiltonian can be simplified as

\[ H = \frac{p^2}{2m} + C \cos(\Phi_L - \Phi_W - 2kz) \]

\[ C = \frac{4\pi r_c^2}{\omega V} \sqrt{I_L I_W} \] (134)

that is, an electron moving in a time independent potential.
We now introduce the scaled variables

\[ W = \frac{2k_m^*}{m} \quad \Omega^2 = \frac{16mr^2}{m} \frac{\omega I_L}{V} \frac{\omega I_W}{V} \]  

(135)

and obtain the pendulum-like Hamilton equations

\[ \frac{dW}{dt} = \Omega^2 \sin \psi \]  

(136)

\[ \frac{d\psi}{dt} = -W. \]

The final step is to derive the small gain formula which we are only quoting, and reference can be made elsewhere for further details:

\[ G = -g^2 \frac{d}{dv} \left( \frac{\sin \psi/2}{\nu/2} \right)^2 \]  

(137)

where \( g \) is the gain coefficient defined as

\[ g = \frac{2\pi}{\gamma} \frac{\lambda L}{I_0} \frac{1}{1+\kappa^2} \left( \frac{\Delta \omega}{\omega_0^2} \right) \]  

(138)

\[ \Sigma = \text{e-beam cross section} \]  

\[ L = \text{undulator length} \]  

\[ \kappa = \text{filling factor} \]  

\[ \I = \text{e-beam peak current} \]  

\[ I_0 = ec/\gamma = \text{Alfven current} \]  

\[ (\Delta \omega/\omega_0) = \text{spontaneous emission bandwidth} \]  

\[ \nu = \frac{\omega - \omega_0}{\omega_0} = \left( \frac{\Delta \omega}{\omega_0} \right)^{-1}, \quad \omega_0 = \frac{2\pi c}{\lambda L}. \]

5.1 FEL coherence properties

A quantum mechanical analysis of FEL interaction is useful to investigate coherence phenomena of FEL radiation [9]. The quantum FEL Hamiltonian can be obtained by classical Hamiltonian replacing the classical vector potential

\[ \hat{A} = a e^{i k z} + a^* e^{-i k z} \]  

(139)

by its quantum counterpart

\[ \hat{A} = \sqrt{\frac{\gamma}{2\epsilon_0 V}} [\hat{e} a e^{i k z} + e^* a^* e^{-i k z}] \]  

(140)
\[ H = \frac{p^2}{2m} + \hbar \omega (a_+^{\dagger} a^* - a a_+^{\dagger}) + \hbar \Lambda (a_+^{\dagger} a_+ e^{-2ikz} + a a_+^{\dagger} e^{2ikz}) \]

\[ \Lambda = \frac{2\pi r_0}{\omega} \cdot c^2 \]  

(141)

The state vector can be written as the tensorial product of radiation and electron Fock spaces:

\[ |\psi> = \exp \left\{ i \left[ \frac{p^2}{2m \hbar} + \omega (n_L + n_w) \right] t \right\} \sum_{k} C_{k} 1_p + 2k|k> \]  

(142)

where \( C_{k} \) is the probability amplitude for the exchange of \( k \) photons between the undulator and the radiation, \( p_0 \) is the initial electron momentum.

From the Schrödinger equation we can derive the following equation for the \( C_{k} \) [9], [16]:

\[ i \frac{dC_{k}}{dt} = (\omega k + \xi k^2)C_{k} + \Lambda (\sqrt{m-k+1} C_{k+1} + \sqrt{m-k+2} C_{k-1}) \]  

(143)

where \( \omega_0 = 2k \omega_0 / m \), \( \hbar \xi = (2m \hbar)^2 / 2m \), \( m \) is initial undulator photons, \( n \) is initial laser photons.

We now present some simplified forms of the same equation.

1) Strong undulator field (\( m \gg k \))

\[ i \frac{dC_{k}}{dt} = (\omega k + \xi k^2)C_{k} + \Lambda' (\sqrt{k+1} C_{k+1} + \sqrt{k} C_{k-1}) \]  

(144)

where \( \Lambda' = \Lambda / \sqrt{m} \).

2) Strong undulator field and spontaneous emission (\( m \gg k \), \( n \approx 0 \))

\[ i \frac{dC_{k}}{dt} = (\omega k + \xi k^2)C_{k} + \Lambda'' (\sqrt{k+1} C_{k+1} + \sqrt{k} C_{k-1}) \]  

(145)

3) Strong undulator and laser fields (\( n, m \gg k \))

\[ i \frac{dC_{k}}{dt} = (\omega k + \xi k^2)C_{k} + \Lambda''' (C_{k+1} + C_{k-1}) \]  

\( \Lambda''' = \Lambda / \sqrt{n \times m} \).  

(146)
To consider the question of FEL coherence we must recall the definition of coherent states. They were introduced by Glauber (1963) [17] and characterized by the following equivalent definitions:

a) A coherent state $|\alpha\rangle$ is an annihilation operator eigenstate

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$ 

b) A coherent state is a minimum uncertainty wave packet.

c) A coherent state is obtained by a unitary shift on the vacuum state

$$|\alpha\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle.$$ 

In terms of photon number states, the coherent state is given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$ 

Photon statistics for a coherent state is clearly Poisson statistics.

In the case of a system whose symmetry group is different from the Heisenberg-Weyl group of radiation fields, only definition c) can characterize a coherent state.

In the $SU_2$ case [18], the definition of the coherent state is

$$|\theta,\phi\rangle = e^{\frac{\xi_j - \xi^*_j}{2}} |\pm j\rangle$$

where $|\pm j\rangle$ is a Dicke state (sharp angular momentum state) with angular momentum state $\pm j$. This case is useful in treating Eq.(143) because according to the Schwinger procedure [19], the symmetry group of two-coupled harmonic oscillators is $SU_2$.

The statistical distribution of Dicke states is now binomial:

$$|\phi,\psi\rangle = \sum_{M=-j}^{j} \left( \begin{array}{c} 2j \\ M-j \end{array} \right) \frac{1}{\sin j\theta} \cos j\frac{\theta}{2} |M\rangle e^{-i(jM)\psi}.$$ 

As in the typical case of interacting quantum fields, the Raman-Nath equations, which are the equations for the quantum system of interacting radiation and electrons, have not been solved. The exception is the case $\epsilon=0$ (no electron recoil) [20].

In this case the solution is a coherent state. In fact a closer examination reveals that it is obtained by a unitary shift on the vacuum state, as we can see if we formally
solve the Raman-Nath equations.

We can therefore say that the electron recoil destroys the coherence in the FEL radiation because the solution for the $C_2^\lambda$ cannot be obtained by a unitary shift on the vacuum state.

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REFERENCES