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**Variational principle and the Dynamical Entropy
of Space Translation**

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Variational principle and the Dynamical Entropy of Space Translation

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Abstract

The dynamical entropy of space translations is used in the variational calculus of the pressure. It is shown that for quantum spin systems on a lattice of an arbitrary dimension, the pressure obtained in this way coincides with the usual one.

1 Introduction

In this note we consider ν -dimensional quantum lattice systems. Let $\mathbf{Z}^\nu$ be the ν -dimensional lattice, and $\mathcal{A}_k (k \in \mathbf{Z}^\nu)$ be copies of $M_d(\mathbb{C})$, the $d \times d$ full matrix algebra. For any bounded region $\Lambda \subset \mathbf{Z}^\nu$, the local algebra $\mathcal{A}_\Lambda$ is given by $\otimes_{x \in \Lambda} \mathcal{A}_x$. It follows from the definition that, for $\Lambda \subset \Lambda'$, we have $\mathcal{A}_{\Lambda'} = \mathcal{A}_\Lambda \otimes \mathcal{A}_{\Lambda' \setminus \Lambda}$, where $\Lambda' \setminus \Lambda$ is the complement of Λ in Λ' . We shall identify $\mathcal{A}_\Lambda$ with $\mathcal{A}_\Lambda \otimes 1_{\Lambda' \setminus \Lambda}$ in $\mathcal{A}_{\Lambda' \setminus \Lambda}$. The definition also implies that if Λ_1 and Λ_2 are disjoint, the elements of $\mathcal{A}_{\Lambda_1}$ commute with those of $\mathcal{A}_{\Lambda_2}$. The C^* -algebra $\mathcal{A}$ under our investigation is the norm completion of the normed algebra $\mathcal{A}_\infty = \cup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda$ associated with finite regions $\Lambda \subset \mathbf{Z}^\nu$. We note that the group $\mathbf{Z}^\nu$ of space translations is a subgroup of the automorphism group of $\mathcal{A}$ and we denote the action of this group by $A \in \mathcal{A}_\Lambda \mapsto \sigma_{\vec{x}} A \in \mathcal{A}_{\Lambda+\vec{x}}$, $\vec{x} \in \mathbf{Z}^\nu$. For each finite subset Λ of $\mathbf{Z}^\nu$, we are given an element $\Phi(\Lambda) \in \mathcal{A}_\Lambda$, called an interaction potential satisfying the following conditions:

- (1) $\Phi(\emptyset) = 0$,
- (2) $\sigma_{\vec{x}} \Phi(\Lambda) = \Phi(\Lambda + \vec{x})$: *translation invariance*,
- (3) $\Phi(\Lambda)^* = \Phi(\Lambda)$: *self-adjointness*.

Given a potential Φ and a finite subset $\Lambda \subset \mathbf{Z}$, the internal Hamiltonian confined to Λ takes the form

$$U_\Lambda \equiv \sum_{X \subseteq \Lambda} \Phi(X).$$

In the following, we consider two different kinds of Banach space of interactions. For the first one, we impose the growth condition on Φ as follows:

$$(4) \|\Phi\|_0 \equiv \sum_{\Lambda \ni 0} \frac{\|\Phi(\Lambda)\|}{|\Lambda|} < \infty.$$

With this norm, the potentials under consideration form a real Banach space $\mathcal{B}_0$.

For the next one, we recall the notion of the standard potential [2]. Let ω be a product state of $\mathcal{A}$, namely $\omega(aa') = \omega(a)\omega(a')$ for $a \in \mathcal{A}_\Lambda, a' \in \mathcal{A}_{\Lambda^c}$ for any finite subset Λ . Let Γ_Λ^ω be the conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_\Lambda$. Namely, for each $a \in \mathcal{A}$, $\Gamma_\Lambda^\omega(a)$ is defined as the element of $\mathcal{A}_\Lambda$ satisfying $\omega(\Gamma_\Lambda^\omega(a)b) = \omega(ab)$ for all $b \in \mathcal{A}_\Lambda$. We assume

$$(4)' \quad \Gamma_I^\omega(\Phi(\Lambda)) = 0 \quad \text{unless } I \supseteq \Lambda : \text{standardness}$$

Finally, for each finite subset $\Lambda \subset \mathbf{Z}^\nu$ we assume the existence of the surface energy connecting inside and outside of Λ , i.e.

$$(5)' \quad W_\Lambda \equiv \left\{ \sum \Phi(X) : X \cap \Lambda \neq \emptyset, X \cap \Lambda^c \neq \emptyset \right\} \in \mathcal{A},$$

where on the right-hand side the finite sum over $X \subset \Lambda'$ is first taken and the van Hove limit $\Lambda' \nearrow \mathbf{Z}^\nu$ is then taken, this limit is assumed to converge in norm. Total interaction Hamiltonian of Λ is defined by

$$H_\Lambda = U_\Lambda + W_\Lambda.$$

We then define

$$\|\Phi\| \equiv \sup\{\|H_n\| : n \in \mathbf{Z}^\nu\} (= \|H_n\|),$$

where $H_n = H_\Lambda$ for the one-point set $\Lambda = \{n\}$ and $\|H_n\|$ is independent of $n \in \mathbf{Z}^\nu$ due to the assumed translation invariance. With this norm, the potentials under consideration form a real Banach space $\mathcal{B}$.

For each $\Phi \in \mathcal{B}_0$, let us define $A_\Phi \in \mathcal{A}$ by

$$A_\Phi \equiv \sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|}.$$

By definition, A_Φ is linear in Φ and

$$\|A_\Phi\| \leq \|\Phi\|_0. \quad (1)$$

For $\Phi \in \mathcal{B}$, the following estimates have been obtained [2]

$$\|U_\Lambda\| \leq \|H_\Lambda\| \leq \sum_{n \in \Lambda} \|H_n\| \leq |\Lambda| \cdot \|\Phi\| \quad (2)$$

$$\lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{\|W_\Lambda\|}{|\Lambda|} = 0 \quad (3)$$

where the limit of $\Lambda \nearrow \mathbf{Z}^\nu$ is the van Hove limit.

Let Tr_Λ denote the canonical trace of $\mathcal{A}_\Lambda$, where the term “canonical” means that $\text{Tr}_\Lambda(e) = 1$ for each minimal projection $e \in \mathcal{A}_\Lambda$. For each finite subset $\Lambda \in \mathbf{Z}^\nu$, the partition function is defined by

$$Z_\Lambda(\beta\Phi) \equiv \text{Tr}_\Lambda (e^{-\beta U_\Lambda}).$$

Let $P_\Lambda(\beta\Phi) \equiv \frac{\log Z_\Lambda(\beta\Phi)}{|\Lambda|}$, then for $\Phi \in \mathcal{B}_0$ (or $\mathcal{B}$) the following basic results are known.

- (i) $P(\beta\Phi) \equiv \lim_{\Lambda \nearrow \mathbf{Z}^\nu} P_\Lambda(\beta\Phi)$ exists in the van Hove limit,
- (ii) $|P(\Phi - \Psi)| \leq \|\Phi - \Psi\|_0$ ($\|\Phi - \Psi\|$),
- (iii) $P(\Phi)$ is a convex function of Φ .

We call $P(\beta\Phi)$ the (thermodynamic) pressure of Φ at β . If ω is a state of $\mathcal{A}$, ω_Λ denotes its restriction to $\mathcal{A}_\Lambda$ for each finite subset of $\Lambda \subset \mathbf{Z}^\nu$. The energy and entropy of this finite system Λ are given by

$$\begin{aligned} E_\Lambda(\omega) &= \omega(U_\Lambda), \\ S_\Lambda(\omega) &\equiv -\text{Tr}_\Lambda D_\Lambda \log D_\Lambda. \end{aligned}$$

Here D_Λ denotes the density matrix of ω_Λ with respect to the canonical trace Tr_Λ on $\mathcal{A}_\Lambda$. Let $\mathcal{A}_{+,1,\text{inv}}^*$ be the set of translation invariant states. We may define the energy density and the entropy density of the translation invariant state $\omega \in \mathcal{A}_{+,1,\text{inv}}^*$ to be

$$e_\Phi(\omega) \equiv \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{E_\Lambda(\omega)}{|\Lambda|}, \quad s(\omega) \equiv \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{S_\Lambda(\omega)}{|\Lambda|}.$$

It has been shown that for Φ in $\mathcal{B}_0$ or $\mathcal{B}$, $e_\Phi(\omega)$ exists and

$$\begin{aligned} |e_\Phi(\omega)| &\leq \|\Phi\|_0(\|\Phi\|), \\ e_{c\Phi+d\Psi}(\omega) &= c \cdot e_\Phi(\omega) + d \cdot e_\Psi(\omega) \quad : \text{linearity}, \\ e_\Phi(\lambda\omega + (1-\lambda)\mu) &= \lambda e_\Phi(\omega) + (1-\lambda)e_\Phi(\mu) \quad : \text{affinity}. \end{aligned}$$

Furthermore, especially in the case of $\Phi \in \mathcal{B}_0$ we have

$$e_\Phi(\omega) = \omega(\mathcal{A}_\Phi).$$

From the strong subadditivity property, it has been shown that the limit entropy density for the translation invariant state exists and is given by the infimum

of over all finite subsets $\Lambda \subset \mathbb{Z}^\nu$. It has also been shown that the function $\omega \in \mathcal{A}_{+,1,inv}^* \mapsto s(\omega)$ is affine and upper semi-continuous with respect to weak*-topology.

The ‘‘Gibbs Variational Principle’’ is used to define the equilibrium states (See for example [5]), that is for any $\Phi \in \mathcal{B}_0$ or $\mathcal{B}$, $P(\beta\Phi)$ is obtained by the following variational equality:

$$P(\beta\Phi) = \sup_{\omega \in \mathcal{A}_{+,1,inv}^*} [s(\omega) - \beta \cdot e_\Phi(\omega)]. \quad (4)$$

Furthermore, if $\rho \in \mathcal{A}_{+,1,inv}^*$ satisfies

$$P(\beta\Phi) = s(\rho) - \beta \cdot e_\Phi(\rho), \quad (5)$$

then ρ is called an invariant equilibrium state for Φ at inverse temperature β . Denote the set of all invariant equilibrium states as $I_{\beta\Phi}$.

In our paper, we use the dynamical entropy of space translations $h_\omega(G(\vec{\sigma}))$ to be defined in the next section instead of entropy density $s(\omega)$. Our main theorem is the following:

Theorem 1.1 (Main Theorem) *Let $\Phi \in \mathcal{B}(\mathcal{B}_0)$, then*

$$P(\beta\Phi) = \sup_{\omega \in \mathcal{A}_{+,1,inv}^*} [h_\omega(G(\vec{\sigma})) - \beta \cdot e_\Phi(\omega)].$$

Narnhofer obtained this theorem for exponentially decreasing potentials, i.e. for some $r > 0$

$$\|\Phi\|_r \equiv \sum_{\Lambda \ni 0} e^{r|\Lambda|} \|\Phi(\Lambda)\| < \infty.$$

We obtain the above result for wider classes of potentials. Moreover, we simplify the proof. (We will mention the Narnhofer’s proof later.) Our strategy to prove the theorem is similar to those of the usual Variational Principle obtained in [20] and [19]. To make use of the methods in these original versions, we just need the well known basic properties of the dynamical entropy such as

- (i)covariance property
- (ii)scaling property in automorphisms
- (iii)affinity in states,

which we will recall in the next section.

2 Dynamical Entropy

We will recall the definition of the dynamical entropy introduced by Connes, Narnhofer, Thirring in [6]. For this purpose, we first explain the quantum relative entropy. Let $\mathcal{A}$ be a finite-dimensional C^* -algebra and let ψ_1 and ψ_2 be states on $\mathcal{A}$. The density matrices corresponding ψ_1 and ψ_2 are denoted by D_1 and D_2 , respectively. The relative entropy of ψ_1 and ψ_2 is defined by

$$S(\psi_1|\psi_2) \equiv \text{Tr} (D_2(\log D_2 - \log D_1)).$$

We then introduce the notion of an abelian model defined in [6]. Let $\mathcal{A}$ be a unital C^* -algebra with a state ψ . Let $\mathcal{C}$ be a finite-dimensional Abelian algebra with a state μ and a positive identity preserving map $\mathbf{P} : \mathcal{A} \mapsto \mathcal{C}$ such that $\mu \circ \mathbf{P} = \psi$. We call any triple $(\mathcal{C}, \mathbf{P}, \mu)$ which satisfies the above condition an abelian model for $(\mathcal{A}, \psi)$. Let $\{e_i\}_{i=1}^r$ be the set of minimal projections of $\mathcal{C}$ such that $\sum_{i=1}^r e_i = Id$. An abelian model then induces a decomposition of the state ψ into a convex combination of states $\hat{\psi}_i$ with weights $\mu_i (\equiv \mu(e_i))$ as follows:

$$\psi(A) = \sum_{i=1}^r \mu_i \hat{\psi}_i(A) \quad A \in \mathcal{A},$$

where $\hat{\psi}_i$ are uniquely determined by

$$\mathbf{P}(A) = \sum_{i=1}^r \hat{\psi}_i(A) e_i.$$

“The Information gain by the measurement” is defined by

$$\mathcal{E}_\mu(\mathbf{P}) = \sum_{i=1}^r \mu_i S(\psi|\hat{\psi}_i).$$

Then “the entropy defect” is defined by

$$s_\mu(\mathbf{P}) = S(\mu) - \mathcal{E}_\mu(\mathbf{P}),$$

where $S(\mu)$ denotes the abelian entropy $-\sum_{i=1}^r \mu_i \log \mu_i$.

From now on we shall assume $\mathcal{A}$ to be a nuclear C^* -algebra. We denote by $CP_1(\mathcal{A})$ the set of all completely positive unital maps from finite-dimensional C^* -algebras into $\mathcal{A}$. Let $\{\gamma_1, \dots, \gamma_k\}$ be a finite subset of $CP_1(\mathcal{A})$ with $\gamma_j : \mathcal{N}_j \mapsto \mathcal{A} (j = 1, \dots, k)$, where each $\mathcal{N}_j$ is a finite-dimensional C^* -algebra. Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be C^* -subalgebras of $\mathcal{C}$ and let $E_j : \mathcal{C} \mapsto \mathcal{C}_j (j = 1, \dots, k)$ be μ -preserving conditional expectations. $(\mathcal{C}_j, \rho_j, \mu_j)$ gives an abelian model for $(\mathcal{N}_j, \psi \circ \gamma_j)$, where $\rho_j = E_j \circ \mathbf{P} \circ \gamma_j$ and $\mu_j = \mu|_{\mathcal{C}_j}$. Then the quadruple $(\mathcal{C}, E_j|_{j=1}^k, \mathbf{P}, \mu)$ is called an abelian model for $(\mathcal{A}, \psi, \{\gamma_1, \dots, \gamma_k\})$. Now the entropy of this abelian model is defined by

$$S(\mu) - \sum_{j=1}^k s_{\mu_j}(\rho_j).$$

The supremum of the entropies of all abelian models for $(\mathcal{A}, \psi, \{\gamma_1, \dots, \gamma_k\})$ is denoted by $H_\psi(\gamma_1, \dots, \gamma_k)$. We shall call $H_\psi(\gamma_1, \dots, \gamma_k)$ the entropy of completely positive unital maps $\{\gamma_1, \dots, \gamma_k\}$. Let θ be an automorphism of $\mathcal{A}$ such

that ψ is invariant under θ . For $\gamma \in CP_1(\mathcal{A})$, the existence of the following limit has been shown by using the subadditivity of $H_\psi(\gamma_1, \dots, \gamma_k)$ and the θ invariance of ψ :

$$h_{\psi, \theta}(\gamma) = \lim_{k \rightarrow \infty} \frac{1}{k} H_\psi(\gamma, \theta \circ \gamma, \dots, \theta^{k-1} \circ \gamma), \quad (6)$$

where θ^i denotes the i -fold repetition of θ . The dynamical entropy of θ with respect to ψ is defined by

$$h_\psi(\theta) = \sup_{\gamma \in CP_1(\mathcal{A})} h_{\psi, \theta}(\gamma) \quad (7)$$

The dynamical entropy for a single automorphism has been generalized to a group generated by several commuting automorphisms [11] as follows. Let $G(\vec{\theta})$ be a group of automorphisms generated by commuting automorphisms $\{\theta_1, \dots, \theta_\nu\}$. We write $\theta_{\vec{x}}$ for $(\theta_1)^{x_1} \circ \dots \circ (\theta_\nu)^{x_\nu}$ where $\vec{x} = (x_1, \dots, x_\nu) \in \mathbb{Z}^\nu$. We consider a set of finite completely positive unital maps $X = \{\gamma_1, \dots, \gamma_m\}$, $\gamma_i \in CP_1(\mathcal{A})$ for $1 \leq i \leq m$. Let $\mathcal{F}$ be the family of all finite subsets of $CP_1(\mathcal{A})$, and by $\mathcal{F}_1$ we denote the all of ‘elementary’ sets $X = \{\gamma\}$ containing only one element. Let Λ be a finite subset of $\mathbb{Z}^\nu$. For $X \in \mathcal{F}$ and $G(\vec{\theta})$, we set

$$X_\Lambda(\vec{\theta}) \equiv \bigcup_{x \in \Lambda} \bigcup_{j=1}^m \theta_{\vec{x}} \circ \gamma_j \in \mathcal{F}.$$

Given $\vec{k} \in \mathbb{Z}^\nu$, let $\Lambda_0(\vec{k})$ be the rectangular parallelepiped of sides $(k_i)_{i=1}^\nu$ with its lower corner at $(0, \dots, 0)$, i.e. $\Lambda_0(\vec{k}) \equiv \{x \in \mathbb{Z}^\nu | 0 \leq x_i < k_i, i = 1, \dots, \nu\}$. Let $\{\Lambda_o(\vec{k}_n)\}_n$ be a monotone increasing sequence of rectangular parallelepipeds with their lower corner $\vec{0}$ such that $\Lambda_o(\vec{k}_n) \nearrow \mathbb{N}^\nu$ as $n \rightarrow \infty$. In the following we use the simple notation $\Lambda_{o,n}$ for $\Lambda_o(\vec{k}_n)$. It is easy to see that the following limit exists as infimum for the same reason as the 1-dimensional case.

$$h_{\psi, G(\vec{\theta})}(X) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_{o,n}|} H_\psi(X_{\Lambda_{o,n}}(\vec{\theta})).$$

The dynamical entropy for the commuting automorphisms group $G(\vec{\theta})$ with respect to an invariant state ψ is defined by

$$h_\psi(G(\vec{\theta})) = \sup_{X \in \mathcal{F}} h_{\psi, G(\vec{\theta})}(X). \quad (8)$$

It is easily shown that the above supremum can be attained even if we take X in $\mathcal{F}_1$. We then recall the following fundamental properties of dynamical entropy which has been also proved in [11] for the dynamical entropy of $\mathbb{Z}^\nu$ -actions by using almost the same way for the 1-dimensional case in [6].

(i)Covariance:

$$h_\psi(G(\vec{\theta})) = h_{\psi \circ \vartheta} \left(\vartheta^{-1} \circ G(\vec{\theta}) \circ \vartheta \right), \quad (9)$$

where ϑ is an arbitrary automorphism of $\mathcal{A}$, and $\vartheta^{-1} \circ G(\vec{\theta}) \circ \vartheta \equiv \{\vartheta^{-1}\theta_i \circ \vartheta | i = 1 \dots \nu\}$.

(ii)Scaling property of automorphisms:

$$h_\psi(G(\vec{\theta})) = p \cdot h_\psi(G_p(\vec{\theta})), \quad (10)$$

where $G_p(\vec{\theta})$ is a subgroup of $G(\vec{\theta})$ with finite index $p \in \mathbb{N}$.

(iii)Affinity in states:

$$h_{\lambda\psi_1+(1-\lambda)\psi_2}(G(\vec{\theta})) = \lambda h_{\psi_1}(G(\vec{\theta})) + (1-\lambda)h_{\psi_2}(G(\vec{\theta})), \quad (11)$$

for $0 \leq \lambda \leq 1$, and $G(\vec{\theta})$ -invariant states ψ_1, ψ_2 .

Note that for the 1-dimensional case, (ii) implies

$$h_\psi(\theta^p) = p \cdot h_\psi(\theta).$$

Now we shall go back to our setting of ν -dimensional quantum lattice systems. The C^* -algebra $\mathcal{A}(=\overline{\cup_\Lambda \mathcal{A}_\Lambda})$ under our investigation is a UHF algebra. We consider ν -dimensional translation group of automorphisms $G(\vec{\sigma})$ of $\mathcal{A}$ generated by $\sigma_1, \dots, \sigma_\nu$, where σ_i is the 1-shift in the direction of the i -coordinate axis. We assume that ψ is a translation invariant state. The non-commutative Kolmogorov-Sinai theorem proved in [6] holds also for the multi-dimensional cases. Therefore

$$h_\psi(G(\vec{\sigma})) = \lim_{n \rightarrow \infty} h_{\psi, G(\vec{\sigma})}(\mathcal{A}_n), \quad (12)$$

where $\{\mathcal{A}_n\}$ is an arbitrary increasing sequence of finite dimensional unital subalgebras of $\mathcal{A}$ such that $\mathcal{A}_n \nearrow \mathcal{A}$ in norm, and we always take the convention that a subalgebra $\mathcal{A}_n$ is standing for the inclusion map $\mathcal{A}_n \mapsto \mathcal{A}$.

In the proof of our theorem we shall compare H_ψ with the von-Neumann entropy $S(\psi)$. It has been shown that H_ψ is always dominated by the entropy S of ψ restricted to the algebra generated by the arguments of H_ψ . More precisely if $\{N_1, \dots, N_k\}$ is a set of finite dimensional subalgebras of $\mathcal{A}$ and they generate a finite dimensional algebra $N = \bigvee_{i=1}^k N_i$, then

$$H_\psi(N_1, \dots, N_k) \leq S_N(\psi). \quad (13)$$

(As above, $N_1, \dots, N_k$ in H_ψ denote the inclusion maps $N_i \mapsto \mathcal{A}$ ($i = 1, \dots, k$), and especially we call this $H_\psi(N_1, \dots, N_k)$ entropy of subalgebras $\{N_1, \dots, N_k\}$.) We recall one of the sufficient condition under which the equality holds in

(13) . Let $\Lambda_1, \dots, \Lambda_k$ be pairwise disjoint finite subsets of Z^ν . We set $\Lambda \equiv \Lambda_1 \cup \dots \cup \Lambda_k$ and $\Lambda^c \equiv Z^\nu \setminus \Lambda$. Let ϕ be a product state $\phi_{\Lambda_1} \otimes \dots \otimes \phi_{\Lambda_k} \otimes \phi_{\Lambda^c}$ on $\mathcal{A}(\equiv \mathcal{A}_{\Lambda_1} \otimes \dots \otimes \mathcal{A}_{\Lambda_k} \otimes \mathcal{A}_{\Lambda^c})$. We have

$$H_\phi(\mathcal{A}_{\Lambda_1}, \dots, \mathcal{A}_{\Lambda_k}) = \sum_{i=1}^k S(\phi_{\Lambda_i}) (= S_\Lambda(\phi)). \quad (14)$$

The detailed estimates of difference between H and S under several kinds of assumptions have been studied in [6].

3 Proof of Main Theorem

Let ω be a translation invariant state on $\mathcal{A}$. Since $e_{\beta\Phi}(\omega) = \beta e_\Phi(\omega)$, it suffices to prove the theorem for $\beta = 1$. From the variational inequality,

$$P(\Phi) \geq s(\omega) - e_\Phi(\omega)$$

It is easily obtained that the entropy density always dominates the dynamical entropy of space translation as follows. Let $\{\Lambda_m\}$ be an increasing sequence of finite subsets of Z^ν such that $\Lambda_m \nearrow Z^\nu$ in the van Hove limit. From the non-commutative version of Kolmogorov-Sinai theorem we have $h_\omega(G(\vec{\sigma})) = \lim_{m \rightarrow \infty} h_{\omega, G(\vec{\sigma})}(\mathcal{A}_{\Lambda_m})$, and the algebra generated by $\{\sigma_{\vec{x}} \mathcal{A}_{\Lambda_m}\}$ ($\vec{x} \in \Lambda_{0,n}$) is equal to $\mathcal{A}_{\Lambda_m \cup \Lambda_{0,n}}$ so that by (13) we have

$$\frac{1}{|\Lambda_{0,n}|} H_\omega(\mathcal{A}_{\Lambda_m}, \dots, \sigma_{\vec{x}} \mathcal{A}_{\Lambda_m}, \dots, \sigma_{\vec{k}_n} \mathcal{A}_{\Lambda_m}) \leq \frac{1}{|\Lambda_{0,n}|} S_{\Lambda_m \cup \Lambda_{0,n}}(\omega). \quad (15)$$

When $n \rightarrow \infty$, the right-hand side of the inequality converges to the entropy density $s(\omega)$ thus we obtain $h_\omega(G(\vec{\sigma})) \leq s(\omega)$. Now we know

$$P(\Phi) \geq h_\omega(G(\vec{\sigma})) - e_\Phi(\omega). \quad (16)$$

We will then construct states ρ_ϵ with $h_{\rho_\epsilon}(G(\vec{\sigma})) - e_\Phi(\rho_\epsilon) \geq P(\Phi) - \epsilon$ for an arbitrary positive number ϵ . Together with the above inequality (16), this proves our theorem.

Given positive integer a , let $\Lambda_o(a)$ be the cube of side a with its lower corner at $(0, \dots, 0)$, i.e. $\Lambda_o(a) = \{x \in Z^\nu | 0 \leq x_i \leq a-1, i = 1, \dots, \nu\}$. Let φ_Λ^c denote the local Gibbs state in Λ , that is,

$$\varphi_\Lambda^c(A) \equiv \frac{\text{Tr}_\Lambda A e^{-U_\Lambda}}{\text{Tr}_\Lambda e^{-U_\Lambda}} \quad (A \in \mathcal{A}_\Lambda).$$

$\sigma_{\vec{m}}^*$ is defined by the relation $\sigma_{\vec{m}}^* \varphi(A) = \varphi(\sigma_{\vec{m}}(A))$, ($A \in \mathcal{A}$, $\varphi \in \mathcal{A}_{+,1}^*$). We define $\varphi_a^c \equiv \bigotimes_{\vec{n} \in Z^\nu} \sigma_{a\vec{n}}^* \varphi_{\Lambda_o(a)}^c \in \mathcal{A}_{+,1,a-in\nu}^*$ which is a periodic state invariant

under $(a\mathbf{Z})^\nu$. From φ_a^c we construct an averaged state $\widehat{\varphi_a^c}$ which is translation invariant:

$$\widehat{\varphi_a^c} \equiv \sum_{\vec{n} \in \Lambda_o(a)} \frac{\sigma_{\vec{n}}^* \varphi_a^c}{|\Lambda_o(a)|} \in \mathcal{A}_{+,1,\text{inv}}^*.$$

Our ρ_ϵ is this state for sufficiently large a . First, we shall show the following lemma to estimate the energy term.

Lemma 3.1 *Let $\widehat{\varphi_a^c}$ and $\varphi_{\Lambda_o(a)}^c$ be as above, for Φ in $\mathcal{B}_0$, or $\mathcal{B}$. Then*

$$\lim_{a \rightarrow \infty} e_\Phi(\widehat{\varphi_a^c}) - |\Lambda_o(a)|^{-1} \varphi_{\Lambda_o(a)}^c(U_{\Lambda_o(a)}) = 0$$

proof.

We prove this lemma for the case $\mathcal{B}_0$ and $\mathcal{B}$ separately.

(i) $\Phi \in \mathcal{B}_0$:

In the following we take from the methods in [20] (and also in [18]), and adapt them to the present situation. We know $e_\Phi(\widehat{\varphi_a^c}) = \widehat{\varphi_a^c}(A_\Phi)$. By definition of $\widehat{\varphi_a^c}$ and A_Φ , we have

$$\begin{aligned} \widehat{\varphi_a^c}(A_\Phi) &= \sum_{\vec{n} \in \Lambda_o(a)} \frac{\sigma_{\vec{n}}^* \varphi_a^c}{|\Lambda_o(a)|} \left(\sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|} \right) \\ &= \sum_{\vec{n} \in \Lambda_o(a)} \sum_{\Lambda \ni 0} \frac{\varphi_a^c(\Phi(\Lambda + \vec{n}))}{|\Lambda_o(a)| \cdot |\Lambda|} \\ &= \sum_Y \frac{|Y \cap \Lambda_o(a)|}{|\Lambda_o(a)| \cdot |Y|} \varphi_a^c(\Phi(Y)), \end{aligned} \tag{17}$$

because a given $Y \subset \mathbf{Z}^\nu$ can be written as $\Lambda + \vec{n}$ with $\Lambda \ni 0$ and $\vec{n} \in \Lambda_o(a)$ exactly $|Y \cap \Lambda_o(a)|$ times.

We note that $\frac{|Y \cap \Lambda_o(a)|}{|Y|} = 1$ for $Y \subset \Lambda_o(a)$ and

$$\begin{aligned} |\Lambda_o(a)|^{-1} \varphi_a^c(U_{\Lambda_o(a)}) &= \sum_{Y \subset \Lambda_o(a)} |\Lambda_o(a)|^{-1} \varphi_a^c(\Phi(Y)) \\ &= \sum_{Y \subset \Lambda_o(a)} \frac{|Y \cap \Lambda_o(a)|}{|\Lambda_o(a)| \cdot |Y|} \varphi_a^c(\Phi(Y)). \end{aligned} \tag{18}$$

Thus it suffices to prove

$$\lim_{a \rightarrow \infty} \left[\sum_{Y \not\subset \Lambda_o(a)} \frac{|Y \cap \Lambda_o(a)|}{|\Lambda_o(a)| \cdot |Y|} \|\Phi(Y)\| \right] = 0$$

or equivalently

$$\lim_{a \rightarrow \infty} \left[\sum_{\substack{\mathfrak{n} \in \Lambda_o(a) \\ Y \ni \mathfrak{n} \\ Y \not\subset \Lambda_o(a)}} \frac{\|\Phi(Y)\|}{|\Lambda_o(a)| \cdot |Y|} \right] = 0. \quad (19)$$

Fix a positive integer r . Let $\hat{\Lambda}_o(a - 2r)$ be the cube, with side $a - 2r$ and with the same center as $\Lambda_o(a)$. We have

$$\begin{aligned} & \sum_{\substack{\mathfrak{n} \in \Lambda_o(a) \\ Y \ni \mathfrak{n} \\ Y \not\subset \Lambda_o(a)}} \frac{\|\Phi(Y)\|}{|\Lambda_o(a)| \cdot |Y|} \\ &= \sum_{\substack{\mathfrak{n} \in \Lambda_o(a-2r) \\ Y \ni \mathfrak{n} \\ Y \not\subset \Lambda_o(a)}} \frac{\|\Phi(Y)\|}{|\Lambda_o(a)| \cdot |Y|} + \sum_{\substack{\mathfrak{n} \in \Lambda_o(a) \setminus \hat{\Lambda}_o(a-2r) \\ Y \ni \mathfrak{n} \\ Y \not\subset \Lambda_o(a)}} \frac{\|\Phi(Y)\|}{|\Lambda_o(a)| \cdot |Y|} \\ &\leq |\Lambda_o(a)|^{-1} \left[|\hat{\Lambda}_o(a-2r)| \cdot \sum_{\substack{Y \ni o \\ \text{diam}(Y) \geq r}} \frac{\|\Phi(Y)\|}{|Y|} + |\Lambda_o(a) \setminus \hat{\Lambda}_o(a-2r)| \cdot \|\Phi\|_o \right]. \end{aligned} \quad (20)$$

Due to $\lim_{a \rightarrow \infty} \frac{|\hat{\Lambda}_o(a-2r)|}{|\Lambda_o(a)|} = 1$, the $\limsup_{a \rightarrow \infty}$ of (20) is bounded by $\sum_{\substack{Y \ni o \\ \text{diam}(Y) \geq r}} \frac{\|\Phi(Y)\|}{|Y|}$ for all r and so we obtain the desired result by taking the limit $r \rightarrow \infty$.

(ii) $\Phi \in \mathcal{B}$:

From (3) , for $\varepsilon > 0$ there exists a positive number a_ε such that

$$\frac{\|W_{\Lambda_o(a)}\|}{|\Lambda_o(a)|} < \varepsilon \quad \text{for } a \geq a_\varepsilon.$$

We fix $a(\geq a_\varepsilon)$ and we take a sufficiently large finite subset Λ in $\mathcal{Z}^\nu$. We denote $\sigma_i(\Lambda_o(a))$ by $\Lambda_i(a)$ for $i \in \mathcal{Z}^\nu$. For each $\vec{n} \in \Lambda_o(a)$, we divide Λ into disjoint regions as follows. Let $\Lambda_{-\vec{n}}^1(\square)$ be the union of all translates $\{\Lambda_{-\vec{n}+a\vec{m}}(a)\}(\vec{m} \in \mathcal{Z}^\nu)$ which are contained in Λ , and let $\partial\Lambda_{-\vec{n}}$ be its complement in Λ . Thus $\Lambda = \Lambda_{-\vec{n}}^1(\square) \cup \partial\Lambda_{-\vec{n}}$.

For the sake of a shorter notation, we assign a numbering $i(\vec{m}) = 1, 2, \dots, q$ for all $\vec{m} \in \mathcal{Z}^\nu$ satisfying $\Lambda_{-\vec{n}+a\vec{m}}(a) \subset \Lambda_{-\vec{n}}^1(\square)$, where q is the total number of such $\vec{m}$, and we denote $\Lambda_{-\vec{n}+a\vec{m}}(a) = \Lambda_i$ with $i = i(\vec{m})$, so that $\Lambda_{-\vec{n}}^1(\square) = \bigcup_{i=1}^q \Lambda_i$. Now we set $\Lambda_{-\vec{n}}^k(\square) \equiv \bigcup_{i=k}^q \Lambda_i$. By the assumed standardness of the potential,

we have

$$\begin{aligned}
U_{\Lambda^1_{-\vec{n}}(\square)} &= U_{\Lambda_1} + \Gamma_{\Lambda^1_{-\vec{n}}(\square)}^\omega(W_{\Lambda_1}) + U_{\Lambda^2_{-\vec{n}}(\square)} = \dots \\
&= \sum_{i=1}^m U_{\Lambda_i} + \sum_{i=1}^q \Gamma_{\Lambda^i_{-\vec{n}}(\square)}^\omega(W_{\Lambda_i}).
\end{aligned}$$

From this equality and the assumption, we obtain

$$\begin{aligned}
& |\Lambda^1_{-\vec{n}}(\square)|^{-1} \|U_{\Lambda^1_{-\vec{n}}(\square)} - \sum_{i=1}^q U_{\Lambda_i}\| \\
&= |\Lambda^1_{-\vec{n}}(\square)|^{-1} \left\| \sum_{i=1}^q \Gamma_{\Lambda^i_{-\vec{n}}(\square)}^\omega(W_{\Lambda_i}) \right\| \\
&\leq |\Lambda^1_{-\vec{n}}(\square)|^{-1} \sum_{i=1}^q \|W_{\Lambda_i}\| \\
&\leq |\Lambda^1_{-\vec{n}}(\square)|^{-1} \left(\sum_{i=1}^q |\Lambda_i| \right) \cdot \varepsilon = \varepsilon.
\end{aligned} \tag{21}$$

From (2) and the standardness of the potential,

$$\|U_{\Lambda} - U_{\Lambda^1_{-\vec{n}}(\square)}\| = \|\Gamma_{\Lambda}^\omega(H_{\partial\Lambda_{-\vec{n}}})\| \leq |\partial\Lambda_{-\vec{n}}| \cdot \|\Phi\| \tag{22}$$

From the definitions of φ_a^c and of U_{Λ_i} , $\sigma_{\vec{n}}^* \varphi_a^c(U_{\Lambda_i}) = \varphi_{\Lambda_o(a)}^c(U_{\Lambda_o(a)})$ for every $i(1 \leq i \leq q)$. Hence we have

$$\begin{aligned}
& \left| |\Lambda|^{-1} \sigma_{\vec{n}}^* \varphi_a^c(U_{\Lambda}) - |\Lambda_o(a)|^{-1} \varphi_{\Lambda_o(a)}^c(U_{\Lambda_o(a)}) \right| \\
&\leq |\Lambda|^{-1} |\sigma_{\vec{n}}^* \varphi_a^c(U_{\Lambda} - U_{\Lambda^1_{-\vec{n}}(\square)})| + |\Lambda^1_{-\vec{n}}(\square)|^{-1} |\sigma_{\vec{n}}^* \varphi_a^c(U_{\Lambda^1_{-\vec{n}}(\square)} - \sum_{i=1}^q U_{\Lambda_i})| \\
&\quad + |\Lambda^1_{-\vec{n}}(\square)|^{-1} \left(1 - \frac{|\Lambda^1_{-\vec{n}}(\square)|}{|\Lambda|} \right) |\sigma_{\vec{n}}^* \varphi_a^c(U_{\Lambda^1_{-\vec{n}}(\square)})| \\
&\leq \varepsilon + 2 \frac{|\partial\Lambda_{-\vec{n}}|}{|\Lambda|} \|\Phi\|,
\end{aligned} \tag{23}$$

where we have used (22), (21) and (2). For every $\vec{n} \in \Lambda_o(a)$, $\frac{\partial\Lambda_{-\vec{n}}}{\Lambda} \rightarrow 0$ as $\Lambda \nearrow \mathbf{Z}^\nu$ in the van Hove limit and since $\Lambda_o(a)$ is a finite subset this convergence is uniformly in $\vec{n} \in \Lambda_o(a)$. From the definition of $\widehat{\varphi}_a^c$ we obtain

$$\|e_{\Phi}(\widehat{\varphi}_a^c) - |\Lambda_o(a)|^{-1} \varphi_{\Lambda_o(a)}^c(U_{\Lambda_o(a)})\| \leq \varepsilon.$$

□

Finally, we consider the dynamical entropy term. Let $G_{a^\nu}(\vec{\sigma})$ be the subgroup of $G(\vec{\sigma})$ which is generated by $\sigma_1^a, \dots, \sigma_\nu^a$. By definition $\sigma_{\vec{n}}^* \varphi_a^c$ is in $\mathcal{A}_{+,1,a-inv}^*$ for each $\vec{n}$, thus we can define $h_{\sigma_{\vec{n}}^* \varphi_a^c}(G_{a^\nu}(\vec{\sigma}))$. By (10) and (11), we have

$$\begin{aligned} & h_{\widehat{\varphi_a^c}}(G(\vec{\sigma})) \\ &= \frac{1}{a^\nu} h_{\widehat{\varphi_a^c}}(G_{a^\nu}(\vec{\sigma})) \\ &= \frac{1}{a^\nu} \sum_{\vec{n} \in \Lambda_o(a)} \frac{1}{|\Lambda_o(a)|} h_{\sigma_{\vec{n}}^* \varphi_a^c}(G_{a^\nu}(\vec{\sigma})). \end{aligned} \quad (24)$$

By (9),

$$h_{\sigma_{\vec{n}}^* \varphi_a^c}(G_{a^\nu}(\vec{\sigma})) = h_{\varphi_a^c}(G_{a^\nu}(\vec{\sigma})). \quad (25)$$

Combining (24) and (25), we have

$$\begin{aligned} h_{\widehat{\varphi_a^c}}(G(\vec{\sigma})) &= \frac{1}{a^\nu} h_{\varphi_a^c}(G_{a^\nu}(\vec{\sigma})) \\ &\geq \frac{1}{a^\nu} h_{\varphi_a^c, G_{a^\nu}(\vec{\sigma})}(\mathcal{A}_{\Lambda_o(a)}), \end{aligned} \quad (26)$$

where the inequality follows from the definition of the dynamical entropy (8). Since φ_a^c is a product state, it follows from (14) that

$$\frac{1}{a^\nu} h_{\varphi_a^c, G_{a^\nu}(\vec{\sigma})}(\mathcal{A}_{\Lambda_o(a)}) = \frac{S_{\Lambda_o(a)}(\varphi_{\Lambda_o(a)}^c)}{|\Lambda_o(a)|}. \quad (27)$$

Thus from (26) and (27),

$$h_{\widehat{\varphi_a^c}}(G(\vec{\sigma})) \geq \frac{S_{\Lambda_o(a)}(\varphi_{\Lambda_o(a)}^c)}{|\Lambda_o(a)|}. \quad (28)$$

There exists a size α such that

$$|P(\Phi) - P_{\Lambda_o(a)}(\Phi)| \leq \frac{\varepsilon}{2} \quad (29)$$

by the existence of the infinite-volume pressure and

$$\left| e_{\Phi}(\widehat{\varphi_a^c}) - |\Lambda_o(a)|^{-1} \varphi_{\Lambda_o(a)}^c(U_{\Lambda_o(a)}) \right| \leq \frac{\varepsilon}{2} \quad (30)$$

by lemma 3.1. Now we use the following identity

$$P_{\Lambda_o(a)}(\Phi) = \frac{S_{\Lambda_o(a)}(\varphi_{\Lambda_o(a)}^c)}{|\Lambda_o(a)|} - |\Lambda_o(a)|^{-1} \varphi_{\Lambda_o(a)}^c(U_{\Lambda_o(a)}). \quad (31)$$

Combining (28), (29), (30) and (31), we obtain

$$h_{\widehat{\varphi_a^c}}(G(\vec{\sigma})) - e_{\Phi}(\widehat{\varphi_a^c}) \geq P(\Phi) - \varepsilon,$$

which proves our theorem. $\square$

4 Discussions

In this section we shall review the related topics. From the definition [6], the procedure of the calculation of the entropy of completely positive unital maps $H_\psi(\gamma_1, \dots, \gamma_k)$ is to choose finite decompositions of the state ψ as closely as possible to the optimal decomposition. For the special cases such as classical states and product states, the optimal decompositions are obvious. But in general, it is hard to find the optimal decompositions for given states. In comparison with the Narnhofer's proof [13] we avoid the difficulties in taking the optimal decompositions of the states by using the approximation as we have shown. As regards our main result, it should be noted that we do not claim that $h_\rho(G(\vec{\sigma})) = s(\rho)$ for translation invariant equilibrium states $\rho \in I_{\beta\Phi}$. The entropy density always dominates the dynamical entropy of space translation thus we know $h_\rho(G(\vec{\sigma})) \leq s(\rho)$. There has been shown some examples for which these two entropies take the same value. For the classical case, $s(\omega) = h_\omega(G(\vec{\sigma}))$ for every $\omega \in \mathcal{A}_{+,1,inv}^*$. For the quantum case it has been shown that this equality holds for the Generalized Markov Chains in [16](see also [3]). So far, it has not been found the example $\omega \in \mathcal{A}_{+,1,inv}^*$ for which strict inequality $h_\omega(G(\vec{\sigma})) < s(\omega)$ holds. We also think that it might be difficult to show that $s(\omega) = h_\omega(G(\vec{\sigma}))$ for every $\omega \in \mathcal{A}_{+,1,inv}^*$ in the quantum case.

This problem is connected to the continuity properties of $h_\omega(G(\vec{\sigma}))$ in $\omega \in \mathcal{A}_{+,1,inv}^*$. It has been shown that for the fixed $X \in \mathcal{F}$, $\psi \mapsto h_{\psi,G(\vec{\sigma})}(X)$ is weak* upper semi-continuous function on $\mathcal{A}_{+,1,inv}^*$ [14]. But unfortunately the weak* upper semi-continuity property of $h_\psi(G(\vec{\sigma})) = \sup_{X \in \mathcal{F}} h_{\psi,G(\vec{\sigma})}(X)$ is not known as yet. If we assume that $h_\omega(G(\vec{\sigma}))$ has the weak* upper semi-continuity, we can derive the affirmative answer for this question as follows. From our main theorem,

$$P(\Phi) = \sup_{\omega \in \mathcal{A}_{+,1,inv}^*} [h_\omega(G(\vec{\sigma})) - \beta \cdot e_\Phi(\omega)].$$

By the assumption of the weak* upper semi-continuity of $h_\omega(G(\vec{\sigma}))$ in $\omega \in \mathcal{A}_{+,1,inv}^*$, we can apply the general result of the Legendre transform to the present case and we obtain for $\rho \in I_{\beta\Phi}$

$$h_\rho(G(\vec{\sigma})) = P(\Phi) + \beta \cdot e_\Phi(\rho) \quad (32)$$

Thus we have

$$h_\rho(G(\vec{\sigma})) = s(\rho) (= P(\Phi) + \beta \cdot e_\Phi(\rho)). \quad (33)$$

Moreover, we have the following strong result under the same assumption that $h_\omega(G(\vec{\sigma}))$ has the weak* upper semi-continuity. Let $\omega \in \mathcal{A}_{+,1,inv}^*$ be an arbitrary translation invariant state. Let $\{\Lambda_m\}$ be an increasing sequence of finite subsets of $\mathbb{Z}^v$ such that $\Lambda_m \nearrow \mathbb{Z}^v$ in the van Hove limit. $\Lambda_0(a)$ is the finite subset as

defined before $\{x \in \mathbf{Z}^\nu | 0 \leq x_i \leq a-1, i = 1, \dots, \nu\}$. We denote the restriction of ω to $\mathcal{A}_{\Lambda_o(a)}$ by $\omega_{\Lambda_o(a)}$. We define $\omega_a \equiv \bigotimes_{\vec{n} \in \mathbf{Z}^\nu} \sigma_{a\vec{n}}^* \omega_{\Lambda_o(a)} \in \mathcal{A}_{+,1,a-inv}^*$ which is a periodic state invariant under $(a\mathbf{Z})^\nu$. From ω_a we construct an averaged state $\widehat{\omega}_a$ which is translation invariant:

$$\widehat{\omega}_a \equiv \sum_{\vec{n} \in \Lambda_o(a)} \frac{\sigma_{\vec{n}}^* \omega_a}{|\Lambda_o(a)|} \in \mathcal{A}_{+,1,inv}^*.$$

We then repeat the discussions in the proof of our min theorem. By the same arguments as in (24) , we have

$$\begin{aligned} & h_{\widehat{\omega}_a}(G(\vec{\sigma})) \\ &= \frac{1}{a^\nu} h_{\omega_a}(G_{a^\nu}(\vec{\sigma})) \\ &= \frac{1}{a^\nu} \sum_{\vec{n} \in \Lambda_o(a)} \frac{1}{|\Lambda_o(a)|} h_{\sigma_{\vec{n}}^* \omega_a}(G_{a^\nu}(\vec{\sigma})). \end{aligned} \quad (34)$$

By (9) ,

$$h_{\sigma_{\vec{n}}^* \omega_a}(G_{a^\nu}(\vec{\sigma})) = h_{\omega_a}(G_{a^\nu}(\vec{\sigma})). \quad (35)$$

Combining (34) and (35) , we have

$$h_{\widehat{\omega}_a}(G(\vec{\sigma})) = \frac{1}{a^\nu} h_{\omega_a}(G_{a^\nu}(\vec{\sigma})) \quad (36)$$

$$\geq \frac{1}{a^\nu} h_{\omega_a, G_{a^\nu}(\vec{\sigma})}(\mathcal{A}_{\Lambda_o(a)}), \quad (37)$$

and we note the following equality

$$\frac{1}{a^\nu} h_{\omega_a, G_{a^\nu}(\vec{\sigma})}(\mathcal{A}_{\Lambda_o(a)}) = \frac{S_{\Lambda_o(a)}(\omega_{\Lambda_o(a)})}{|\Lambda_o(a)|}. \quad (38)$$

Thus

$$h_{\widehat{\omega}_a}(G(\vec{\sigma})) \geq \frac{S_{\Lambda_o(a)}(\omega_{\Lambda_o(a)})}{|\Lambda_o(a)|}.$$

As $a \rightarrow \infty$ the right hand side converges to the entropy density $s(\omega)$, and by definition $\widehat{\omega}_a \rightarrow \omega$ in the weak* topology. Thus our assumption yields

$$h_\omega(G(\vec{\sigma})) = \limsup_a h_{\widehat{\omega}_a}(G(\vec{\sigma})) \geq s(\omega). \quad (39)$$

Since the inverse inequality always holds, thus we conclude

$$h_\omega(G(\vec{\sigma})) = s(\omega). \quad (40)$$

The difficulty of this problem is connected to the fundamental difference between the quantum state and the classical probability in the description of composite systems. To explain this feature, we will raise one simple example and then consider the meaning of the difference between the algebraic entropy H and the von Neumann entropy S . Let $|s\rangle \equiv \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $\psi_{1,2}$ be its vector state on $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. The restriction of $\psi_{1,2}$ to $M_2(\mathbb{C}) \otimes 1$ denoted by ψ_1 is a mixed state whose density matrix is $\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We easily see that $S(\psi_1) = \log 2 > S(\psi_{1,2}) = 0$. The former is a mixed state on $M_2(\mathbb{C})$ with greater uncertainty than the later which is a pure state on $M_4(\mathbb{C})$. In contradiction to the classical case, $S_\Lambda(\psi)$ is not generally monotonic in regions Λ for the quantum case. This property comes from quantum mechanical correlations between regions. However the algebraic entropy has the monotone property in regions, that is for every $X, Y \in \mathcal{F}$ the inequality $H_\psi(X) \leq H_\psi(X \cup Y)$ holds, and this is one of the crucial points of [6]. For the above example, we have $H_{\psi_{1,2}}(M_2(\mathbb{C}) \otimes 1) = H_{\psi_{1,2}}(M_2(\mathbb{C}) \otimes M_2(\mathbb{C})) = 0$.

Let ρ be a translation invariant pure state, i.e. ρ is in $\mathcal{A}_{+,1,inv}^*$ and does not have non-trivial decompositions in $\mathcal{A}_{+,1}^*$. From the definition we have $h_\rho(G(\vec{\sigma})) = 0$. Now the following question arises. Whether or not $s(\rho)$ is equals to 0? There has been a subclass of states on quantum spin chains(1-dimensional quantum lattice systems) called "quantum Markov chains" [1]. It has been shown in [10] that the entropy density of a purely generated C^* -finitely correlated state vanishes. (See also [8] and [9] for the reference of the C^* -finitely correlated states.) They are defined by giving an explicit formula for every finite volume expectation in terms of certain auxiliary objects. Roughly speaking, they are constructive states and their quantum correlations are not so strong. If there exists the translation invariant pure state which has non-vanishing entropy density, it can be said that this state has strongly long range quantum mechanical correlations.

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