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THE PERIODIC PARABOLIC EIGENVALUE PROBLEM
WITH $L^\infty$ WEIGHT

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ABSTRACT. In this paper we study existence, uniqueness and simplicity of the principal eigenvalue for the Neumann and the Dirichlet periodic parabolic eigenvalue problem with a bounded, possibly discontinuous, weight and suitable regularity conditions on the coefficients.

1. Introduction.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^{2+\theta}$ boundary, $0 < \theta < 1$, let \( \{a_{i,k}(x,t)\}_{1 \leq i, k \leq n}; \{a_j(x,t)\}_{1 \leq j \leq n} \) be two families of $(\theta, \theta/2)$ Hölder continuous functions on $\Omega \times \mathbb{R}$. Suppose $a_{i,k}(x,t), a_j(x,t)$ are $T$-periodic functions in $t$, satisfying the symmetry condition $a_{i,k} = a_{k,i}$ and such that for some $c > 0$ and all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$

$$\sum_{i,k} a_{i,k}(x,t)\xi_i\xi_j \geq c \sum_i \xi_i^2.$$

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We consider the periodic parabolic boundary eigenvalue problem
\[
\begin{align*}
\partial u/\partial t - \sum a_{i,k}(x,t)D_{i,k}u - \sum a_j(x,t)D_ju &= \lambda m(x,t)u \\
Bu &= 0 \\
u(x,t) &= u(x,t + T) \text{ for } (x,t) \in \Omega \times IR
\end{align*}
\]
where \(Bu = u|_{\partial \Omega \times IR}\) or \(Bu = \partial u/\partial \nu\) along \(\partial \Omega \times IR\). The case \(m \in C^{\theta,\theta/2}(\Omega \times IR)\), \(m(x,t)\) \(T\)-periodic in \(t\), is solved, for \(Bu = u|_{\partial \Omega \times IR}\) by Beltramo-Hess in [4] and for general boundary conditions (that includes the Neumann condition), in [3] by Beltramo. They find necessary and sufficient conditions for the existence, uniqueness and simplicity of the principal eigenvalue. In [3], the key for existence result is that
\[
\int_0^T \sup_{x \in \Omega} m(x,t)dt > 0 \tag{1.2}
\]
implies the existence of a Hölder continuous function \(c : [0,T] \rightarrow IR\) such that \(\int_0^T c(t)dt > 0\) and such that in a suitable tubular subregion of \(\Omega \times [0,T]\) \(m(x,t) \geq c(t)\). In this paper, we show that, under the additional assumption \(D_i a_{i,j} \in C(\Omega \times IR)\) these results can be extended for an arbitrary \(T\)-periodic function \(m \in L^\infty(\Omega \times IR)\). The main difficulty is that such a \(c\) may not exist. However we prove that (1.2) is equivalent to have \(m\) with positive integral in a suitable tubular subregion of \(\Omega \times IR\). This is sufficient to obtain the desired results.

2. Notation and Preliminaries.

We set, for \(u \in C^{2,1}(\Omega \times IR)\), \(Lu = \partial u/\partial t + A(x,t,D)u\), where
\[
A(x,t,D) = -\sum a_{i,k}(x,t)D_{i,k}u - \sum a_j(x,t)D_ju
\]

Let \(a(x,t)\), \(f(x,t)\) be two \(T\)-periodic in \(t\) functions belonging to \(C^{\theta,\theta/2}(\Omega \times IR)\), \(0 < \theta < 1\). We start recalling some well known facts concerning the existence of solutions for the parabolic boundary problem
\[
\begin{align*}
(L + aI)u &= f \text{ in } \Omega \times IR \\
Bu &= 0 \\
u(x,0) &= u_0(x)
\end{align*}
\]
with \( Bu = \partial u / \partial v \) or \( Bu = u|_{\partial \Omega \times \partial \Omega} \).

For \( p > 1 \), let \( W^{2,p}_B(\Omega) = \{ f \in W^{2,p}(\Omega) : Bf = 0 \} \). Let \( E \) be a vector space of functions on \( \Omega \times \mathbb{R} \), we set \( E_T = \{ f \in E : f(x,t) = f(x,t + T) \) a.e. \( (x,t) \in \Omega \times \mathbb{R} \} \) and \( E_B = \{ f \in E \cap \text{Dom}(B) : Bf = 0 \} \). The norm on \( L^p_T(\Omega \times \mathbb{R}) \) will be the norm \( \| f \|_{L^p_T(\Omega \times \mathbb{R})} = \left( \int_{\Omega \times (0,T)} |f|^p \right)^{1/p} \).

We fix, for the whole paper, \( n + 2 < p < \infty \). Let \( X = L^p(\Omega) \). We consider \( A_\alpha(t) : W^{2,p}_B(\Omega) \subseteq X \rightarrow X \), \( t \in \mathbb{R} \), given by

\[
A_\alpha(t)u = -\sum a_{i,k}(\cdot,t)D_{i,k}u - \sum a_j(\cdot,t)D_ju + a(\cdot,t)u.
\]

Each \( A_\alpha(t) \) is a closed, linear and densely defined operator, with domain independent of \( t \). Moreover for \( k \) large enough, say \( k \geq 1 + \| a \|_\infty \), we set \( A = A_{a+k}(0) \). For \( 0 \leq \alpha \leq 1 \) let \( A^\alpha \) be defined as in [7]. Let \( X_\alpha \) be the domain of \( A^\alpha \). For \( x \in X_\alpha \) we set \( \| x \|_\alpha = \| A^\alpha x \|_{L^p(\Omega)} \). Provided with this norm \( X_\alpha \) is a Banach space. Let \( \| \cdot \|_{\alpha, \beta} \) denotes the norm in the space of the bounded linear operators from \( X_\alpha \) into \( X_\beta \), \( 0 \leq \alpha, \beta \leq 1 \). Then we have

\[
X_\alpha \subseteq X_\beta \text{ for } 0 \leq \beta \leq \alpha \leq 1, \; X_0 = L^p(\Omega), \; X_1 = W^{2,p}_B(\Omega)
\]

and for \( \beta < \alpha \) the inclusion \( i_{\alpha,\beta} : X_\alpha \rightarrow X_\beta \) is a compact operator. Moreover for \( 1/2 + n/(2p) < \alpha \leq 1 \) we have \( X_\alpha \subseteq C^{1+\gamma}_B(\Omega) \) for some \( 0 < \gamma = \gamma(\alpha) < 1 \) where \( C^{1+\gamma}_B(\Omega) \) denotes the subspace of the elements in \( C^{1+\gamma}(\Omega) \) satisfying the boundary condition and this inclusion is compact. [cf. [2], p. 16; [7], p. 33].

The inhomogeneous linear evolution equation

\[
\begin{cases}
\frac{du}{dt} + A_{a+k}(t)u(t) = f(t) & f \in C^0([0,T + \omega], X), \; 0 < \theta \leq 1 \\
u(0) = u_0 & u_0 \in X
\end{cases}
\]

has an unique solution \( u \) satisfying

\[
\begin{cases}
u \in C([0, T + \omega], X) \cap C^1((0, T + \omega], X) & \text{for } u_0 \in X \\
u \in C^1([0, T + \omega], X) & \text{if } u_0 \in X_1.
\end{cases}
\]

Moreover, for \( 0 \leq t \leq T + \omega \), \( u(t) \) is given by

\[
u(t) = U_{a+k}(t,0)u_0 + \int_0^t U_{a+k}(t,\tau)f(\tau)d\tau \tag{2.1}
\]
where \( U_{a+k} \in B(X) \), \( 0 \leq \tau \leq t \leq T + \omega \), is the associated evolution operator.

We denote \( \Delta = \{ (t, \tau) \in [0, T + \omega] \times [0, T + \omega] : 0 \leq \tau \leq t \leq T + \omega \} \) and we consider \( U_a(t, \tau) = e^{k(t-\tau)}U_{a+k}(t, \tau) \). Known properties of \( U_{a+k} \) (see [2], lemma 2.1) imply, for \( (t, \tau) \in \Delta' = \{ (t, \tau) \in \Delta : \tau < t \} \), that

\[
\|U_a(t, \tau)\|_{\alpha, \beta} \leq c' \alpha, \beta, \gamma)(t - \tau)^{-\gamma} \quad \text{for} \quad 0 \leq \alpha \leq \beta < 1, \beta - \alpha < \gamma < 1
\]

(2.2)

And for \( 0 \leq \alpha < \beta \leq 1, 0 \leq \gamma < \beta - \alpha \), and \((t, \tau), (s, \tau) \in \Delta\)

\[
\|U_a(t, \tau) - U_a(s, \tau)\|_{\beta, \alpha} \leq c' \alpha, \beta, \gamma |t - s|^\gamma
\]

(2.3)

We put, for \( 2^{-1} + (2p)^{-1} \) \( n < \alpha \leq 1 \), \( K_{a, \alpha} = U_a(T, 0)|_{X_\alpha} : X_\alpha \to X_\alpha \).

**Remark 2.1.** We observe that \( f \in L^p(\Omega), f > 0 \) and \((t, \tau) \in \Delta\) imply \( U_a(t, \tau)f \) belongs to the interior of the positive cone in \( C^{1+\gamma}_B(\overline{\Omega}) \), ([7], lemma 13.4).

3. Auxiliary Lemmas.

For \( \lambda > 0 \) in \( IR \) it is natural to have a generalized solution operator

\[
(IL + \lambda)^{-1} : L^p_T(\Omega \times IR) \to L^p_T(\Omega \times IR)
\]

compact and positive. Moreover the restriction to \( C^{1+\delta/2}_T(\Omega \times IR) \) coincides with the classical solution operator. Our aim is to prove that the same result holds for \((IL + a)^{-1}\) with \( a(x, t) \in L^\infty_T(\Omega \times IR) \) such that \( 0 < \delta < a(x, t) < d < \infty \), for some \( \delta, d \in IR \).

Since \( p > n+2 \), we can fix, from now on, \( 0 < \alpha < 1 \) such that \( \frac{1}{2} + \frac{n}{2p} < \alpha < 1 \) and \( \frac{1}{1-\alpha} < p < \alpha \).

We will need the following

**Lemma 3.1.** Suppose as above \( Bu = \partial u/\partial v \) or \( Bu = u|_{\partial \Omega \times IR} \). Let \( a \in C^{0,\theta/2}_{\overline{\Omega} \times IR}, 0 < \theta < 1, a(x, t) \) \( T \)-periodic in \( t \) satisfying

\[
a \geq 0 \quad \text{and} \quad a \neq 0 \quad \text{if} \quad Bu = \partial u/\partial v|_{\partial \Omega \times IR}
\]

\[
a \geq 0 \quad \text{if} \quad Bu = u|_{\partial \Omega}.
\]
Let $X_0 = L^p(\Omega)$ and $X_1 = W^{2,p}_B(\Omega)$ in the preceding construction. Then there exists $0 < \gamma < 1$ such that the operator

$$S_{\alpha} : L^p_T(\Omega \times \mathbb{R}) \to C^\gamma([0, T + \omega], X\alpha)$$

defined by

$$S_{\alpha}(g)(t) = U_{\alpha}(t, 0)[I - K\alpha]^{-1} \left( \int_0^T U_{\alpha}(T, \tau)g(\tau)d\tau \right) + \int_c^t U_{\alpha}(t, \tau)g(\tau)d\tau$$

is an injective, positive, and bounded operator.

**Proof.** We fix $\beta$ such that $1 > \beta > \alpha$ and $1/(1 - \beta) < p$, also we fix $\delta$ such that $0 < \delta < \beta - \alpha$, and $\gamma', 1 > \gamma' > \beta$, such that $p > 1/(1 - \gamma')$. We set

$$S_{\alpha,1}(g)(t) = \int_c^t U_{\alpha}(t, \tau)g(\tau)d\tau$$

$$S_{\alpha,2}(g)(t) = U_{\alpha}(t, 0)[I - K_{\alpha,\alpha}]^{-1} \left( \int_0^T U_{\alpha}(T, \tau)g(\tau)d\tau \right)$$

We note that the integrals exist in the Bochner sense. The strong continuity of the evolution operator implies the measurability of the application from $[0, T + \omega]$ into $X\alpha$ given by $\tau \to U_{\alpha}(t, \tau)g(\tau)$. (2.2) and Hölder inequality give us

$$\sup_{t \in [0, T + \omega]} \int_c^t \|U_{\alpha}(t, \tau)g(\tau)\|\alpha d\tau < c\|g\|_{L^p_T(\Omega \times \mathbb{R})}.$$ 

Also, for $0 \leq s \leq t \leq T + \omega$

$$\|(S_{\alpha,1}g)(t) - (S_{\alpha,1}g)(s)\|\alpha \leq \int_0^s \|U_{\alpha}(t, s) - U_{\alpha}(s, s)\|\alpha, \beta \|U_{\alpha}(s, \tau)\|\alpha, \beta \|g(\tau)\|\alpha d\tau$$

$$+ \int_s^t \|U_{\alpha}(s, \tau)\|, \alpha \|g(\tau)\|\alpha d\tau.$$ 

A straightforward computation using (2.2) and (2.3) shows that, for some $c > 0$, $\varepsilon > 0$

$$\|(S_{\alpha,1}g)(t) - (S_{\alpha,1}g)(s)\|\alpha \leq c|t - s|^\varepsilon \|g\|_{L^p_T(\Omega \times \mathbb{R})}.$$
$K_a : K_\alpha \to X_\alpha$ is a compact, and strongly positive operator with spectral radius $0 < \rho(K_a) < 1$ ([7], Remark 14.1 and Lemma 14.2), so $(I - K_a)^{-1} : X_\alpha \to X_\alpha$ is bounded, then $(S_{a,2g})(t)$ is well defined. We have

$$
\| (S_{a,2g})(t) - (S_{a,2g})(s) \|_\alpha \\
\leq \| U_a(t, 0) - U_a(s, 0) \|_{\beta, \alpha} \| (I - K_a)^{-1} \|_{\beta, \beta} \int_0^T \| U_a(T, \tau) \|_{\beta, 0} \| g(\tau) \|_0 d\tau \\
\leq c |t - s|^\delta \ell \| g \|_{L^p_T(\Omega \times \mathbb{R})}.
$$

Also

$$
\| (S_{a,2g})(t) \|_\alpha \leq \| (S_{a,2g})(t) - (S_{a,2g})(0) \|_\alpha + \| (S_{a,2g})(0) \|_\alpha
$$

then

$$
\sup_{t \in [0, T + \omega]} \| (S_{a,2g})(t) \|_\alpha \leq c \| g \|_{L^p_T(\Omega \times \mathbb{R})}.
$$

So, for some $\gamma \in (0, 1)$, $S_a : L^p_T(\Omega \times \mathbb{R}) \to C^\gamma([0, T + \omega], X_\alpha)$ is bounded.

The positivity assertion follows from remark 2.1.

To prove the injectivity we note that for $g \in L^p_T(\Omega \times \mathbb{R})$, $S_a(g) = 0$ implies $S_a(g)(t) = 0$ in $C(\overline{\Omega})$ for all $t$, $t = 0$ gives $(I - K_a)^{-1}(\int_0^T U_a(T, \tau) g(\tau) d\tau) = 0$ and so $\int_0^t U_a(t, \tau) g(\tau) d\tau = 0$ for $0 \leq t \leq T \omega$. Then for $s < t$

$$
0 = \int_s^t U_a(t, \tau) g(\tau) d\tau \\
= U_a(t, s) \int_0^s U_a(s, \tau) g(\tau) d\tau + \int_s^t U_a(t, \tau) g(\tau) d\tau \\
= \int_s^t U_a(t, \tau) g(\tau) d\tau = 0.
$$

So $U_a(t, \tau) g(\tau) = 0$ a.e. $\tau \in [0, t]$, for all $0 < t < T + \omega$, then $g = 0$.

We note that, for $g \in C^{0, \delta/2}_T(\overline{\Omega} \times \mathbb{R})$ and $t \in (0, \omega)$, $S_a(g)(t + \omega) = S_a(g)(t)$ and so, by density, the same holds for $g \in L^p_T(\Omega \times \mathbb{R})$. So $S_a(g)$ has an unique $T$-periodic extension to $\overline{\Omega} \times \mathbb{R}$, we will denote this extension also by $S_a(g)$.

**Corollary 3.2.** Under the assumption of the Lemma 3.1 $S_a : L^p_T(\Omega \times \mathbb{R}) \to L^p_T(\Omega \times \mathbb{R})$ is a compact operator. Moreover, there exists $\gamma''$, $0 < \gamma'' < 1$, such that

$$
S_a : C^{1+\gamma''}_T, \gamma''(\overline{\Omega} \times \mathbb{R}) \to C^{1+\gamma''}_T, \gamma''(\overline{\Omega} \times \mathbb{R})
$$

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is a compact operator.

**Proof.** \(1/2 + n/(2p) < \alpha < 1\) implies that there exists \(0 < \sigma < 1\) such that \(X_\alpha \subseteq C^{1+\sigma}(\overline{\Omega})\). Moreover, for some \(0 < \gamma'' < 1\) we have

\[
C^\gamma_T(\Omega \times IR, X_\alpha) \subseteq C^{1+\gamma''}(\overline{\Omega} \times IR) \subseteq L^{p_T}(\Omega \times IR)
\]

with continuous inclusions and the last inclusion is a compact operator by Ascoli Arzela theorem. \(\blacksquare\)

**Remark 3.3.** We set \(Y = C^{1+\gamma''}(\overline{\Omega} \times IR)\). Then \(S_a : L^{p_T}(\Omega \times IR) \to Y\) is a strongly positive operator. Indeed, for a positive \(g\) in \(L^{p_T}(\Omega \times IR)\), \(S_ag\) belongs to \(Y\), moreover for \(t \in IR\) remark 2.1 and the definition of \(S_a\) imply that, for the Neumann boundary condition, \(S_a(g)(t)\) is a never zero function in \(C(\overline{\Omega})\), so \(S_a(g)\) belongs to the interior of the positive cone in \(C(\overline{\Omega} \times IR)\). For the Dirichlet boundary condition, we note that \(S_a(g)(t)\) belongs to the interior of the positive cone in \(C^{1+\gamma''}(\overline{\Omega})\) and \(\partial(S_a(g))/\partial\nu\) is a continuous and never zero function on \(\partial\Omega \times IR\), so \(S_a(g)\) belongs to the interior of the positive cone in \(C^{1+\gamma''}(\overline{\Omega} \times IR)\).

In the sequel, Krein Rutman Theorem refers to the version stated in [1], Th. 3.2.

**Remark 3.4.** Under the hypothesis of the Lemma 3.1 the spectral radius of the operator \(S_a : L^{p_T}(\Omega \times IR) \to L^{p_T}(\Omega \times IR)\) agrees with the spectral radius of its restriction \(S_a : Y \to Y\).

Indeed, the spectrum of \(S_a : L^{p_T}(\Omega \times IR) \to L^{p_T}(\Omega \times IR)\) is the point spectrum (except by the zero element), and every eigenfunction belongs to \(Y\), so both spectra agree (except perhaps by the zero element).

Krein Rutman theorem, corollary 3.2 and remark 3.3 imply that these spectral radius agree with a positive eigenvalue and no other eigenvalue has positive eigenfunction.

**Remark 3.5.** Let \(\lambda\) be a positive real number; for \(a = \lambda\) we consider the bounded operator \(S_\lambda : L^{p_T}(\Omega \times IR) \to Y\). We observe that \(W = S_\lambda(L^{p_T}(\Omega \times IR))\)
is independent of \( \lambda \). Moreover, for \( \lambda, \mu \in \mathbb{R}^{>0} \) we have \( S_{\lambda}^{-1} - \lambda I = S_{\mu}^{-1} - \mu I \) on \( W \).

**Definition 3.6.** We define \( \mathbb{IL} : W \rightarrow L^p_T(\Omega \times \mathbb{R}) \) by

\[
\mathbb{IL} = S_{\lambda}^{-1} - \lambda I, \quad \lambda > 0.
\]

\( \mathbb{IL} \) is an extension of the differential operator \( L \), such that \( \mathbb{IL} + \lambda : W \rightarrow L^p_T(\Omega \times \mathbb{R}) \) is a bijective operator with positive inverse. We consider \( W \) endowed with the \( Y \)-topology. It follows that \( \mathbb{IL} : W \rightarrow L^p_T(\Omega \times \mathbb{R}) \) is a closed operator.

Let \( P \) be the positive cone in \( Y \) and let \( T_1, T_2 \) be operators on \( Y \), we say \( T_1 \ll T_2 \) if \( (T_2 - T_1)(P) \subseteq (P)^{\circ} \).

**Lemma 3.7.** Suppose \( a \in L^\infty_T(\Omega \times \mathbb{R}) \) satisfies \( \delta < a(x, t) < d \) for some positive constants \( 0 < \delta < d \) and \( W, Y \) as in remarks 3.6 and 3.3 respectively. Then

1. \( \mathbb{IL} + a : W \rightarrow L^p_T(\Omega \times \mathbb{R}) \) is a bijection with continuous inverse.
2. \( (\mathbb{IL} + a)^{-1} : Y \rightarrow Y \) is a strongly positive and compact operator with positive spectral radius \( r \).
3. \( (\mathbb{IL} + a)^{-1} : L^p_T(\Omega \times \mathbb{R}) \rightarrow L^p_T(\Omega \times \mathbb{R}) \) is a compact operator and its spectral radius agrees with \( r \).
4. This spectral radius is an eigenvalue with positive eigenfunction and no other eigenvalue has positive eigenfunction.

**Proof.** We choose \( \eta \in \mathbb{R} \), \( \eta > d \) and we set

\[
T_i : L^p_T(\Omega \times \mathbb{R}) \rightarrow L^p_T(\Omega \times \mathbb{R}), \quad i = 1, 2, 3
\]

given by

\[
T_1 = (\eta - d)S, \\
T_2 = S_{\eta}(\eta - a), \\
T_3 = (\eta - \delta)S_{\eta},
\]

where \( \eta - a \) denotes the operator multiplication by \( \eta - a \). Each \( T_i \) is a positive and compact operator, then the spectrum \( \sigma(T_i) \) is the point spectrum (except
perhaps by the zero element). For \( i = 1,2,3 \ T_i(L^p_T(\Omega \times \mathbb{R})) \) is contained in \( Y \),
then the spectrum \( \sigma(T_i) \) agrees with the spectrum of the restriction \( T_i|_Y : Y \rightarrow Y \) (except perhaps by the zero element). Also, we note that these restrictions are strongly positive operators. Let \( r_i \) denotes the spectral radius of \( T_i \). Now \( 0 < \eta - d < \eta - a < \eta - \delta \) and then, as operators on \( Y \), \( T_1 \ll T_2 \ll T_3 \).

Suppose the Neumann condition, since \(( IL + \eta)(1) = \eta 1\), the Krein Rutman thorem says that \( 1/\eta \) is the spectral radius of \( S_\eta \). The same theorem gives us \( r_1 < r_2 < r_3 \), and so \( 0 < r_2 < 1 \). For the Dirichlet condition, let \( \lambda_0, u_0 \) be the principal eigenvalue and the positive eigenfunction associated, respectively for \( L \), i.e. \(( IL + \eta)u_0 = (\eta + \lambda_0)u_0 \). So \( 1/(\eta + \lambda_0) \) is the spectral radius of \( S_\eta \), then \( 0 < r_2 < 1 \). From this we obtain, in both cases

\[
(I L + a)^{-1} = [I - (IL + \eta)^{-1}(\eta - a)]^{-1}(IL + \eta)^{-1}
\]

which implies (1)-(4).

Suppose \( a \) as in Lemma 3.7. We set

\[
S_a = (IL + a)^{-1} : L^p_T(\Omega \times \mathbb{R}) \rightarrow L^p_T(\Omega \times \mathbb{R})
\]

Note that, for \( a \in C^{0,\theta/2}(\Omega \times \mathbb{R}) \), \( S_a \) agrees with the operator defined in the statement of the Lemma 3.1.

**Remark 3.8.** Suppose the Neumann boundary condition. We consider

\[
(I L + 1)^{-1} : L^p_T(\Omega \times \mathbb{R}) \rightarrow L^p_T(\Omega \times \mathbb{R})
\]

The Krein Rutman theorem implies that

\[
(I L + 1)^{-1} : L^p_T(\Omega \times \mathbb{R}) \rightarrow L^p_T(\Omega \times \mathbb{R})
\]

has a positive eigenvector \( \Psi \) with eigenvalue 1. We normalize \( \Psi \) such that \( \langle \Psi, 1 \rangle = 1 \).

**Remark 3.9.** Let \( m(x,t) \) be a \( T \)-periodic in \( t \) function in \( L^\infty_T(\Omega \times \mathbb{R}) \) satisfying \( \|m\|_\infty \leq 1/2 \). Suppose \( \lambda \in \mathbb{R}^{>0} \), then (by Lemma 3.7)

\[
S_{\lambda(1-m)} : L^p_T(\Omega \times \mathbb{R}) \rightarrow L^p_T(\Omega \times \mathbb{R})
\]

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is a compact and positive operator with positive spectral radius \( \rho_m(\lambda) \). We define \( \mu : \mathbb{R}^+ \to \mathbb{R} \) by \( \mu_m(\lambda) = \rho_m(\lambda)^{-1} - \lambda \) for \( \lambda > 0 \) and \( \mu_m(0) = 0 \). It is known that, for a Holder continuous \( m \), \( \mu_m \) is a concave function. Now we extend this result to a bounded \( m \).

**Lemma 3.10.** Let \( m \) be a function in \( L^\infty_T(\Omega \times \mathbb{R}) \). Then \( \mu_m \) is a concave function on \([0, \infty)\) and \( \mu_m \) is analytic on \((0, \infty)\).

**Proof.** Without lost of generality we can suppose \( ||m||_{\infty} \leq 1/2 \). We consider the following norm on \( W \).

\[
\|f\|_G = \|f\|_{C^{1,1}, \gamma, \gamma'}(\Omega \times [0,T]) + \|(I + 1)f\|_{L^T_p(\Omega \times \mathbb{R})}
\]

\( W_{\|, \|_G} \) is a Banach space. We consider \( T_0 : W_{\|, \|_G} \to L^T_p(\Omega \times \mathbb{R}) \) given by

\[
T_0 = \mathcal{L} + \lambda(1 - m),
\]

\( T_0 \) is bijective and bicontinuous. Let \( K \) be the inclusion \( K : W_{\|, \|_G} \to L^T_p(\Omega \times \mathbb{R}) \). \( K \) is compact, so \( T_0 - (\mu_m(\lambda) + \lambda)K \) is a compact perturbation of an isomorphism and then it is a Fredholm operator with zero index. Lemma 3.7 and the Krein Rutman theorem imply that \( \dim \ker (T_0 - (\mu_m(\lambda) + \lambda)K) = 1 \) and if \( u_0 \) is a generator of \( \ker (T_0 - (\mu_m(\lambda) + \lambda)K) \) then \( u_0 \notin \mathcal{R}(T_0 - (1 - \mu_m(\lambda) + \lambda)K) \).

The Crandall Rabinowitz lemma (see [5], Lemma 1.3, p. 163) implies that \( \mu_m(\lambda) \) is a real analytic function of \( \lambda \) for \( \lambda > 0 \).

We choose \( \{m_j\}_{j \in \mathbb{N}} \) a sequence in \( C^\infty(\Omega \times \mathbb{R}) \), with \( \text{supp}(m_j) \subseteq K_j \times \mathbb{R} \), for some compact subset \( K_j \) of \( \Omega \), and satisfying \( ||m_j||_{\infty} \leq 1/2 \) and such that \( m_j \) converges to \( m \) in the \( L^p \) sense. Each \( \mu_{m_j} \) is a concave function on \([0, \infty)\), ( [7], Lemma 15.2). We set \( T_j : W_{\|, \|_G} \to L^T_p(\Omega \times \mathbb{R}) \) given by

\[
L_j = \mathcal{L} + \lambda(1 - m_j)
\]

so \( T_j - T_0 \) tends to zero in \( B(W_{\|, \|_G} \to L^T_p(\Omega \times \mathbb{R}) \). Now \( T_0 u_0 = (\mu_m(\lambda) + \lambda)u_0 \). The Crandall Rabinowitz lemma implies that there exists \( \alpha_j(\lambda) \) and \( u_j \) satisfying \( T_j u_j = \alpha_j(\lambda)u_j \) and such that \( u_j \to u_0 \) in \( W_{\|, \|_G} \) and \( \alpha_j(\lambda) \to \mu_m(\lambda) + \lambda \) as \( j \) tends to \( \infty \), so \( u_j \gg 0 \) for a large enough \( j \). By the Krein Rutman
theorem $\alpha_j(\lambda) = \mu_{m_j}(\lambda) \to +\lambda$. So $\lim_{j \to \infty} \mu_{m_j}(\lambda) = \mu_m(\lambda)$, for $\lambda > 0$. Also $\mu_{m_j}(0) = \mu_m(0)$. Then $\mu_m(\lambda)$ is a concave function on $[0, \infty)$.

**Remark 3.11.** The Crandall Rabinowitz lemma implies that for $\lambda > 0$ $\mu_m(\lambda) + \lambda$ is a $K$-simple eigenvalue of the operator $IL + \lambda(1 - m)$. Now, for $u \in W, ILu - \lambda mu - \mu_m(\lambda)u = T_0u - (\mu_m(\lambda) + \lambda)Ku$. Suppose $\mu_m(\lambda) = 0$, let $M$ be the operator $M : W \to L^p_T(\Omega \times \mathbb{R})$ given by $Mu = mu$. Then, as in [4], lemma 3.7, $\lambda$ is an $M$-simple eigenvalue of $IL$.

4. Main results.

In this section we will assume that the coefficients $a_{i,j}, 1 \leq i, j \leq n$ belong to $C^1(\overline{\Omega} \times \mathbb{R})$. Let $m$ be a function in $L^\infty(\Omega \times [a, b])$ such that $\|m\|_\infty \leq 1$. We set $m^\sim : [a, b] \to \mathbb{R}$ defined by $m^\sim(t) = \text{ess sup}_{x \in \Omega} m(x, t)$. Let $\pi$ denote the projection $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by $\pi(x, t) = t$. For $B \subseteq \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$ we put $B_t = \{x \in \mathbb{R}^n : (x, t) \in B\}$. Also we set, for a domain $\Omega$ and for $\delta > 0$ $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\}$.

**Lemma 4.1.** Let $m$ be a function in $L^\infty(\Omega \times (a, b))$. Suppose $c \in \mathbb{R}$ such that

$$\int_a^b m^\sim(t)dt > c.$$ 

Given $\delta > 0$ such that $\Omega_\delta \neq \emptyset$, there exists a finite disjoint set $\{Q_r\}_{1 \leq r \leq N}$ of congruent open cubes in $\mathbb{R}^{n+1}$ with edges of length $\ell$ and parallel to the coordinates axis satisfying

1. $\ell \leq \delta/(2(n + 1))$, $Q_r \subseteq \Omega_{\delta/2} \times [a, b]$, $1 \leq r \leq N$.
2. $\{\pi(Q_r)\}_{1 \leq r \leq N}$ is disjoint.
3. $\sum_{1 \leq r \leq N} |\pi(Q_r)| = b - a$.
4. $\int_{r=1}^N \int_{Q_r} m(x, t)dxdt > c \ell^n$.

**Proof.** Without lost of generality we assume that $\|m\|_\infty \leq 1$. For $k \in \mathbb{N}$ we define $m^\sim_k(t) = \text{ess sup}_{x \in \Omega_k} m(x, t)$. Each $m^\sim_k$ is a measurable function on $[a, b]$. We
have \( m_j^\sim(t) \leq m_{j+1}^\sim(t) \) and \( \lim_{j \to \infty} m_j^\sim(t) = m^\sim(t) \). So

\[
\lim_{j \to -\infty} \int_a^b m_j^\sim(t)dt > c
\]

We fix \( k \in N \) large enough such that \( \int_a^b m_k^\sim(t)dt > c \) and \( k > 1/\delta \). Let 
\[
E(\eta) = \{(x, t) \in \Omega_{1/k}x[a, b] : m(x, t) \geq m_k^\sim(t) - \eta\}.
\]
Also we set \( (E(\eta))^d = \{(x, t) \in E : (x, t) \text{ is a density point of } E_\eta\} \).

We fix \( \alpha \in (0, 1/2) \). Then we consider for \( r \in N \) the set \( E(\eta)^{(r)} \) of the points in \((E(\eta))^d\) such that \(|Q \cap E(\eta)|/|Q| \geq 1 - \alpha\) for each open cube \( Q \) containing \((x, t)\) with diameter less than \(1/r\) and edges parallel to the coordinate axis. It is easy to see that \( E(\eta)^{(r)} \) is a measurable set. Also \( E(\eta)^{(r)} \subset E(\eta)^{(s)} \) for \( r < s \) and \((E(\eta))^d \subset \bigcup_{r \in N} E(\eta)^{(r)}\).

Moreover, we have \(|(E(\eta))t| \neq 0 \) a.e. \( t \in [a, b] \), so \(|(E(\eta)^d)t| \neq 0 \) a.e. \( t \in [a, b] \) and then \(|\pi(E(\eta)^d)| = b - a \). So \( \lim_{r \to -\infty} |\pi(E(\eta)^{(r)})| \geq |\pi(E(\eta)^d)| = b - a \). Then \( \lim_{r \to -\infty} |\pi(E(\eta)^{(r)})| = b - a \).

Given \( \varepsilon > 0 \), we fix \( r > 2k \) such that \( |\pi(E(\eta)^{(r)})| \geq b - a - \varepsilon \), then we choose \( \ell, \ 0 < \ell < 1/(r(n + 1)) \) such that \( N\ell = b - a \) for some \( N \in N \).

Let \( \{t_i\}_{0 \leq i \leq N} \) be the partition of \([a, b]\) given by \( t_i = a + i\ell \), \( 1 \leq i \leq N \). For \( 1 \leq i \leq N \), we take a cube \( Q_i \) with edges parallel to the coordinate axis and of length \( \ell \), chosen as follows: If the strip \( R^n \times (t_{i-1}, t_i) \) meets \( E(\eta)^{(r)} \) we take \( Q_i \) such that \( Q_i \cap E(\eta)^{(r)} \neq \emptyset \) and \( \pi(Q_i) = (t_{i-1}, t_i) \). In the other cases, we choose \( Q_i \) such that \( Q_i \cap \Omega_{1/k} \neq \emptyset \). Since \( E(\eta)^{(r)} \subset \Omega_{1/k} \) and \( \text{diam}(Q_i) < 1/(2k\sqrt{n + 1}) \) we have \( Q_i \subset \Omega_{1/(2k)} \times (t_{i-1}, t_i) \), \( 1 \leq i \leq N \). Let \( I = \{i : 1 \leq i \leq N \} \) and \((R^n \times (t_{i-1}, t_i)) \cap E(\eta)^{(r)} \neq \emptyset\} \) and let \( I^c \) be its complement. Since \( |\pi(E(\eta)^{(r)})| \geq b - a - \varepsilon \), \( I^c \) satisfies \( \sum_{i \in I^c} (t_i - t_{i-1}) < \varepsilon \).

We have, for \( i \in I \)

\[
\int_{Q_i} m(x, t)dxdt = \int_{Q_i \cap E(\eta)} m(x, t)dxdt + \int_{Q_i \cap E(\eta)^c} m(x, t)dxdt.
\]
Now
\[
\int_{Q_i \cap E(\eta)} m(x, t) dx \, dt \\
\geq \int_{Q_i \cap E(\eta)} m_k^\sim(t) dx \, dt - \eta |Q_i \cap E(\eta)| \\
= \int_{t_{i-1}}^{t_i} m_k^\sim(t)|((Q_i \cap E(\eta))_t| - |(Q_i)_t|) dt + \int_{t_{i-1}}^{t_i} m_k^\sim(t)|(Q_i)_t| dt \\
- \eta |Q_i \cap E(\eta)| \\
\geq \int_{t_{i-1}}^{t_i} |(Q_i \cap E(\eta))_t| - |(Q_i)_t| dt + \ell^n \int_{t_{i-1}}^{t_i} m_k^\sim(t) dt - \eta \ell^{n+1} \\
= |Q_i \cap E(\eta)| - |Q_i| + \ell^n \int_{t_{i-1}}^{t_i} m_k^\sim(t) dt - \eta \ell^{n+1} \\
\geq -\alpha \ell^{n+1} - \eta \ell^{n+1} + \ell^n \int_{t_{i-1}}^{t_i} m_k^\sim(t) dt
\]
on the other hand
\[
\left| \int_{Q_i \cap E(\eta)^c} m(x, t) dx \, dt \right| \leq |Q_i \cap E(\eta)^c| = |Q_i| - |Q_i \cap E(\eta)| \\
\leq |Q_i|(1 - (1 - \alpha)) = \alpha \ell^{n+1}
\]
So
\[
\sum_{i \in I} \int_{Q_i} m(x, t) dx \, dt \geq \ell^n \sum_{i \in I} \int_{t_{i-1}}^{t_i} m_k^\sim(t) dt - \#(I^c)\alpha \ell^{n+1} - \#(I) \eta \ell^{n+1}
\]
where \#(I) means cardinal of I and, since \( \ell \#(I^c) \leq \varepsilon \)
\[
\sum_{i \in I^c} \left| \int_{Q_i} m(x, t) dx \, dt \right| \leq \#(I^c) \ell^{n+1} \leq \varepsilon \ell^n
\]
Hence
\[
\sum_{i=1}^{N} \int_{Q_i} m(x, t) dx \, dt \\
\geq \ell^n \int_a^b m_k^\sim(t) dt - 2\alpha \ell^{n+1} N - \varepsilon \ell^n - \sum_{i \in I^c} \left| \int_{t_{i-1}}^{t_i} m_k^\sim(t) dt \right| - \#(I) \eta \ell^{n+1} \\
\geq \ell^n \int_a^b m_k^\sim(t) dt - 2\alpha \ell^{n+1} N - 2\varepsilon \ell^n - \eta(b-a) \ell^n.
Finally $\sum_{i=1}^{N} \int_{Q_i} m(x, t) dx dt \geq c \ell^n$ for $\alpha, \eta$ and $\varepsilon$ small enough. ■

Remark 4.2. By the absolute continuity of the indefinite integral, in Lemma 4.1, $Q_1$ and $Q_N$ can be chosen with the same projection on $\mathbb{R}^n$. Also for $\delta$ small enough, we can replace each $Q_i$ by $Q_i^\sim$ where $Q_i^\sim$ is the parallelepiped with the same basis as $Q_i$ and such that $\pi(Q_i^\sim) = (t_{i-1} + \delta, t_i - \delta)$.

Let $A, B$ two sets, we will denote with $A\Delta B$ their symmetric difference $(A - B) \cup (B - A)$.

Remark 4.3 Suppose $\Omega_\varepsilon$ connected, let $\{Q_i\}_{i=1}^{N}$ be a family of congruent open cubes in $\mathbb{R}^{n+1}$ with edges of length $\ell < \varepsilon / 2n$ and parallel to the coordinates axis satisfying $\bigcup_{1 \leq i \leq N} Q_i \subseteq \Omega_\varepsilon \times [a, b]$ and $\bigcup_{1 \leq i \leq N} \Pi(Q_i) = [a, b]$, then there exists a tube $B = \{ (\gamma(t) + \Omega_0, t), 0 \leq t \leq T \} \subseteq \Omega \times [a, b]$ with $\gamma \in C^\infty([0, T])$, $\gamma(j)(0) = \gamma(j)(T)$ for all $j$, and $\Omega_0$ a domain with $C^\infty$ boundary such that $\left| \left( \bigcup_{1 \leq i \leq N} Q_i \right) \Delta B \right| \leq \delta$.

Lemma 4.4 Let $m$ be a function in $L^\infty(\Omega \times \mathbb{R})$, $m(x, t)$ $T$-periodic in $t$, suppose

$$P(m) = \int_{0}^{T} \text{ess sup}_{x \in \Omega} m(x, t) dt > 0.$$ 

Then there exist $\gamma \in C^2(\mathbb{R}, \Omega)$ a periodic curve in $\Omega$ and a domain $\Omega_0$ in $\mathbb{R}^n$ with $C^\infty$ boundary such that the tube $B = \{ (\gamma(t) + z, t) : z \in \Omega_0, 0 \leq t \leq T \}$ satisfies: $B \subseteq \Omega \times [0, T]$ and $\int_{B} m(x, t) dx dt > 0$.

Proof. We can assume $\|m\|_\infty \leq 1$. Since $\Omega$ has regular boundary, there exists $\varepsilon > 0$ such that $\Omega_\varepsilon$ is a non empty and connected set. Let $\{Q_i\}_{i=1}^{N}$ be the family of cubes with edges of length $\ell$, provided by lemma 4.1 such that $\sum_{i=1}^{N} \int_{Q_i} m(x, t) dx dt > \ell^n P(m) / 2$, for this family and $\delta = 4^{-1} \ell^n P(m)$ we consider the tube $B$, provided by remark 4.3. Then

$$\left| \int_{B} m(x, t) dx dt - \sum_{i=1}^{N} \int_{Q_i} m(x, t) dx dt \right| \leq 2 \left| B \Delta \left( \bigcup_{1 \leq i \leq N} Q_i \right) \right| < 4^{-1} \ell^n P(m).$$

So $\int_{B} m(x, t) dx dt \geq \sum_{i=1}^{N} \int_{Q_i} m(x, t) dx dt - 4^{-1} \ell^n P(m) \geq 4^{-1} \ell^n P(m) > 0$ ■

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Theorem 4.5. Let $m$ be a $T$-periodic function in $L^\infty(\Omega \times IR)$.

(a) Suppose $P(m) > 0$ and $\langle \Psi, m \rangle < 0$. Then there exist $\lambda > 0$, and $w > 0$, $w \in C^{1+\gamma', \gamma}_T(\overline{\Omega} \times IR)$ solution of the periodic Neumann eigenvalue problem

$$\mathcal{L} w = \lambda m w$$
$$\partial w / \partial n|_{\partial \Omega \times IR} = 0.$$ 

(b) Suppose $P(m) > 0$. Then there exist $\lambda > 0$, and $w > 0$, $w \in C^{1+\gamma', \gamma}_T(\overline{\Omega} \times IR)$ solution of the periodic Dirichlet eigenvalue problem

$$\mathcal{L} w = \lambda m w$$
$$w|_{\partial \Omega \times IR} = 0$$

Proof. First, we treat the case Dirichlet boundary condition. We take $m_j \in C^\infty(\Omega \times IR)$, $T$-periodic with $\text{supp}(m_j) \subseteq K_j \times IR$ for some compact $K_j \subseteq \Omega$, and such that $\lim_{j \to \infty} m_j = m$ in $L^p_T(\Omega \times IR)$. We may suppose $\|m\|_\infty \leq 1/2$. If the tube $B$ provided by lemma 4.4 is a cylinder $C = \Omega_0 \times [0, T]$ the function $\mu^{c}_{m_j}(\lambda)$ defined by

$$Lu_j^c - \lambda m_j u_j^c = \mu^{c}_{m_j}(\lambda)u_j^c \quad \text{on} \quad \Omega_0 \times IR \quad (4.1)$$

$$u_j^c \in C^{2,1}(\overline{\Omega_0} \times IR), \quad u_j^c|_{\partial \Omega_0 \times IR} = 0$$

$$u_j^c > 0 \quad \text{in} \quad \Omega_0 \times IR \quad \text{and} \quad T - \text{periodic}$$

is such that $\mu^{c}_{m_j}(\eta) < 0$ for some $\eta > 0$ independent of $j$. This holds because from $\int_C m(x,t) dx dt > 0$ (lemma 4.4), there exists $\varphi \in C^\infty_\varepsilon(\Omega_0)$, $\varphi > 0$, $\int_C \varphi^2(x) dx = 1$ and $c > 0$ such that $\int_C m_j(x,t) \varphi^2(x) dx dt > c > 0$ for all $j$. Also $D_i a_{i,j} \in C_T(\overline{\Omega} \times IR)$, so we can apply Prop. 3.1 in [6], p. 110, to obtain that the principal eigenvalues $\lambda^{c}_{m_j}$ given by

$$Lv_j^c = \lambda^{c}_{m_j}(m_j) v_j^c \quad \text{in} \quad \Omega_0 \times IR \quad (4.2)$$

$$v_j^c \in C^{2,1}(\overline{\Omega_0} \times IR), \quad v_j^c|_{\partial \Omega_0 \times IR} = 0$$

$$v_j^c > 0 \quad \text{in} \quad \Omega_0 \times IR \quad \text{and} \quad T - \text{periodic}$$
are uniformly bounded above by \( \eta \), and from the concavity of \( \mu_{m_j}^c(\lambda) \) we obtain \( \mu_{m_j}^c(\eta) < 0 \) for all \( j \). We normalize \( v_j^c \) by \( \|v_j^c\|_{L^\infty(C)} = 1 \). From (4.2) and the compactness of \((IL + 1)^{-1}\) it follows that there exist (modulo a subsequence) \( v^c = \lim_{j \to \infty} v_j^c \) in \( L^p(C) \) and \( \mu_{m}^c(\eta) = \lim_{j \to \infty} \mu_{m_j}^c(\eta) \). \( v^c \) is a solution of a Dirichlet problem in \( C \) of the type (4.1) with weight \( m^c = m|_C \) and eigenvalue \( \mu_{m}^c(\eta) \). We denote \( v_j \) and \( v \) the extensions of \( v_j^c \) and \( v^c \) respectively, by zero to \( \Omega \times IR \). From the maximum principle applied to \( w_j = \eta(L + \eta)^{-1}(1 + m_j)v_j \) ([7], p. 43) we obtain
\[
\eta(\eta L + \eta)^{-1}((1 + m)v) \geq v
\] (4.3)

Let \( S_\eta : Y \to Y \) be the operator defined by \( S_\eta u = \eta(\eta L + \eta)^{-1}((1 + m)u) \), and let \( \rho, u_\eta \) be its spectral radius and a positive eigenfunction associated respectively. So, by (4.3) and the Krein rutman theorem, \( \rho \geq \eta \). Since \( S_\eta u_\eta = \rho u_\eta \) we have \((\eta L + \lambda^\sim(1 - m))^{-1}u_\eta = (2\rho^{-1}\eta - \eta)^{-1}u_\eta \), where \( \lambda^\sim = \rho^{-1}\eta \) and so \( \mu_m(\lambda^\sim) = \eta\rho^{-1} - \eta < 0 \). Also \( \mu_m(0) > 0 \). Then we have a solution \( u^D \in W \), \( \lambda^D > 0 \) of the Dirichlet problem
\[
IL(u^D) = \lambda m u^D \text{ in } \Omega \times IR
\]
\[
u^D > 0 \text{ in } \Omega \times IR \text{ and } T - \text{periodic, } u^D|_{\partial\Omega \times IR} = 0.
\]

If the tube \( B \) is not a cylinder, by the change of coordinates \((y, t) = \Phi(x, t) = (x - \gamma(t), t)\) we have a similar problem to (4.1) in a cylinder \( C \) with a new operator \( L^\Phi \) and a new weight \( m^\Phi \) with \( \int_C m^\Phi(x, t)dxdt > 0 \). We denote \( v_j \) and \( v \), defined on \( B \), extended by 0 to \( \Omega \times IR \) corresponding to the functions \( v_j^c \) and \( v^c \) defined in the cylinder \( C = \Phi(B) = \Omega_0 \times IR \). So we obtain (4.3) on \( \Omega \times IR \) and we get the solution \( u^D \) in \( \Omega \times IR \). We may remark that \( \mu_{m_j}^N(\lambda) \leq \mu_{m_j}^D(\eta) \) (the supra index \( N, D \) refers to the Neumann or Dirichlet condition). So we have \( \mu_{m_j}^N(\eta) < 0 \) for all \( j \). This gives that \( \mu_{m_j}^N(\epsilon) \leq 0 \), but \( \mu_{m_j}^N(0) = 0 \). Now the condition \( \langle \Psi, m \rangle < 0 \) gives \( d\mu_{m_j}^N/d\lambda|_{\lambda=0} > 0 \) which gives \( \mu_{m_j}^N(\epsilon) > 0 \) for small enough \( \epsilon > 0 \). Existence and uniqueness of the principal eigenvalues \( \lambda^D > \lambda^N > 0 \) follows from the concavity of \( \mu_{m}^N(\lambda) \) and \( \mu_{m}^N(\lambda) \).

**Theorem 4.6.** Under the hypothesis of the theorem 4.1 the principal eigenvalue is an \( M \)-simple eigenvalue.

**Proof.** Follows from remark 3.11.
Remark 4.7. Since for a $T$-periodic function $m \in L^\infty(\Omega \times IR), \mu_m$ is real analytic and concave, with the same proof given for the case $m \in C^{\theta,\theta/2}(\overline{\Omega} \times IR), \theta > 0$ (see [7], Theorems 16.1 and 16.3) the following results holds.

Let $m$ be a $T$-periodic function, $m \in L^\infty(\Omega \times IR)$ and let $\underline{m}(t) = \text{ess inf } m(x,t), \overline{m}(t) = \text{ess sup } m(x,t).$ Suppose that there exists a positive eigenvalue $\lambda$ with a positive eigenfunction $u_\lambda \in \text{Dom}(IL)$ associated, solution of the periodic parabolic boundary eigenvalue problem $IL u = \lambda m u, Bu = 0$. Then if the boundary condition is the Dirichlet condition we have $P(m) > 0$, and for the Neumann condition we have

1. $\underline{m} \neq \overline{m}$ in $L^\infty(IR)$ implies $P(m) > 0$ and $\langle \Psi, m \rangle < 0$.
2. $\underline{m} = \overline{m}$ in $L^\infty(IR)$ (i.e. $m$ is function of $t$ alone) implies

$$\int_0^T m(t)dt = 0.$$
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