Is bi-maximal mixing compatible with the large angle MSW solution of the solar neutrino problem?

C. Giunti

INFN, Sezione di Torino, and Dipartimento di Fisica Teorica, Università di Torino, Via P. Giuria 1, I-10125 Torino, Italy

Abstract

It is shown that the large angle MSW solution of the solar neutrino problem with a bi-maximal neutrino mixing matrix implies an energy-independent suppression of the solar $\nu_e$ flux. The present solar neutrino data exclude this solution of the solar neutrino problem at 99.6% CL.
The possibility that the neutrino mixing matrix $U$ has the bi-maximal mixing form

$$U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}}
\end{pmatrix}$$

(1)

has attracted a large attention [1] after the presentation at the Neutrino '98 conference of the Super-Kamiokande evidence in favor of atmospheric neutrino oscillations with large mixing [2].

Neutrino bi-maximal mixing is capable of explaining in a elegant way the atmospheric neutrino anomaly [2, 3, 4, 5], through $\nu_\mu \rightarrow \nu_\tau$ oscillations due to $^1 \Delta m^2_{31} \sim 10^{-3} \text{eV}^2$ and the solar neutrino problem (SNP) [6, 7, 8, 9, 10] through $\nu_e \rightarrow \nu_\mu, \nu_\tau$ oscillations in vacuum due to $\Delta m^2_{21} \sim 10^{-10} \text{eV}^2$ [11].

As noted in [12], the results of the recent analysis of solar neutrino data presented in [13] seem to imply$^2$ that neutrino bi-maximal mixing may be also compatible at 99\% CL with the large mixing angle (LMA) MSW [14] solution of the SNP [15] (see Fig. 2 of [13]).

Here I would like to notice that this conclusion seems to be in contradiction with the exclusion at 99.8\% CL of an energy-independent suppression of the solar $\nu_e$ flux presented in the same paper [13] (see section IV.D).

The reason of this incompatibility is that bi-maximal mixing with the $\Delta m^2_{21} \sim 10^{-5} - 10^{-4} \text{eV}^2$ corresponding to the LMA solution of the SNP implies an energy-independent suppression by a factor 1/2 of the solar $\nu_e$ flux.

This can be seen following the simple reasoning presented in [16]. The mixing of the neutrino states in vacuum is given by (see, for example, [17])

$$|\nu_\alpha\rangle = \sum_{k=1,2,3} U^*_{\alpha k} |\nu_k\rangle \quad (\alpha = e, \mu, \tau),$$

(2)

where the states $|\nu_\alpha\rangle$ ($\alpha = e, \mu, \tau$) describe neutrinos produced in weak interaction processes and the states $|\nu_k\rangle$ ($k = 1, 2, 3$) describe neutrinos with definite masses $m_k$.

In the bi-maximal mixing scenario the numbering of the massive neutrinos is the usual one, i.e. such that $m_1 \leq m_2 \leq m_3$, and $\Delta m^2_{31} \sim 10^{-3} \text{eV}^2$ for the solution of the atmospheric neutrino anomaly. If $\Delta m^2_{21} \sim 10^{-5} - 10^{-4} \text{eV}^2$ for the LMA solution of the SNP, we have $\Delta m^2_{32} \simeq \Delta m^2_{31} \sim 10^{-3} \text{eV}^2$.

Solar neutrinos have energy $E \sim 1 \text{MeV}$ and the ratio $\Delta m^2_{31}/E \simeq \Delta m^2_{32}/E \sim 10^{-9} \text{eV}$ is much larger than the matter induced potential $V \lesssim 10^{-11} \text{eV}$ in the interior of the sun. Hence, the evolution equation of the heaviest massive neutrino $\nu_3$ is decoupled from that of the two light neutrinos $\nu_1$ and $\nu_2$ (see, for example, [18]). Taking also in account that in the case of bi-maximal mixing $U_{e3} = 0$, one can see that an electron neutrino

$^1$ $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$ is the difference between the squared masses of the two massive neutrinos $\nu_i$ and $\nu_j$. In the bi-maximal mixing scenario there are three massive neutrinos, $\nu_1, \nu_2$ and $\nu_3$.

$^2$I want to emphasize from the beginning that I do not want to criticize at all the beautiful paper [13]. I am only concerned with the interpretation of its results.
is created in the core of the sun as a superposition of the two light mass eigenstates $\nu_1$ and $\nu_2$ and, whatever happens during his propagation in the interior of the sun, its state when it emerges from the surface of the sun is a linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$:

$$|\nu\rangle_s = \sum_{k=1,2} a_k |\nu_k\rangle,$$

with

$$|a_1|^2 + |a_2|^2 = 1.$$

Since the massive neutrino states $|\nu_k\rangle$ propagate as plane waves, the state describing the neutrino detected on the Earth is

$$|\nu\rangle_E = \sum_{k=1,2} a_k e^{-iE_kL} |\nu_k\rangle,$$

where $L$ is the distance from the surface of the Sun to the detector on the Earth. The survival probability of solar electron neutrinos is then given by $P_{\nu_e \rightarrow \nu_e} = |\langle \nu_e |\nu\rangle_E|^2$:

$$P_{\nu_e \rightarrow \nu_e} = \left| \sum_{k=1,2} a_k e^{-iE_kL} \langle \nu_e |\nu_k\rangle \right|^2 = \left| \sum_{k=1,2} a_k e^{-iE_kL} U_{ek} \right|^2.$$

Taking now into account the explicit values $U_{e1} = 1/\sqrt{2}$ and $U_{e2} = -1/\sqrt{2}$ in the case of bi-maximal mixing and the fact that the neutrinos are extremely relativistic, we have

$$P_{\nu_e \rightarrow \nu_e} = \frac{1}{2} \left| a_1 - a_2 \exp \left( -i \frac{\Delta m^2_{21} L}{2E} \right) \right|^2.$$

In the case of the LMA solution of the SNP $\Delta m^2_{21} \sim 10^{-5} - 10^{-4} \text{eV}^2$ and the survival probability (7) oscillates with an oscillation length $4\pi E/\Delta m^2_{21} \sim 10^7 \text{cm}$ that is about one million times smaller than the Sun–Earth distance. Hence, the oscillations are not observable on the Earth because of averaging over the energy spectrum and only the average probability

$$\langle P_{\nu_e \rightarrow \nu_e} \rangle = \frac{1}{2} \left( |a_1|^2 + |a_2|^2 \right) = \frac{1}{2}$$

is observable. We have obtained the announced result: the LMA solution of the SNP in the bi-maximal mixing scenario implies an energy-independent suppression of the solar $\nu_e$ flux of a factor $1/2$.

Therefore, we have the apparent paradox that an energy-independent suppression of the solar $\nu_e$ flux seems to be allowed at 99% CL by Fig. 2 of Ref. [13] and is excluded at 99.8% CL in Section IV.D of the same paper. Notice that the two conclusions are based on the same set of data and the same theoretical calculation of the neutrino flux produced by thermonuclear reactions in the core of the sun [19].

The fact that the two cases refer to the same physical situation, i.e., an energy-independent suppression of the solar $\nu_e$ flux, is also shown by the $\chi^2$ calculated in the
two cases. The $\chi^2$ of the right border of the LMA region\textsuperscript{3} in Fig. 2 of Ref. [13] is $4.3 + 9.2 = 13.5$, whereas the $\chi^2$ calculated in Section IV.D of the same paper for an energy-independent suppression of the solar $\nu_e$ flux by a factor 0.48 is 12.0. Since this is the best fit for an energy-independent suppression of the solar $\nu_e$ flux, a value of $\chi^2 = 13.5$ for a suppression factor 0.5 looks plausible.

The solution of the apparent paradox explained above lies in a correct statistical interpretation of the allowed LMA region in Fig. 2 of Ref. [13] and of the exclusion in Section IV.D of the same paper. The two cases have different statistical meanings.

The allowed regions in Fig. 2 of Ref. [13] are obtained under the assumption that the neutrino masses and mixing parameters are not known. In this case a general neutrino oscillation formula is used in the fit, with the neutrino masses and mixing angles considered as free parameters. The best fit in the LMA region happens to have a $\chi^2_{\text{min}} = 4.3$, which corresponds to a CL of 3.8% with 1 DOF. Hence, a LMA solution is allowed at 3.8% CL and one can draw a 99% CL region corresponding to the parameters that have $\chi^2 \leq \chi^2_{\text{min}} + 9.2$.

The statistical analysis discussed in Section IV.D of Ref. [13] assumes that the solar $\nu_e$ flux is suppressed by a constant factor that is the free parameter to be determined by the fit. It happens that the best fit has $\chi^2_{\text{min}} = 12.0$, which corresponds to a CL of 0.2% with 2 DOF. Hence, the hypothesis is excluded at 99.8% CL and no allowed region of the free parameter can be drawn.

Since the two statistical analyses start from different assumptions, it is clear that they answer different questions and their conclusions cannot be compared. Moreover, it is important to notice that the test of the maximal mixing scenario with $\Delta m^2_{21} \sim 10^{-6} - 10^{-4} \text{eV}^2$ does not correspond to either of the two statistical analyses. Indeed, if this scenario is assumed, we know that the solar $\nu_e$ flux is suppressed by an energy-independent factor 0.5 and there is no parameter to fit. Hence we test the hypothesis under consideration on the basis of its $\chi^2$. The $\chi^2 \simeq 13.5$ indicated by Fig. 2 of Ref. [13] implies a CL of 0.4% with 3 DOF. Therefore, the hypothesis is rejected at 99.6% CL.

Notice that this exclusion is based only on the values of the elements $U_{\alpha 1}$, $U_{\alpha 2}$ and $U_{\alpha 3}$ of the neutrino mixing matrix. This means that also other types of neutrino mixing matrix, as those discussed in [20], are incompatible with the LMA solution of the SNP.

In conclusion, I would like to emphasize that the allowed regions of the neutrino oscillation parameters calculated in the usual way (i.e. as Fig. 2 of Ref. [13]) cannot be used to test a definite model (as the bi-maximal mixing model) because they have been obtained under different assumptions\textsuperscript{4}. In order to obtain allowed regions appropriate for model testing one must use the procedure described in [21, 22], i.e. one must consider each point of the parameter space as a model and perform a goodness of fit testing with it. I think that it would be very useful if both types of allowed regions will be presented in future papers.

\textsuperscript{3}If $U_{\alpha 3} = 0$, we have $\sin^2 2\theta = 4|U_{\alpha 1}|^2 |U_{\alpha 2}|^2$ (see [11]) and $\sin^2 2\theta = 1$ corresponds to $|U_{\alpha 1}| = |U_{\alpha 2}| = 1/\sqrt{2}$, as in the bi-maximal mixing matrix (1).

\textsuperscript{4}They are useful if one wants to know the allowed range of the mixing parameters for other purposes.
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References


