Magnetic Monopole Content of Hot Instantons

R. C. Brower**, D. Chen\textsuperscript{b}, J. Negele\textsuperscript{b}, K. Orginos\textsuperscript{c} and C-I Tan\textsuperscript{d}

\textsuperscript{a}Physics Department, Boston University, 590 Commonwealth Ave, Boston, MA 02215, USA
\textsuperscript{b}Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
\textsuperscript{c}Physics Department, University of Arizona, Tucson, AZ 85721, USA
\textsuperscript{d}Physics Department, Brown University, Providence, RI 02912, USA

We study the Abelian projection of an instanton in $R^3 \times S^1$ as a function of temperature ($T$) and non-trivial holonomic twist ($\omega$) of the Polyakov loop at infinity. These parameters interpolate between the circular monopole loop solution at $T=0$ and the static `t Hooft-Polyakov monopole/anti-monopole pair at high temperature.

1. INTRODUCTION

Although many qualitative features of QCD are well described by a vacuum state dominated by an instanton “liquid”, confinement appears to be an exception \cite{1}. Instead, magnetic monopoles are thought to be the crucial ingredient. This raises the question of how magnetic degrees of freedom can be incorporated into (or reconciled with) an instanton “liquid”. A recent step in this direction was taken by Brower, Orginos and Tan (BOT)\cite{2} who studied in detail the magnetic content of a single isolated instanton, defining magnetic currents via the Maximally Abelian (MA) projection. They found a marginally stable direction for the formation of a monopole loop. Now with the more general calorion solution of T. Kraan and P. van Baal \cite{3}, and K. Lee and C. Lu \cite{4}, this analysis can be extended to an isolated SU(2) instanton at finite temperature ($T$) with a non-trivial holonomy ($\omega$) for the Polyakov loop.

The resultant picture that emerges is appealing (see Fig. 1). For $\omega = 0$, the small monopole loop at the core of a cold instanton grows in size as one increases the temperature and is transformed into a single static `t Hooft-Polyakov monopole at infinite temperature, as noted earlier by Rossi \cite{5}. Note that the other quadrants of Fig. 1 can be found by applying $Z_2$ center symmetry ($\omega \rightarrow \omega + \frac{1}{2}$) and monopole to anti-monopole charge conjugation ($\omega - \frac{1}{2} \rightarrow \frac{1}{2} - \omega$).

The MA projection provides a fully gauge and Lorentz invariant definition of monopole currents by introducing an auxiliary adjoint Higgs field, $\phi(x)$, fixed at the classical minimum,

$$ G = \frac{1}{2} \int [(D_\mu(A)\phi)^2 + \lambda(\phi^2 - 1)^2] d^4x , \tag{1} $$

in a fixed background gauge field, $A_\mu$. This yields the Abelian projected field strength,

$$ f_{\mu\nu} = n \cdot F_{\mu\nu} - n \cdot D_\mu n \times D_\nu n , \tag{2} $$

\footnote{Talk presented by R.C. Brower. Supported by the US DOE. Computations were done at the Theoretical Physics Computing Facility at Brown University.}
with its U(1) monopole current,
\[
k_\mu = \frac{1}{4\pi} \partial_\mu j_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho\sigma} \partial_\nu n \cdot \partial_\rho n \times \partial_\sigma n ,
\]
where \( n(x) \equiv \vec{\phi}(x)/|\phi(x)| \). It is conventional to identify the MA gauge by the rotation \( \Omega(x) \) in the coset \( SU(2)/U(1) \) that aligns \( n \) along the 3 axis. We have extended the conventional Abelian projection (\( \lambda = \infty \)) to a continuous family including the analytically more tractable BPS limit (\( \lambda = 0 \)), where the difficult problem of minimizing the MA functional \( G \) reduces to an eigenvector problem for the Higgs field, \( D_\mu (A)^2 \delta E = E \delta E \).

2. COLD MONOPOLE LOOP

We begin at the origin of Fig. 1, where there is a single isolated instanton at zero temperature. A trivial, but essential, observation is that the singular gauge instanton in the ‘t Hooft ansatz,
\[
A_\mu^a = \bar{\eta}^a_{\mu\nu} \partial_\nu \log (1 + \rho^2 / x^2),
\]
is also the MA projection which minimizes the Higgs action \( G \). In the BPS limit, this is equivalent to having a zero eigenvalue solution,
\[
\vec{\phi}_0(x) = \frac{x^2}{x^2 + \rho^2} [0, 0, 1],
\]
for the Higgs field aligned with the 3-axis. Consequently there is no magnetic content to the MA projection. This would be the entire story except that there is another zero eigenvalue that implies a flat direction for the formation of an infinitesimal monopole loop.

The geometry of this loop is interesting. The gauge singularity at the origin is caused by a rotation, \( g(x) = x_\mu x_\nu / |x| \) and \( \tau_\mu = (1, i\tau) \). Locally it is advantageous to “unwinding” this singularity to a distance \( R \) further reducing the MA functional \( G \). This almost wins, creating a monopole loop of radius \( R \), which only slightly increases \( G \), \( \delta G / G \approx R^4 \log(R) \). Almost any local disturbance, due to a nearby instanton for example, will stabilize the loop [2]. The second zero mode vector in the BPS limit is easily constructed using conformal invariance.
\[
\vec{\phi}_1(x) = \frac{1}{x^2(x^2 + \rho^2)} [\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta],
\]
where \( \alpha = \varphi - \psi , \cos \beta = (v^2 - u^2)/(u^2 + v^2) \), with \( x_1 + i x_2 = u e^{i\varphi}, x_3 + i x_0 = v e^{i\psi} \).

The superposition of the two zero modes produces a loop. Finally note that the topological charge is related to the magnetic charge, through a surface term on the boundary of the loop, \( \Sigma \),
\[
Q = \int_\Sigma \frac{Tr[\Omega_R d\Omega_R^* \wedge \Omega_R d\Omega_R^* \wedge \Omega_R d\Omega_R^*]}{24\pi^2},
\]
where \( \Omega_R \) is the singular gauge transformation providing a Hopf fibration for the infinitesimal “non-contraction” loop.

3. HOT BPS MONOPOLES

At low temperature with \( \omega = 0 \) the MA projection is similar. The periodic instanton in the ‘t Hooft ansatz,
\[
A_\mu^a = \bar{\eta}^a_{\mu\nu} \partial_\nu \log \left( 1 + \frac{\pi T \rho^2 \sin^2(\pi T r)}{2r(\sin^2(\pi T r) + \sin^2(\pi T t))} \right),
\]
is again equivalent to the MA projection. “Unwinding” the periodic copies of the singularities at \( x = 0 \) is now accomplished by \( g = X_\mu \tau_\mu/|X| \); \( X_\mu = [\tan(\pi Tr), \tau \tanh(\pi Tr)] \), leaving a monopole loop. However surprisingly at infinite temperature, or equivalently \( \rho = \infty \) as noted by Rossi, the instanton is gauge equivalent to the static ‘t Hooft-Polyakov monopole solution. With \( n(x) = \tilde{r} \), this is the correct MA projection (or unitary gauge). For the case of the BPS limit, the solution is simply \( \vec{\phi}(x) = \vec{\phi}(x) \), as one might expect. Consequently the MA projection correctly identifies the standard static monopole.

4. CALORON T-\( \omega \) PLANE

For \( \omega \neq 0 \), we encounter the full complexity of the new caloron solution [3],
\[
A_\mu = \tau \bar{\eta}^a_{\mu\nu} \partial_\nu \log \phi(x) + (\tau^+ \bar{\eta}^a_{\mu\nu} \partial_\nu \chi(x) + c.c.) \psi(x),
\]
in the singular gauge. In the limit of \( T \to \infty \), we have verified that MA projection now gives a pair of ‘t Hooft-Polyakov BPS monopole/antimonopole separated by distance \( D = \pi \rho^2 T \) as expected. However, now the singular gauge caloron no longer satisfies the MA projection and it is difficult to find the MA projection analytically.
Thus we have minimized G numerically in the interior of the $T-\omega$ phase plane of Fig. 1 by placing the functional on a grid. On symmetry grounds, one can prove the existence of cylindrical solutions with $\alpha = \varphi + \tilde{\alpha}(u,z,t)$, $\beta(u,z,t)$. The Dirac sheet is located by a jump in $\beta(t,x)$ by $\pi$. In Fig. 2, we give the profile for $\beta(u,z,t)$ in the $z$-$t$ plane slicing through the instanton centered at $u = 0$.

To explore further the transition from the monopole loop to a pair of monopole lines, we plot in Fig. 3 the area of the minimal spanning Dirac sheet. At $\rho T \simeq 0.56$, there is a clear transition separating the two regimes.

Based on the absence of a loop for a single isolated instanton [2], we anticipate that the formula for the size of the loop (or separation of the lines) must involve a new length scale, $L$. This scale represents the distance to nearby perturbations such as the anti-instanton presented in Ref. [2]. For the single caloron plotted here, the new scale is $L = \beta = 1/T$. This suggests a simple scaling form: $R \sim \rho(\rho T)^\gamma$ with $\gamma > 0$. Indeed in Fig. 3 for $(\rho T)^2 < 0.32$, we do see a positive curvature for the area, $A/\beta^2 \sim (RT)^2 \sim (\rho T)^{2+2\gamma}$, i.e., $\gamma > 0$, consistent with our expectation. On the other hand, at high temperatures, the monopole/anti-monopole trajectories are known [3,4] to be separated asymptotically by $D = \pi \rho(\rho T)^\gamma$ with $\gamma = 1$, which is also confirmed by a linear fit to $A = D\beta \simeq \pi \rho^2$ for $(\rho T)^2 > 0.3$.

Finally it is interesting to note that the kinematical “transition” seen in Fig. 3 is near to the Yang-Mills deconfinement temperature for a typical instanton size of $\frac{1}{\pi}$ fermi. However, a serious analysis of deconfinement dynamics and its possible relations to the monopole content of the caloron is left to future investigations.

**REFERENCES**

1. Talks by J. W. Negele and D. Chen at this conference.